Reduction of Grassmannian mesh friezes.

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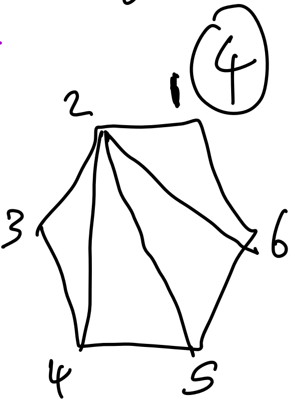
__Aim__: give a categorical proof of reduction results for Grassmannian mesh friezes

Consider a Coxeter-Conway frieze, F, arising from a triangulation T of a convex polygon P_n:

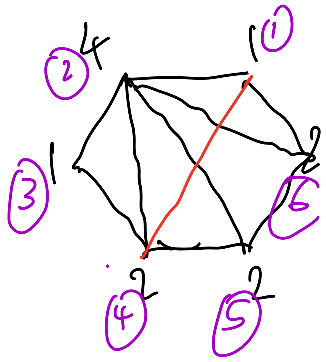
$$\overset{12}{1}\quad \overset{23}{1}\quad \overset{34}{1}\quad \overset{45}{1}\quad \overset{56}{1}\quad \overset{16}{1}\quad \overset{12}{1}\quad \overset{23}{1}\quad \overset{34}{1}\quad \overset{45}{1}$$
$$\overset{13}{4}\quad \overset{24}{1}\quad \overset{35}{2}\quad \overset{46}{2}\quad \overset{15}{2}\quad \overset{26}{1}\quad \overset{13}{4}\quad \overset{24}{1}\quad \overset{35}{2}$$
$$\overset{14}{3}\quad \overset{25}{1}\quad \overset{36}{3}\quad \overset{14}{3}\quad \overset{25}{1}\quad \overset{36}{3}\quad \overset{14}{3}\quad \overset{25}{1}$$
$$\overset{46}{2}\quad \overset{15}{2}\quad \overset{26}{1}\quad \overset{13}{4}\quad \overset{24}{1}\quad \overset{35}{2}\quad \overset{46}{2}\quad \overset{15}{2}\quad \overset{26}{1}$$
$$\overset{45}{1}\quad \overset{56}{1}\quad \overset{16}{1}\quad \overset{12}{1}\quad \overset{23}{1}\quad \overset{34}{1}\quad \overset{45}{1}\quad \overset{56}{1}\quad \overset{16}{1}\quad \overset{12}{1}$$



n sides.

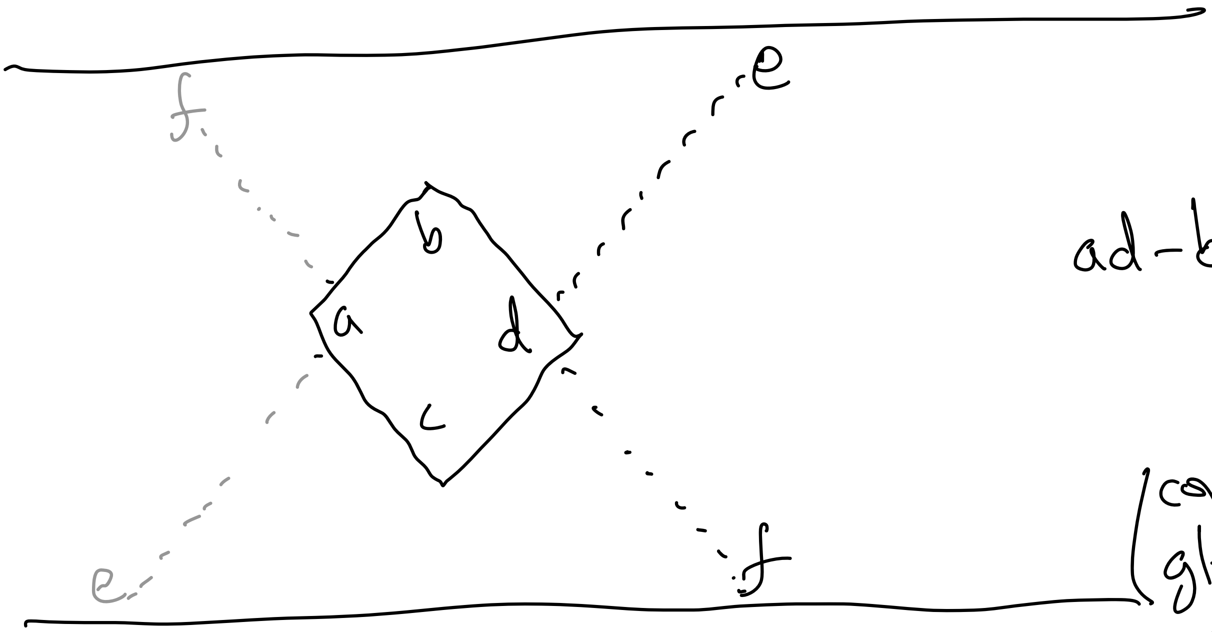
We can restrict F to a subpolygon of P.

$$\overset{12}{1}\quad \overset{23}{1}\quad \overset{34}{1}\quad \overset{14}{3}\quad \overset{12}{1}\quad \overset{23}{1}\quad \overset{34}{1}\quad \overset{14}{3}$$
$$\overset{13}{4}\quad \overset{24}{1}\quad \overset{13}{4}\quad \overset{24}{1}\quad \overset{13}{4}\quad \overset{24}{1}\quad \overset{13}{4}\quad \overset{24}{1}$$
$$\overset{14}{3}\quad \overset{12}{1}\quad \overset{23}{1}\quad \overset{34}{1}\quad \overset{14}{3}\quad \overset{12}{1}\quad \overset{23}{1}\quad \overset{34}{1}$$



We then obtain, by a result of Cuntz-Holm-Jørgensen, a tame frieze pattern with coefficients (see also Maldonaldo).

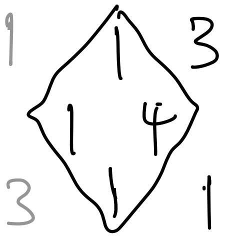
This must have integer entries, non-zero coefficients, 3x3 adjacent determinants vanish, and satisfy :



$ad - bc = ef.$

(conditions ⇒ glide symmetry → other e & f.)

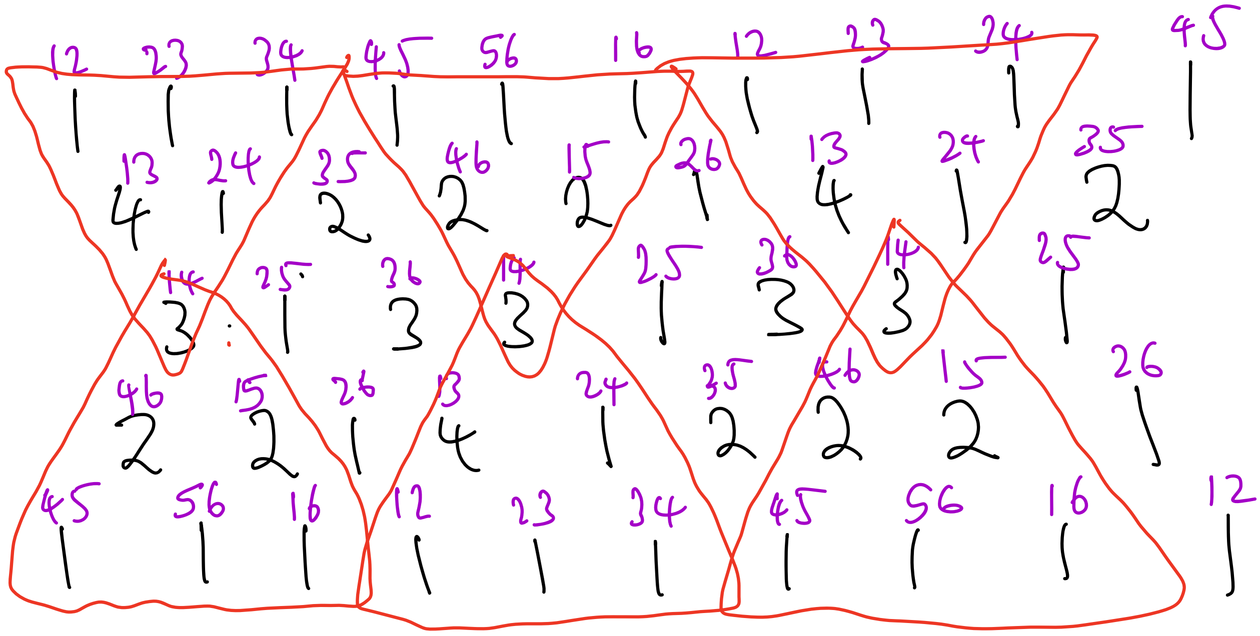
e.g.



$1 \times 4 - 1 \times 1 = 3 \times 1 ..$

Thus, it is the same as a Coxeter-Conway frieze pattern, if all the coefficients are equal to 1 and entries positive. Can this be interpreted categorically?

Going back to the initial Coxeter-Conway frieze pattern:



We can consider removing 25, 35, 26, 36, i.e. the arcs crossing 13: in this we case we obtain a second frieze in addition. Thus we are taking all the entries "orthogonal" to 14 in a geometric sense.

We can interpret this categorically by passing to the Grassmannian cluster category, introduced by Jensen-King-Su

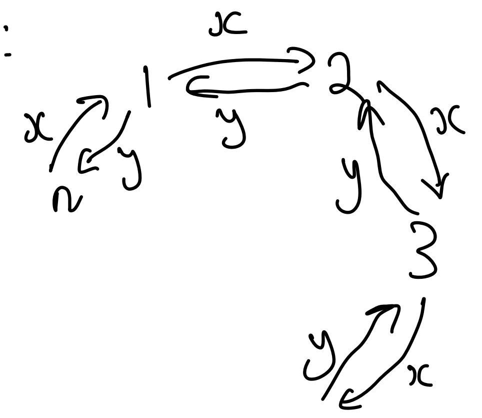
Their aim was to model the Scott cluster algebra structure of the Grassmannian $Gr(k,n)$ (c.f. Khrystyna Serhiyenko's talk).

$\mathbb{C}[Gr(k,n)]$ = homogeneous coordinate ring of Grassmanian.

$\overset{\cup}{p_I}$ Plücker coordinate $I \in \binom{[n]}{k}$

Initial categorical models were found by Geiss-Leclerc-Schröe but they missed just one cluster variable. This gap was closed by Jensen-King-Su, with a new idea:
2016

Q:



$$A(k,n) = \frac{kQ}{\left(\begin{array}{l} xy = yx \\ y^k = x^{n-k} \end{array}\right)}.$$

$\mathcal{C}(k,n)$ is the category of maximal Cohen-Macaulay modules over $A(k,n) = \{M : \mathrm{Ext}^1(M,\Lambda)=0\}$.

The defining condition forces all projective modules to become injective in $\mathcal{C}(k,n)$, so it becomes a Frobenius exact category:

- exact (notion of exact sequences)
- enough projectives ($\forall M \exists P \twoheadrightarrow M$, P proj.)
- enough injectives
- projectives & injectives coincide

This models the cluster structure of $\mathbb{C}[Gr(k,n)]$ as in Khrystyna Serhiyenko's talk.

Plücker coordinates, in particular $\leftrightarrow$ rank 1 modules
$p_I \leftrightarrow M_I$ rigid.

$\text{rank } M = \text{len}(M \otimes_Z K)$
K = field of fractions of Z = centre of $A(k,n)$.

Grassmannian mesh friezes.

Baur–Faber–Gratz–Serhiyenko–Todorov studied friezes arising from Grassmannian cluster categories.

$\mathcal{C}(k,n)$ $\overset{[JKS]}{\text{has}}$ $\overset{\boxed{k\leq n/2}}{\text{finite type}}$ $(|\text{Ind } \mathcal{C}(k,n)| < \infty) \iff$
$(k,n) = (2,n), (3,6), (3,7), (3,8)$.

Then a <u>mesh frieze</u> for $Gr(k,n)$ is a function
$F: \text{Ind } \mathcal{C}(k,n) \to \mathbb{N}$ (positive integers) such that

$F(P) = 1$ $\forall$ projective-injective modules P
and, whenever

$$0 \to A \to \bigoplus_{i=1}^{\overset{\text{index}}{l}} B_i \to C \to 0$$

is an Auslander-Reiten sequence in $\mathcal{C}(k,n)$, $F(A)F(C) = \prod_{i=1}^{l} F(B_i) + 1$.

Thus a mesh frieze is drawn on the Auslander-Reiten quiver, with vertices $\iff$ ind $\mathcal{C}(k,n)$.
If $k=2$, then ind $\mathcal{C}(k,n) = \{M_I : I \text{ a 2-subset of } \{1,\dots,n\}\}$
& we see that the vertices $\iff$ diagonals in P_n.
$\iff$ entries in fundamental domain of a CC frieze.
In fact, if a CC-frieze L for P_n is drawn periodically, it coincides with a mesh frieze of $\mathcal{C}(2,n)$.

Now fix $M \in \mathrm{Ind}\,\mathcal{C}(k,n)$.

Set $M^{\perp} = \{X \in \mathcal{C}(k,n) : \overset{\mathrm{Ext}^1(M,X)}{\overset{\shortparallel}{\mathrm{Ext}^1(X,M)}} = 0\}$

<u>Theorem</u> (Iyama–Yoshino, key special case). 2008.

$\mathcal{C}$ 2-Calabi-Yau triangulated category: (2-CY: Hom-finite & $\mathrm{Ext}^1(X,Y) \simeq D\,\mathrm{Ext}^1(Y,X)$ functorially)

$M \in \mathcal{C}$ rigid (i.e. $\mathrm{Ext}^1_{\mathcal{C}}(M,M)=0$)

Then $\dfrac{M^{\perp}}{\mathrm{add}(M)}$ is 2-Calabi-Yau triangulated.

Applying this, we obtain

$\dfrac{M^{\perp}}{\mathrm{add}\,M} \simeq$ stable category of a smaller Grassmannian cluster category.

BFGST 2018: If $F(M)=1$ then there is an induced frieze F_M on $\dfrac{\mathcal{C}(k,n)}{\mathcal{P}}$ (extended by 1 on projectives) obtained by restricting F. – combinatorial proof.

We can check that this is the same as restricting F to a subpolygon as in Cuntz-Holm-Jørgensen (2020) ← later paper, if $k=2$.

Iyama-Yoshino's result works on the level of the triangulated category, but $\mathcal{C}(k,n)$ is Frobenius.
Its stable category is triangulated but we want to include the frozen variables (consecutive Plückers).

Solution work with $M^{\perp} \subseteq \mathcal{C}(k,n)$.

This turns out to be Frobenius exact again, by a result of Buan-Iyama-Reiten-Scott. In fact, we have:

Buan-Iyama-Reiten Scott - exact case,
Theorem 1 Faber-M-Pressland - extriangulated case.
sense of Nakaoka-Palu.

$\mathcal{F}$ — stably 2-Calabi-Yau, Krull-Schmidt Frobenius extriangulated category. $M \in \mathcal{F}$ rigid.
Then $M^{\perp_1}$ is a stably 2-Calabi-Yau, Krull-Schmidt Frobenius extriangulated category, with projective-injectives $\mathrm{add}(M) \oplus \mathcal{P}(\mathcal{F})$.

Theorem 2.

$\mathcal{F}$ - stably 2-Calabi-Yau Frobenius extriangulated, Krull-Schmidt.

$M \in \mathcal{F}$ rigid.

Then $\exists$ triangle equivalence

$$\underline{M^{\perp_1}} \simeq M^{\perp_1}_{\underline{\mathcal{F}}} / \mathrm{add}(M).$$

This gives an "explanation" of Iyama-Yoshino's result in the triangulated case: a triangulated category is Frobenius extriangulated with projectives zero.
Then $M^{\perp_1}$ is extriangulated with projectives $\mathrm{add}(M)$.
So we see stable category of $M^{\perp_1}$ is IY reduction $\frac{M^{\perp_1}}{\mathrm{add}(M)}$.

From this, we can obtain a categorical proof of the restriction result using cluster characters.
We can also obtain restricted friezes on other finite Grassmannian cluster categories: the coefficients then appear in unexpected places.

Cluster Characters.

Let $\mathcal{F}$ be a stably 2-Calabi-Yau Frobenius extriangulated category which is Krull-Schmidt.

$T \in \mathcal{F}$ cluster-tilting object (≅ cto of $M^{\perp_1}$.) containing M as summand.

Assume $\mathrm{End}_{\mathcal{F}}(T)$ has finite global dimension.

By Fu-Keller (Wang-Wei-Zhang &) ∃ cluster character $X_T^{\mathcal{F}}$ on $\mathcal{F}$.

$M \in \mathcal{F}$ direct summand of T. (arguments go through). cf Pressland '22, 6.8

Then $M^{\perp_1}$ is also exact and Krull-Schmidt.

Theorem ⇒ it is stably 2-Calabi Yau Frobenius extriangulated

⟶ cluster character, $X_T^{M^{\perp_1}}$

Theorem [FMP]

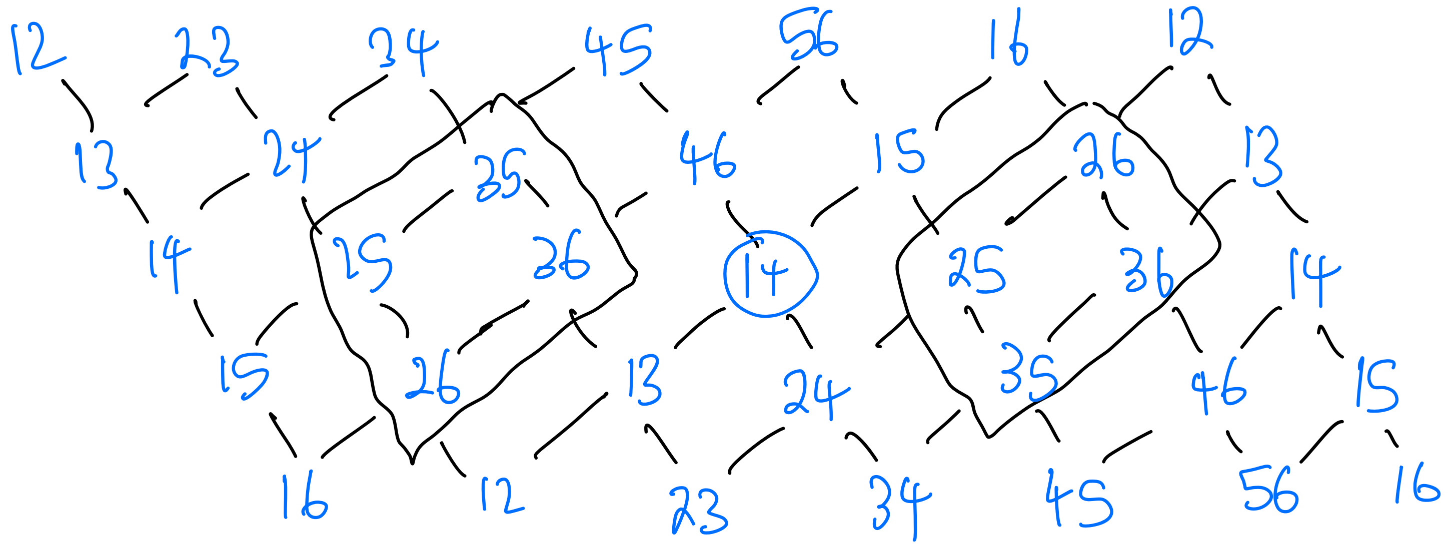
In the above situation,

$$X^T_{M^{\perp_1}} = X^T_{\mathcal{F}}\Big|_{M^{\perp_1}}.$$

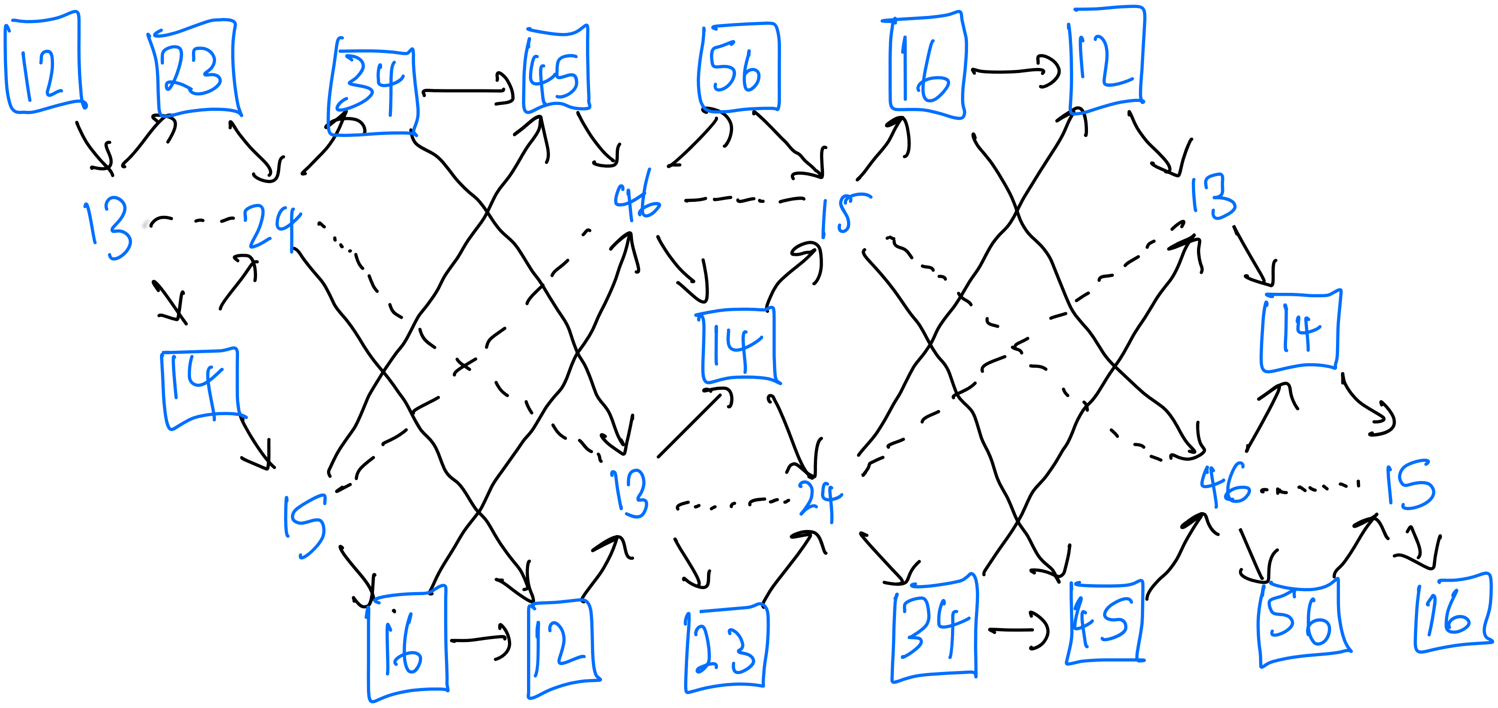
Idea is then that a mesh frieze is given by specialising a cluster character (choose agreement on slice = cluster & specialise it). So restriction is cluster character on $M^{\perp_1}$ and thus specialises also to a mesh frieze there.

Gr(2,n) Example.

The AR-quiver of $\mathcal{C}(2,n)$ is :



Deleting the indecomposables not in $M_{14}^{\perp_1}$, we get the AR-quiver of $M_{14}^{\perp_1}$:



This can be redrawn as follows: