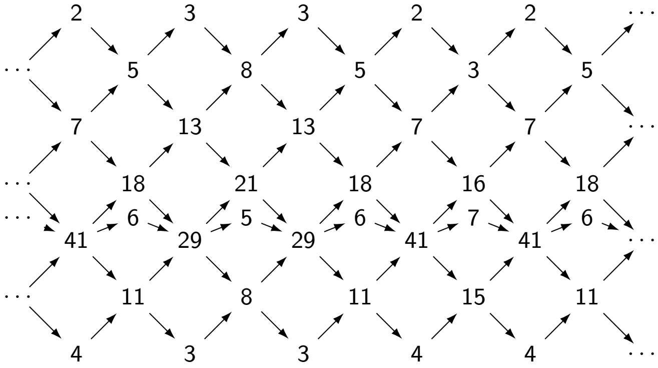
Greg Muller May 16th, 2025

The finiteness ~~and enumeration~~ of Dynkin friezes



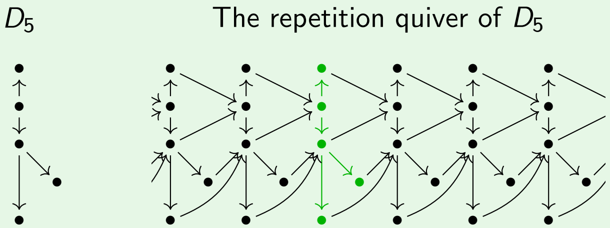
Friezes of type Q

A preliminary definition

The **repetition quiver** of an acyclic quiver Q consists of

- $\mathbb{Z}$-many copies of Q (informally called **slices**), with
- arrows from each slice to the next, opposite each arrow in Q.

Example: The repetition quiver of type D_5



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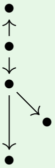

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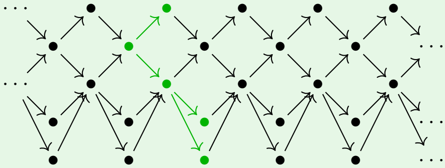

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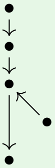

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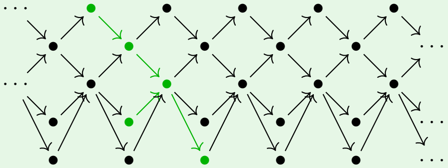

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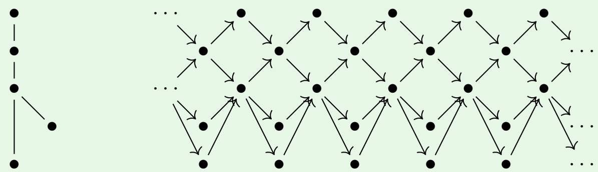

Any two orientations of a tree have equivalent repetition quivers.

Definition: A frieze of type Q

A **frieze of type Q** (or **Q-frieze**) puts of a positive integer on each vertex of the repetition quiver of Q, satisfying the **mesh relations**: The product of any two horizontally adjacent values equals

$$1 + \prod \text{intermediate values}$$

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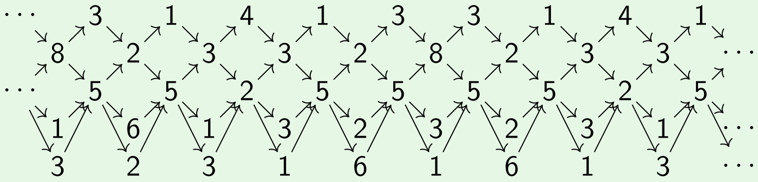

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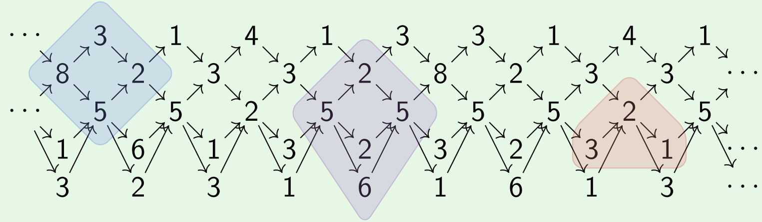
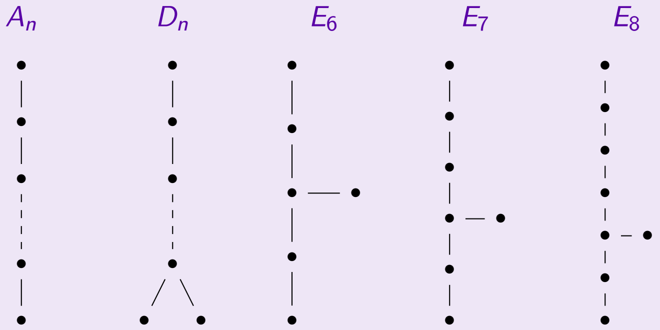


$8 \cdot 2 = 1 + 3 \cdot 5$ $\quad 5 \cdot 5 = 1 + 2 \cdot 2 \cdot 6$ $\quad 3 \cdot 1 = 1 + 2$

We will be most interested in friezes of the following types.

The (simply laced) Dynkin diagrams



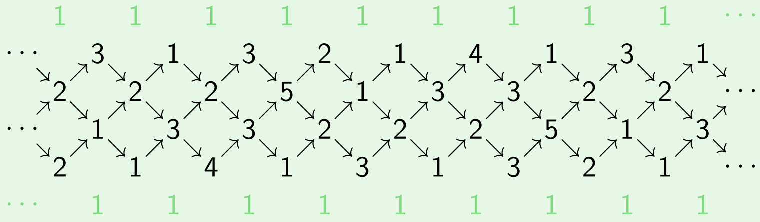
The non-simply-laced Dynkin diagrams B_n, C_n, F_4, and G_2 require defining friezes of labelled quivers (which is possible but clunky).

Prop: Relation to (original) friezes

A frieze of type A_n is equivalent to a Conway-Coxeter frieze with n-many non-trivial rows.

Just delete the arrows and add rows of 1s on the top and bottom.

Example: A frieze of type A_4



Theorem [Conway-Coxeter, 73]

A_n-friezes are in bijection with triangulations of an $(n+3)$-gon.

In particular, there are finitely many A_n-friezes.

Natural question

For which acyclic quivers Q are there only finitely many Q-friezes?

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Answer [GM22,M23]

An acyclic quiver admits finitely many friezes iff it is Dynkin.

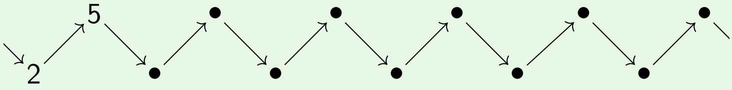
If Q is not Dynkin, there are infinitely many via cluster algebras.
If Q is Dynkin, there are finitely many Q-friezes via two proofs:

- A non-constructive proof via the geometry of cluster algebras.
- An explicit bound on values via Cartan matrices.

Connection to cluster algebras

If we put non-zero values on the initial slice of the repetition quiver, we can compute the other values via the mesh relations.

Example (Type A_2)



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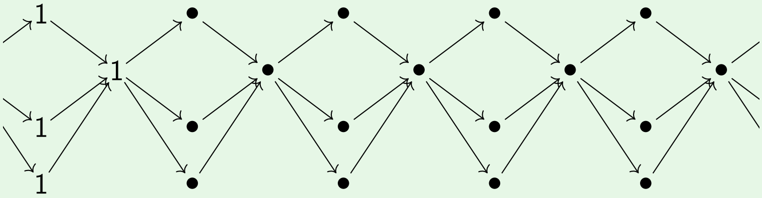

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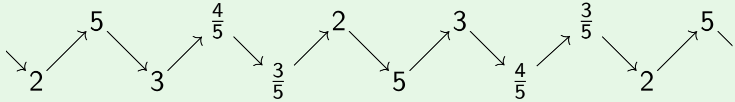

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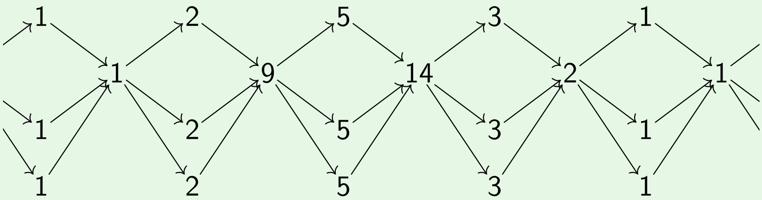

Note the resulting values will positive but not necessarily integers.

Let's derive formulas for the other values in terms of the initial ones by putting generic invertible variables on the initial slice.

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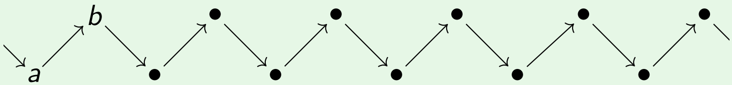

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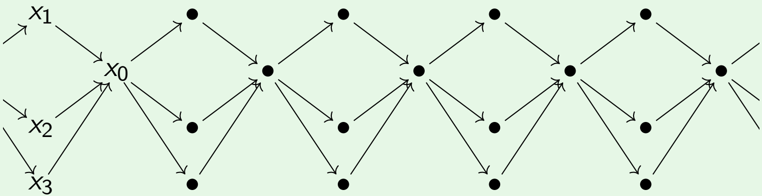

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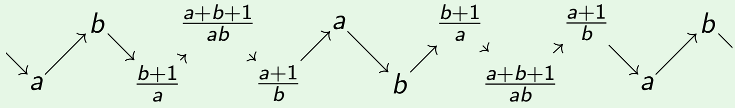

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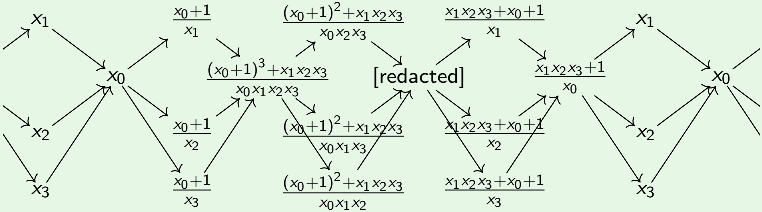


Hey, these are **cluster variables** in the cluster algebra $\mathcal{A}(Q)$ of Q!

Perspective: $\mathcal{A}(Q)$-valued friezes

We can interpret the previous diagrams as Q-friezes with values in the cluster algebra $\mathcal{A}(Q)$, rather than in the positive integers.

The preceding construction can be repeated for any acyclic quiver, and the resulting values will still be cluster variables.

Claim

For each Q, there is a unique frieze of type Q with values in $\mathcal{A}(Q)$ whose values on the initial slice are the initial cluster variables.

We will call this the **generic frieze of type Q**.

The generic Q-frieze is 'universal' in that any (positive integral) Q-frieze can be obtained as a specialization of its values.

Definition: Frieze points

A **frieze point** of a cluster algebra $\mathcal{A}$ is a ring homomorphism

$$f : \mathcal{A} \to \mathbb{Z}$$

which sends every cluster variable to a positive integer.

Prop: Friezes = frieze points

1. If f is a frieze point of $\mathcal{A}(Q)$, then applying f to the generic Q-frieze produces a (positive integral) Q-frieze.
2. Every (positive integral) Q-frieze can be constructed this way.

Why?

By [BNI20], $\mathcal{A}(Q)$ is generated by the cluster variables on two adjacent slices, with relations generated by the mesh relations.

Prop: The existence of unitary points

Given a cluster $\{x_1, x_2, ..., x_n\}$ in $\mathcal{A}(Q)$, there is a unique frieze point $\mathcal{A}(Q) \to \mathbb{Z}$ which sends each x_i to 1.

Why?

The Laurent phenomenon! (Positivity helps but is not needed)

Corollary

Every cluster in $\mathcal{A}(Q)$ determines a frieze of type Q by applying the corresponding frieze point to the generic frieze.

Frieze constructed in this way are sometimes called **unitary friezes**.

Thm: The classification of finite-type cluster algebras [FZ]

A cluster algebra $\mathcal{A}$ has finitely many clusters if and only if $\mathcal{A} \simeq \mathcal{A}(Q)$ for some Dynkin quiver Q.

Since ...

$$\{\text{clusters in } \mathcal{A}(Q)\} \simeq \{\text{unitary } Q\text{-friezes}\} \subseteq \{Q\text{-friezes}\}$$

...this immediately implies the following.

Corollary

If Q is not Dynkin, there are infinitely many friezes of type Q.

Finiteness via superunitary regions

Motivational Problem

Count A_2-friezes without using Coxter and Conway's bijection.

We've seen an A_2-frieze corresponds to a choice of a, b such that

$$\searrow a \nearrow b \searrow \tfrac{b+1}{a} \nearrow \tfrac{a+b+1}{ab} \searrow \tfrac{a+1}{b} \nearrow a \searrow b \nearrow \tfrac{b+1}{a} \searrow \tfrac{a+b+1}{ab} \nearrow \tfrac{a+1}{b} \searrow a \nearrow b \searrow$$

consists entirely of positive integers.

Equivalent Problem

Find all integers a, b for which

$$a \qquad b \qquad \frac{b+1}{a} \qquad \frac{a+b+1}{ab} \qquad \frac{a+1}{b}$$

are positive integers.

This arithmetic problem embeds into a geometric problem via a simple observation.

Positive integers are greater than or equal to 1.

So, any a, b defining an A_2-frieze will also solve the following.

Weaker Problem

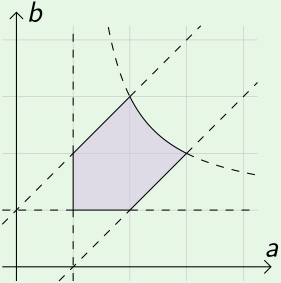
Find all real numbers a, b so that

$$a \geq 1 \qquad b \geq 1 \qquad \frac{b+1}{a} \geq 1 \qquad \frac{a+b+1}{ab} \geq 1 \qquad \frac{a+1}{b} \geq 1$$

Five inequalities

We can plot the region carved out by these five inequalities:

$$a \geq 1$$
$$b \geq 1$$
$$b + 1 \geq a$$
$$a + b + 1 \geq ab$$
$$a + 1 \geq b$$



Any choice of (a, b) that gives a frieze must lie in this region.

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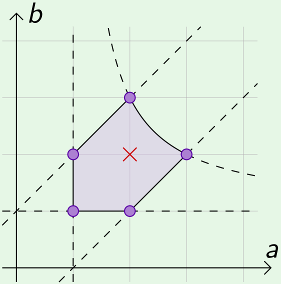
$$\begin{aligned} a &\geq 1 \\ b &\geq 1 \\ b+1 &\geq a \\ a+b+1 &\geq ab \\ a+1 &\geq b \end{aligned}$$



Any choice of (a, b) that gives a frieze must lie in this region.

There are 6 choices of integers (a, b) in this 'dented pentagon'. Plugging into the cluster variables, **5** of them give friezes:

$$(1,1) \quad (1,2) \quad (2,1) \quad (2,3) \quad (3,2)$$

Definition: The superunitary region

The **superunitary region** of a cluster algebra $\mathcal{A}$ of rank r is the subset of $\mathbb{R}^r_{>0}$ on which every cluster variable is ≥ 1.

Etymology: 'Super' = 'greater than' and 'unitary' = 'related to 1'.

This definition assumes every cluster variable has been written as a Laurent polynomial in a distinguished initial cluster.

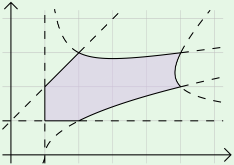
Invariant characterization

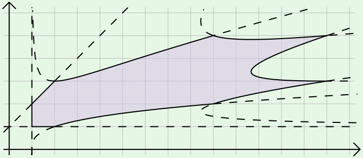
Superunitary region is homeomorphic to space of homomorphisms

$$\mathcal{A} \to \mathbb{R}$$

sending every cluster variable into $[1, \infty) \subset \mathbb{R}$.

Superunitary regions in Dynkin types B_2/C_2 and G_2





These regions are compact, so they give finitely many possible a, b.

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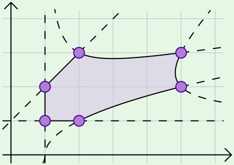
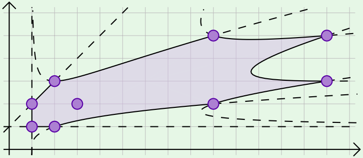




These regions are compact, so they give finitely many possible a, b. Checking these yields **6** and **9** points corresponding to friezes.

The ‘corners’ of the superunitary region are precisely where a cluster is equal to 1, so they correspond to unitary friezes.

Theorem [Gunawan-M, on arXiv 2022]

The superunitary region of each Dynkin type has a face-preserving homeomorphism to a polytope (the generalized associahedron).

Since polytopes are compact, and the friezes correspond to certain integer-valued points (a discrete subset), a topology exercise says...

Corollary (First proof of finiteness)

There are finitely many friezes of each Dynkin type.

Proof via the Cartan 'trick'

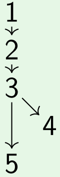
The next proof is much simpler, but requires explicit formulas.

The Cartan matrix of an acyclic quiver

Let Q be acyclic, and index the vertices by $1, 2, ..., n$ so arrows are increasing. Then the **Cartan matrix** of Q is the $n \times n$ matrix with

$$\mathsf{C}_{i,j} := \left\{ \begin{array}{cc} 2 & \text{if } i = j \\ -(\#\text{ of arrows between } i \text{ and } j) & \text{if } i \neq j \end{array} \right\}$$

Example (Type D_5)

$Q =$ 

$$\mathsf{C} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{bmatrix}$$

Formula for mesh relations

Let $F_{i,k}$ denote the value in the ith vertex of the kth slice of a given frieze F. Then the mesh relations can be written as

$$F_{i,k}F_{i,k+1} = 1 + \prod_{j<i} F_{j,k+1}^{-C_{i,j}} \prod_{j>i} F_{j,k}^{-C_{i,j}}$$

We can derive a pair of bounds that only involve multiplication.

Multiplicative bounds

$$\prod_{j<i} F_{j,k+1}^{-C_{i,j}} \prod_{j>i} F_{j,k}^{-C_{i,j}} < F_{i,k}F_{i,k+1} \leq 2\prod_{j<i} F_{j,k+1}^{-C_{i,j}} \prod_{j>i} F_{j,k}^{-C_{i,j}}$$

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Multiplicative bounds

$$1 < F_{i,k}F_{i,k+1} \prod_{j<i} F_{j,k+1}^{C_{i,j}} \prod_{j>i} F_{j,k}^{C_{i,j}} \leq 2$$

$$1 < F_{i,k} F_{i,k+1} \prod_{j<i} F_{j,k+1}^{C_{i,j}} \prod_{j>i} F_{j,k}^{C_{i,j}} \leq 2$$

Next, to make these expressions linear, we apply $\log_2$.

$$0 < \log_2(F_{i,k}) + \log_2(F_{i,k+1}) + \sum_{j<i} C_{i,j} \log_2(F_{j,k+1}) + \sum_{j>i} C_{i,j} \log_2(F_{j,k}) \leq 1$$

Finally, since F is periodic, we can remove the distinction between k and $k+1$ by averaging over a fundamental domain.

If F has period p, then

$$0 < \frac{1}{p} \sum_{k=1}^{p} \left(2 \log_2(F_{i,k}) + \sum_{j \neq i} C_{i,j} \log_2(F_{j,k}) \right) \leq 1$$

$$1 < F_{i,k} F_{i,k+1} \prod_{j<i} F_{j,k+1}^{C_{i,j}} \prod_{j>i} F_{j,k}^{C_{i,j}} \leq 2$$

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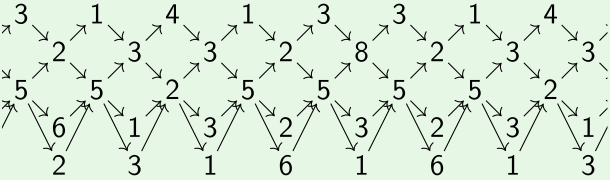
If F has period p, then

$$0 < \sum_{j=1}^{n} C_{i,j} \sum_{k=1}^{p} \frac{\log_2(F_{i,k})}{p} \leq 1$$

The Cartan trick [M 2023]

If $\vec{v}$ is the vector of average $\log_2$ of rows of a Dynkin frieze, each entry of the product with the Cartan matrix $C\vec{v}$ is between 0 and 1.

Example (Type D_5)



1.034
1.634
2.058
1.192
1.192

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1.034 \\ 1.634 \\ 2.058 \\ 1.192 \\ 1.192 \end{bmatrix} = \begin{bmatrix} 0.434 \\ 0.176 \\ 0.096 \\ 0.327 \\ 0.327 \end{bmatrix}$$

Corollary

The average $\log_{@}$s of the ith row of a Q-frieze is at most the sum of the ith row of C^{-1}.

Example

$$C^{-1} = \begin{bmatrix} 1 & 1 & 1 & \frac{1}{2} & \frac{1}{2} \\ 1 & 2 & 2 & 1 & 1 \\ 1 & 2 & 3 & \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{3}{2} & \frac{5}{4} & \frac{3}{4} \\ \frac{1}{2} & 1 & \frac{3}{2} & \frac{3}{4} & \frac{5}{4} \end{bmatrix} \quad \text{row sums} \begin{bmatrix} 4 \\ 7 \\ 8 \\ 5 \\ 5 \end{bmatrix} \quad \text{ave. } \log_2 \begin{bmatrix} 1.034 \\ 1.634 \\ 2.058 \\ 1.192 \\ 1.192 \end{bmatrix}$$

Corollary (Second proof of finiteness)

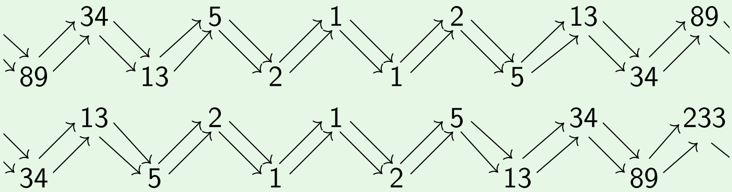
An entry in the ith row of a frieze of type Γ is at most 2^{pb_i}, where p is the period and b_i is the sum of the ith row of C^{-1}.

What's next?

Let's consider the simplest non-Dynkin quiver.

Example (Type $A_1^{(1)}$)

Every frieze of type $\bullet \rightrightarrows \bullet$ is a **shift** of one of the following two.



There are infinitely many friezes, but only on a technicality.

Natural question

For which acyclic quivers are there finitely many friezes up to shift?

My best guess

A quiver has finitely many friezes up to shift iff it is affine Dynkin.

How would one show this?

As before, big theorems in cluster algebras eliminate most cases, so proving this can be reduced to proving two conjectures.

- affine Dynkin $\Rightarrow$ finitely many friezes up to shift.
- wild rank 2 (i.e. $bc > 4$) $\Rightarrow$ infinitely many friezes up to shift.

A test case for the second part is the following.

Open problem

Show the 3-Kronecker admits infinitely many friezes up to shift.

There is a richer generalization in the language of cluster algebras.

Definition: The cluster modular group

The **cluster modular group** of a cluster algebra $\mathcal{A}$ is the set of automorphisms which send cluster variables to cluster variables.

Natural question

Which cluster algebras have finitely many frieze points up to the cluster modular group?

My best guess

A cluster algebra has finitely many frieze points up to the cluster modular group iff it is mutation-finite but not wild rank 2; that is,

- Dynkin, affine Dynkin, extended affine Dynkin,
- a marked surface cluster algebra,
- X_6, or X_7.

A curious consequence of the enumeration

New friezes from old

There are several ways to construct a Q'-frieze from a Q-frieze.

- Folding along nice symmetries of Q.
- Extending (by 1) along an embedding of Q into Q'.

A dearth of atomic friezes

The only non-trivial Dynkin friezes that cannot be constructed by a repeatedly folding and extending other Dynkin friezes are:

- The D_n-friezes corresponding to non-trivial divisors of n.
- The four distinct shifts of the E_8-frieze on the intro slide.

Natural question

What is so special about this E_8-frieze?