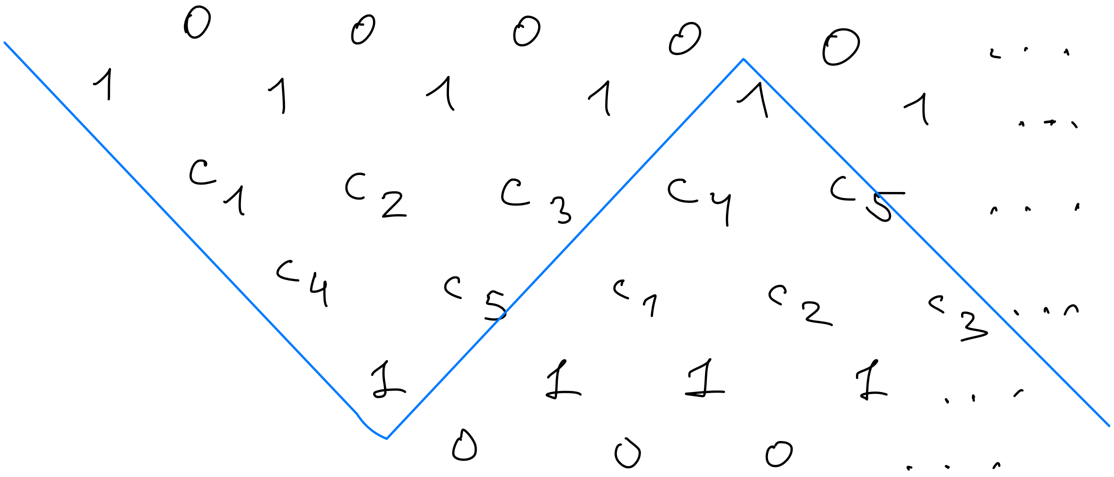
Frieze patterns from a geometric point of view:

projective geometry & difference equations

CIRM, May 2025

Lecture 1. Coxeter friezes & geometry of the projective line

Coxeter frieze:
Simplest example



$c_1 c_2 = c_4 + 1$

$c_2 c_3 = c_5 + 1$

. . .

Coxeter '71 (Acta Arithmetica)

Two main motivations:

- Pentagramma mirificum;
- negative continued fractions.

Goal of the short course:

develop this!

Concrete

ex :

0 0 0 0 0

1 1 1 1 1 1

1 3 1 2 2 ①

2 2 1 3 1

1 1 1 1

0 0 0 0

-1 -1 -1

-1 -3

-2 -2

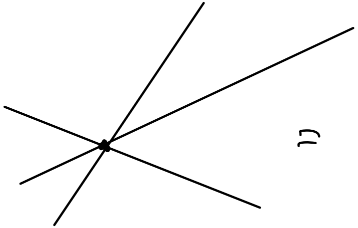
a) The (real) projective line $\mathbb{RP}^1$

complex, ...

$\mathbb{R} \cup \{\infty\}$ equipped w/ the action of $PSL(2,\mathbb{R})$

$\overset{\cup}{x}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = \frac{ax+b}{cx+d} \qquad \begin{matrix} PSL(2,\mathbb{R}) \\ \shortparallel \\ SL(2,\mathbb{R}) / \{\pm Id\} \end{matrix}$$

geometrically  $= \mathbb{P}^1$ set of lines through the origin

- transitive
- 3-transitive !!! $\{x_1, x_2, x_3\} \to \{0, 1, \infty\}$

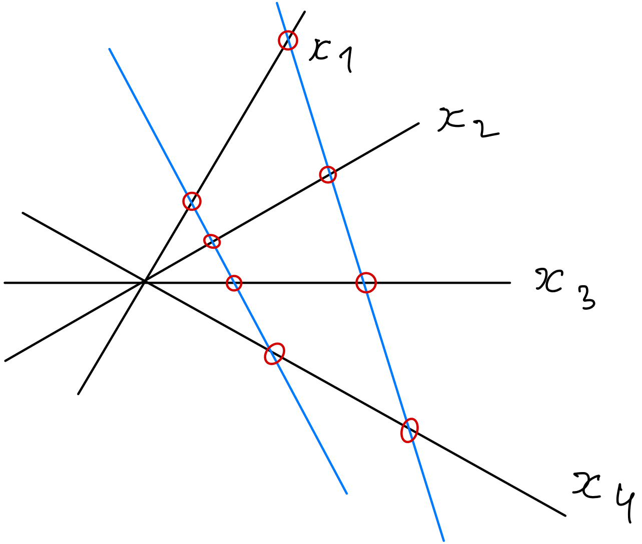
4 points x_1, x_2, x_3, x_4 have one invariant:

the cross-ratio:

$$[x_1, x_2, x_3, x_4] := \frac{(x_1 - x_4)(x_2 - x_3)}{(x_1 - x_2)(x_3 - x_4)}$$

to visualize :

4 pts on blue lines have the same cross-ratio



5 points $\{x_1, \dots, x_5\} \big/ PSL_2 = \mathcal{M}_{0,5}$ dim = 2

5 cross-ratios $c_1 = [x_2, x_3, x_4, x_5]$

invariants but not independent! $c_2 = [x_3, x_4, x_5, x_1]$

$\dots$

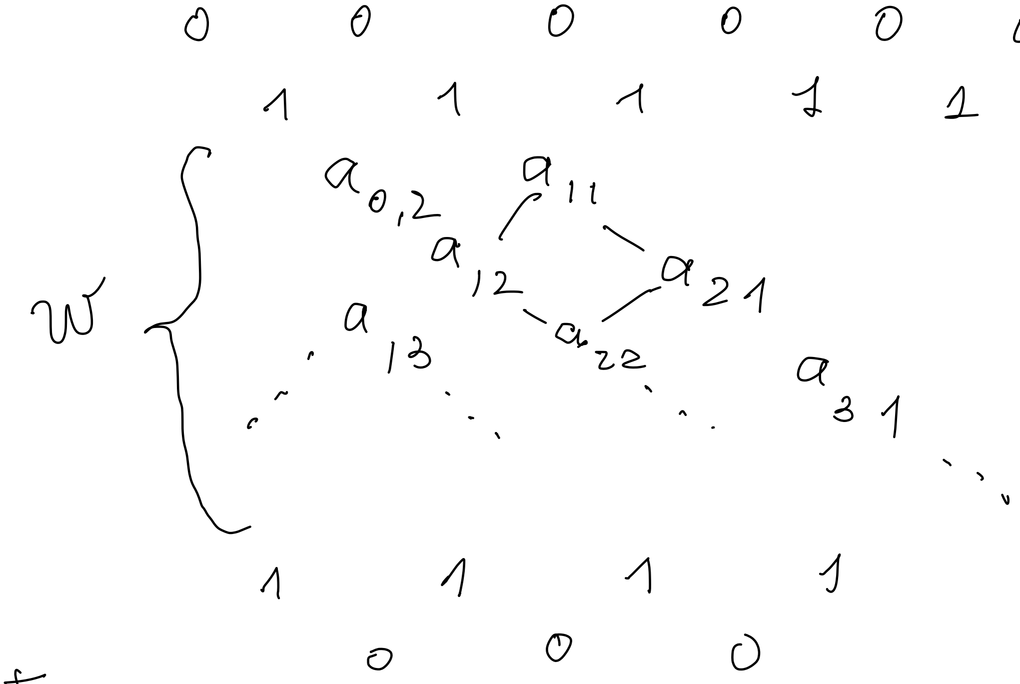
Gauss' "pentagramma mirificum"

$$\boxed{c_i \, c_{i+1} = c_{i+3} + 1} \quad \Longleftrightarrow$$

$(c_1, c_2, c_3, c_4, c_5)$ form a frieze.

B) A general statement

- The set of tame friezes of width w $\mathcal{F}_w$



$\mathcal{F}_w$

$a_{ij} \in \mathbb{R}$

$\mathcal{F}_w$ - alg. variety

- Moduli space of configurations of n points in $\mathbb{P}^1$ **very classical!**

$$\mathcal{M}_{0,n} = \Big\{ \underset{x_i \neq x_{i+1}}{\overset{n\text{-tuple}}{(x_1, \dots, x_n)}} \Big\} \Big/ PSL(2,\mathbb{R})$$

$x : \mathbb{Z} \to \mathbb{P}^1$; cycl. ordered $x_{i+n} = x_i$

Thm [M.G., O., T] 2012 $\mathcal{M}_{0,n} \cong \mathcal{F}_{n-3}$, provided n odd

Rmk: n even $\mathcal{F}_{n-3} \longrightarrow \mathcal{M}_{0,n}$

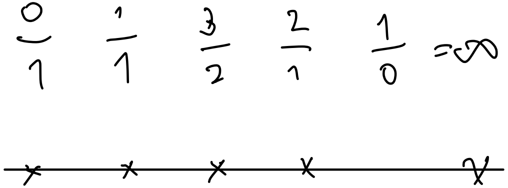
canonical projection

Rmk: Correspondence

$$\{\text{integer Cox. friezes}\} \longleftrightarrow \{\text{Farey } n\text{-gons}\}$$

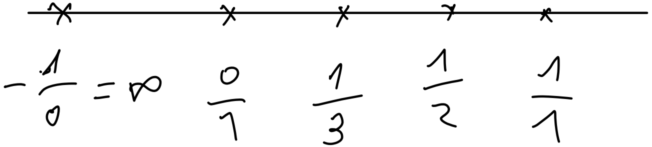
is a particular case of Thm

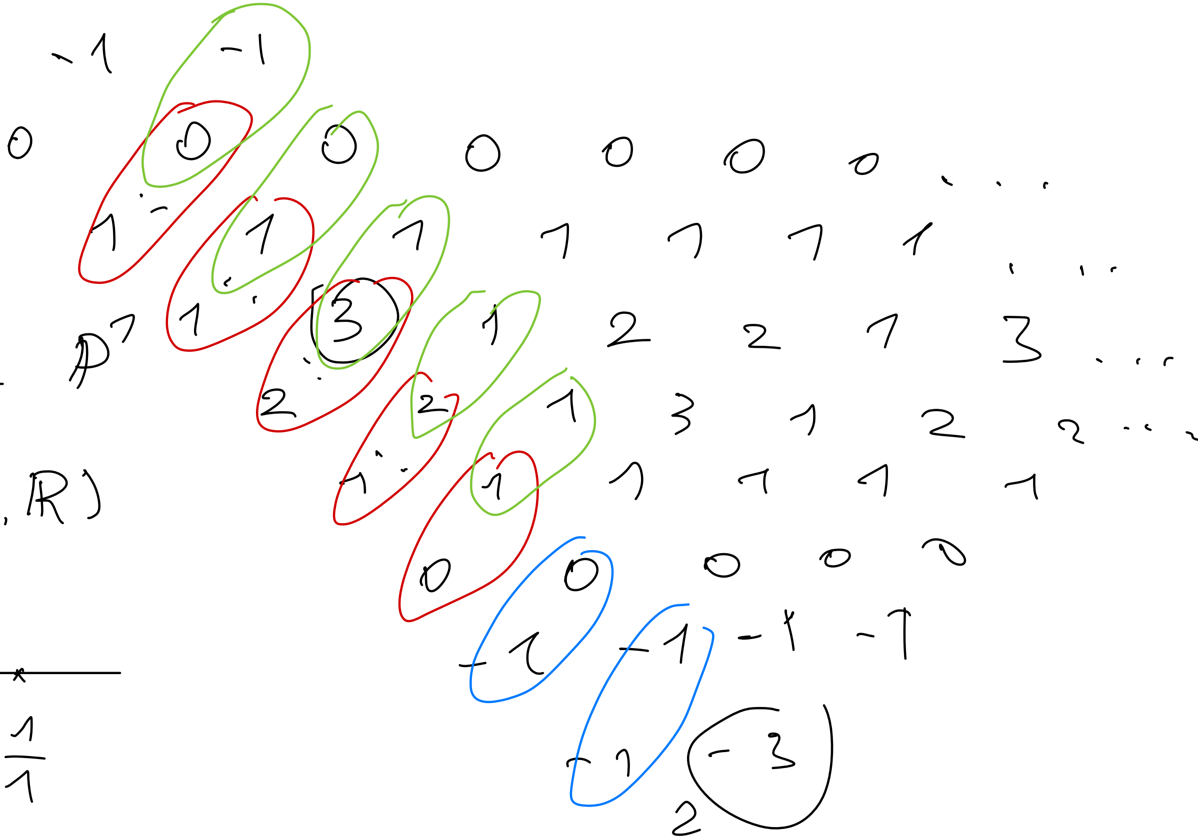
Ex.



modulo

$\|\wr\ PSL(2,\mathbb{R})$





- cross-ratios in a frieze

$$\begin{array}{ccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & \cdots & \\
1 & & 1 & & 1 & & 1 & & 1 & & \cdots \\
 & a_0 & & a_1 & & a_2 & & a_3 & & & \\
 & & \boxed{c_0 = a_1 a_2 - 1} & & \boxed{c_1} & & \boxed{c_2} & & & \xleftarrow{\quad} & [x_i, x_{i+1}, x_{i+2}, x_{i+3}] \\
 & & \ddots & & \ddots & & \cdots & & & &
\end{array}$$

appear in

Second row!

(see survey Morier-Genoud)

c) Linear difference operators/equations

Given: (a_i), $i \in \mathbb{Z}$ sequences of (real) numbers

$$(*) \quad V_i = a_i V_{i-1} - V_{i-2}$$

Discrete
- Sturm-Liouville
- Hill
- Schrödinger ...

sequence (V_i) - solution

$$\mathcal{E}_n = \{ \text{eq } (*) \text{ with } n\text{-antiperiodic sol} \}$$

$(V_{i+n} = -V_i)$ for all i $\Rightarrow$ $\{ a_{i+n} = a_i$ +MORE! $\}$

(*) is a discrete version of $\boxed{v''(x) = u(x)\, V(x)}$ (**)

coeff. "potential" solution

$u(x) \rightsquigarrow u_i$

$v(x) \rightsquigarrow V_i$

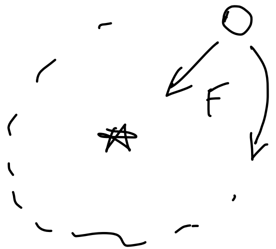
$v'(x) \rightsquigarrow V_i - V_{i-1}$

$V''(x) \rightsquigarrow V_i - 2V_{i-1} + V_{i-2}$; $a_i = u_i + 2$

(**) Sturm-Liouville, Schrödinger, Hill eq.

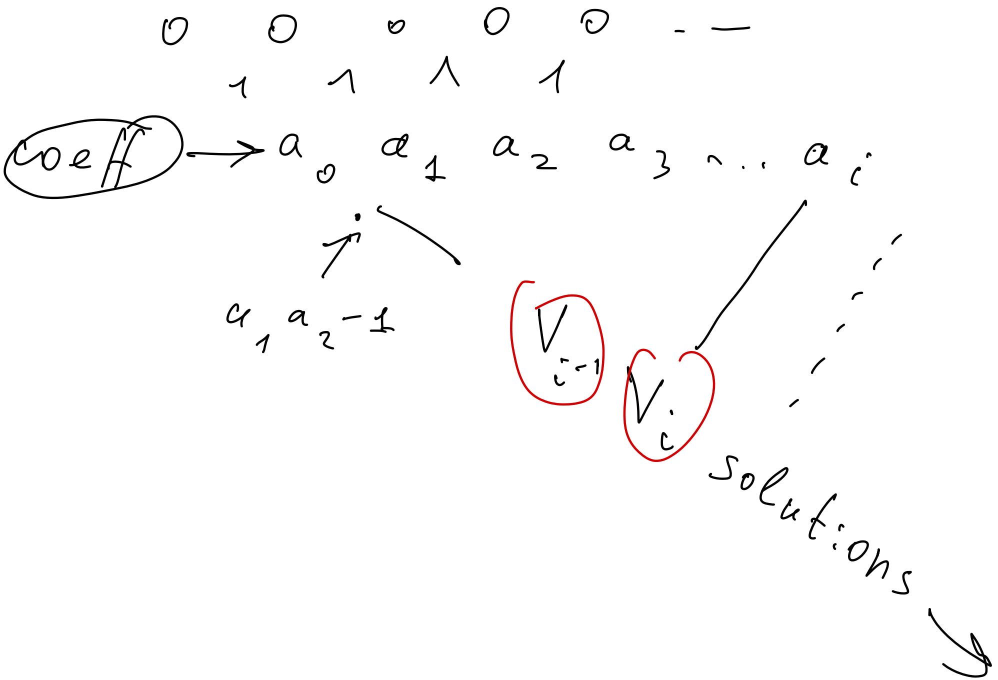
- alg. geometry
- dynamics

...



motion
central
force

Friezes => differ. eqns



$$V_i = a_i V_{i-1} - V_{i-2}$$

At every diagonal!

(Coxeter) &

(Coxeter-Conway)

Thm. $\mathcal{F}_{n-3} \cong \mathcal{E}_n$ as alg. varieties.

Ex

V_0 0 0 0 0

V_1 1 1 1 1 1 1 ...

V_2 1 3 1 2 2 1 ... 5-periodic (a_i)

V_3 2

V_4 1

V_5 0

V_6 . (V_i)

5-antiper. !

$V_0 = 0$, $V_1 = 1$

$V_2 = 1$

$V_3 = 3 \cdot 1 - 1 = 2$

$V_4 = 1 \cdot 2 - 1 = 1$

$V_5 = 2 \cdot 1 - 2 = 0$

$V_6 = -1$

d) Triality

(not enough explored)

$$\begin{tikzcd} \mathcal{F}_{n-3} \arrow[r, phantom, "\cong"] \arrow[dr, "*"'] & & \mathcal{E}_n \arrow[dl] \\ & \mathcal{M}_{0,n} & \end{tikzcd}$$

//? n odd

- * Counting friezes (over $\mathbb{F}_q$)
- constructing friezes
- coordinates on moduli spaces
- dynamical systems

...

- superfriezes

- q- numbers

e) A variant of tame friezes:

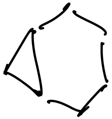
positive integers →

$$\begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ a_0 & a_1 & a_2 & a_3 & a_4 & \cdots & \\ \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \\ 1 & 1 & 1 & 1 & 1 & 1 & \end{array}$$

$n-3$

may be zero or negative!

<u>Thm</u> (V.O.) These friezes are classified by "3d-dissections"



3, 6, 9-gons; counted in [Conley-Ovs]

Lecture 2. Generalised friezes

a) Moduli spaces of polygons in $\mathbb{P}^{K-1}$

$$v : \mathbb{Z} \to \mathbb{P}^{K-1}$$

n-gon $v_{i+n} = v_i$; $\langle v_i, v_{i+1}, \dots, v_{i+K-1} \rangle$ not in a hyperplane!

$v \sim w$ if $A \in PSL(K, \mathbb{R})$,

$v = A \circ w$

$\mathcal{C}_{K,n}$ = moduli space of n-gons
(Gelfand-MacPherson, R Schwartz, ...)

Assume: (K, n) coprime!

b) Difference equations

$$V_i = a_i^1 N_{i-1} - a_i^2 V_{i-2} + \dots (-1)^{K-1} a_i^k V_{i-K} + (-1)^k V_{i-k}$$

$$\boxed{a_{i+n} = a_i} \quad ; \quad \boxed{V_{i+n} = (-1)^{k-1} V_i}$$

$\Xi_{k,n}$ - alg. variety

Ex. (Next after Sturm-Liouville)

$$V_i = a_i V_{i-1} - b_i V_{i-2} + V_{i-3}$$

"Pentagramma mirificum reincarnated"

Ex. (a_i), (b_i) n-periodic

$$W_i = a_i W_{i-1} - b_i W_{i-2} + W_{i-3}$$

$$\Downarrow$$ "forgetting b"

Question: $V_i = a_i V_{i-1} - V_i$

Suppose $W_{i+n} = W_i$ n-periodic is $V_{i+n} = -V_i$ n-antiperiodic

NO!

Except for $n=5$ **YES!**

c) SL_k - friezes (Bergeron-Reutenauer 2010)

$k-1 \left\{ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & \\ 0 & 0 & \\ \ldots & & \end{array} \right.$

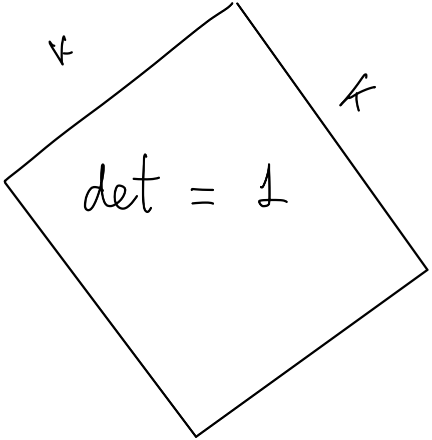
tame! $(k+1) \times (k+1)$ minors $= 0$.

$1 \quad 1 \quad 1 \quad 1 \quad 1$

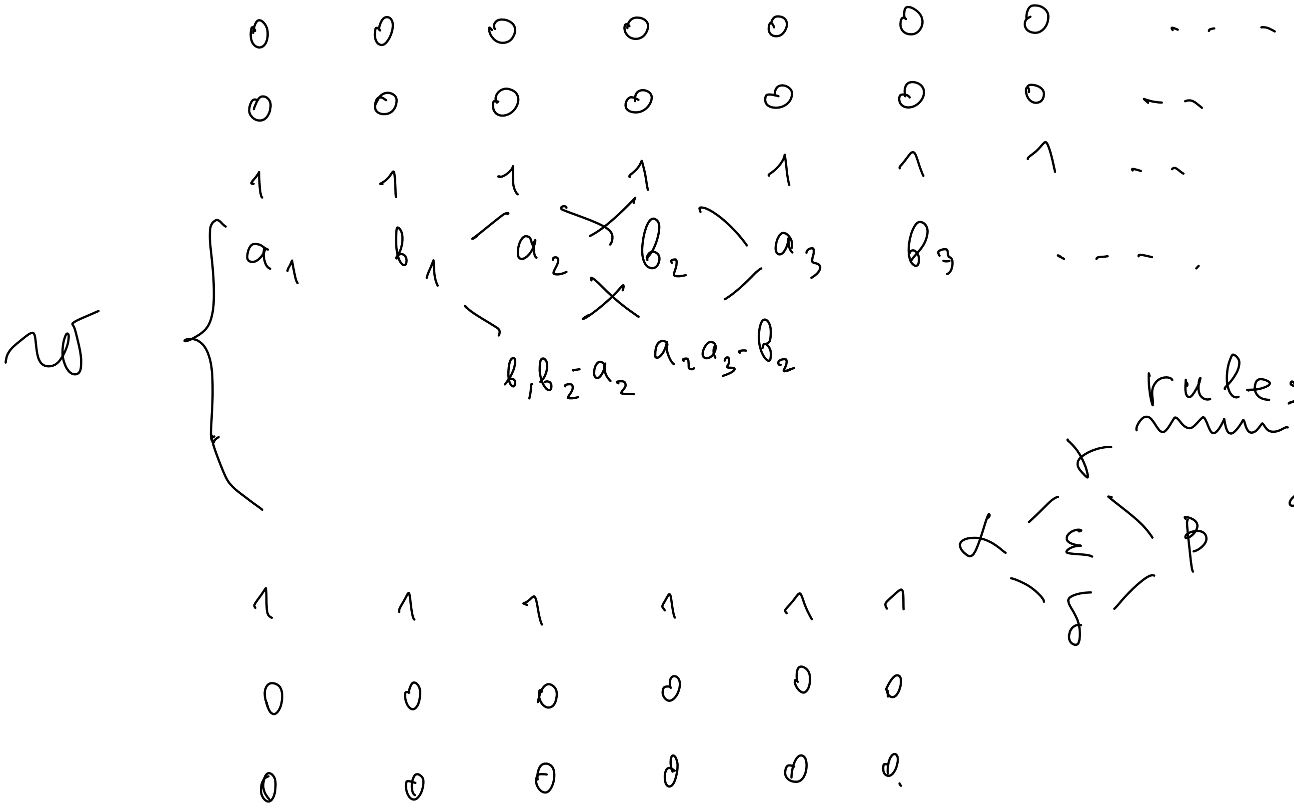
$a_{12} \quad a_{11}$

$a_{12} \quad a_{21}$

$\ldots$

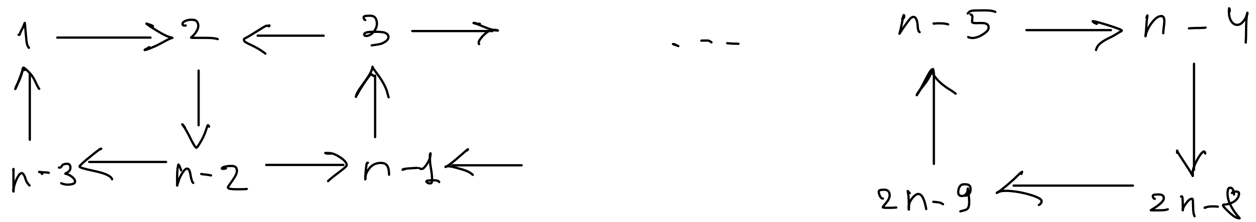


Special case: SL_3-friezes $\equiv$ 2-friezes



$$\alpha\beta-\gamma\delta=\varepsilon$$

- Space of 2-friezes – cluster manifolds associated with the quiver



Thm. (M·G, Tab, V.O) 2-frieze is closed iff the eq. $V_i = a_i V_{i-1} - B_i V_{i-2} + V_{i-3}$ has periodic solutions $V_{i+n} = V_i$.

Thm (M-G, Tab, V.O.)
2014

Triality:

(I)
$$\begin{tikzcd}
\mathcal{F}_{k,n} \arrow[rr, phantom, "\cong"] \arrow[d, phantom, "\wr\wr ?"] \arrow[dr] & & \mathcal{E}_{k,n} \arrow[dl, "\text{Old Projective geometry}"] \\
\text{if}(r,n)=1 \ \text{coprime!} & \mathcal{X}_{k,n} &
\end{tikzcd}$$

(II) If $(k,n)=d$, fiber of dim$=d-1$

Explanation: $\mathcal{F}_{k,n} \hookrightarrow Gr_{k,n}$

embedding into Grassmannian

$\Delta_I = 1$ = Plucker coords, minors with consecutive columns

$\mathcal{X}_{k,n} \simeq G_{k,n} / T^{n-1}$ (Gelfand-MacPherson)

d) An application: <u>Gale</u> <u>duality</u>

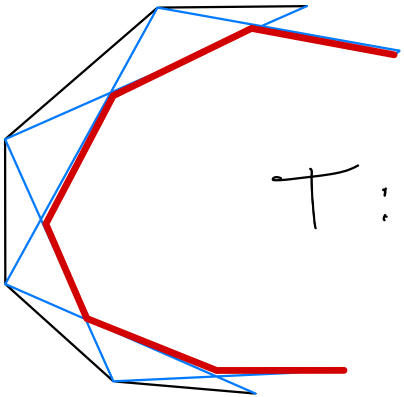
$$k + l = n \qquad (Gr_{k,n} \simeq Gr_{l,n})$$

Thm $\mathcal{F}_{k,n} \simeq \mathcal{F}_{l,n}$

D. Gale: $\zeta_{k,n} \simeq \zeta_{l,n}$.

NB: When $n = k+2$, obtain Sturm-Liouville Eq by "forgetting" all of the coeffs. except the first one.

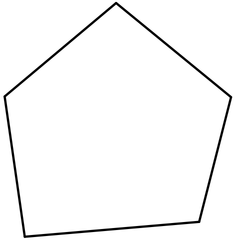
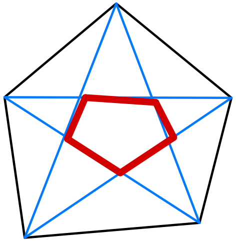
e) The Pentagram map



T a discrete dynamical system

$$T : \mathcal{C}_{K,n} \longrightarrow \mathcal{C}_{K,n}$$

"Pentagramma mirificum" again!

Ex: $T :$  $\longrightarrow$ 

$n = 5$

identity map! $T = Id$ (Conway)

f) <u>Legendrian</u> <u>configurations</u>

"standard"

$$\begin{tikzcd}(\mathbb{R}^{2m}, \omega) \arrow[d] \\ \mathbb{RP}^{2m-1}\end{tikzcd}$$

symplectic space

contact

ω - skew-symmetric Bilinear form
$$\left(\begin{array}{c|c} & \begin{matrix}1&&&\\&1&&\\&&1&\\&&&\ddots\,1\end{matrix} \\ \hline \begin{matrix}-1&&\\&-1&\\&&\ddots\,-1\end{matrix} & \end{array}\right)$$

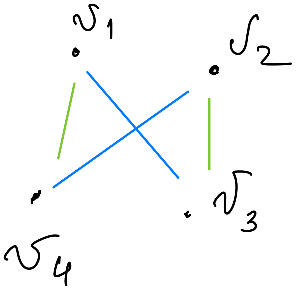
$v: \mathbb{Z} \to \mathbb{R}^{2m}$ n-periodic

m consecutive: $\langle v_i, v_{i+1}, \dots v_{i+m-1} \rangle$ span a Lagrangian subspace

2m consecutive <u>not</u> in a hyperplane!

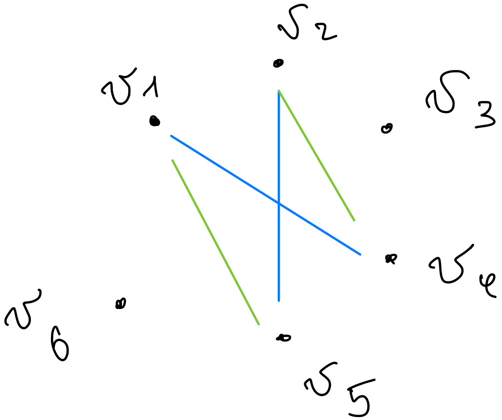
- The cross-ratio $\{v_1, v_2, v_3, v_4\}$

$$[v_1,v_2,v_3,v_4] = \frac{\omega(v_1,v_3)\,\omega(v_2,v_4)}{\omega(v_1,v_4)\,\omega(v_2,v_3)}$$



$Sp(2n,\mathbb{R})$-invariant of 4 points

Example Configurations of 6 points in $\mathbb{R}^4$.



$$c_1 = \frac{\omega(v_1, v_4)\,\omega(v_2, v_5)}{\omega(v_1, v_5)\,\omega(v_2, v_4)}$$

$c_2 = \ldots$

$c_3 = \ldots$

Thm (Conley-Ovs)

$$\frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3} = 1$$

" Hexagramma mirificum "...

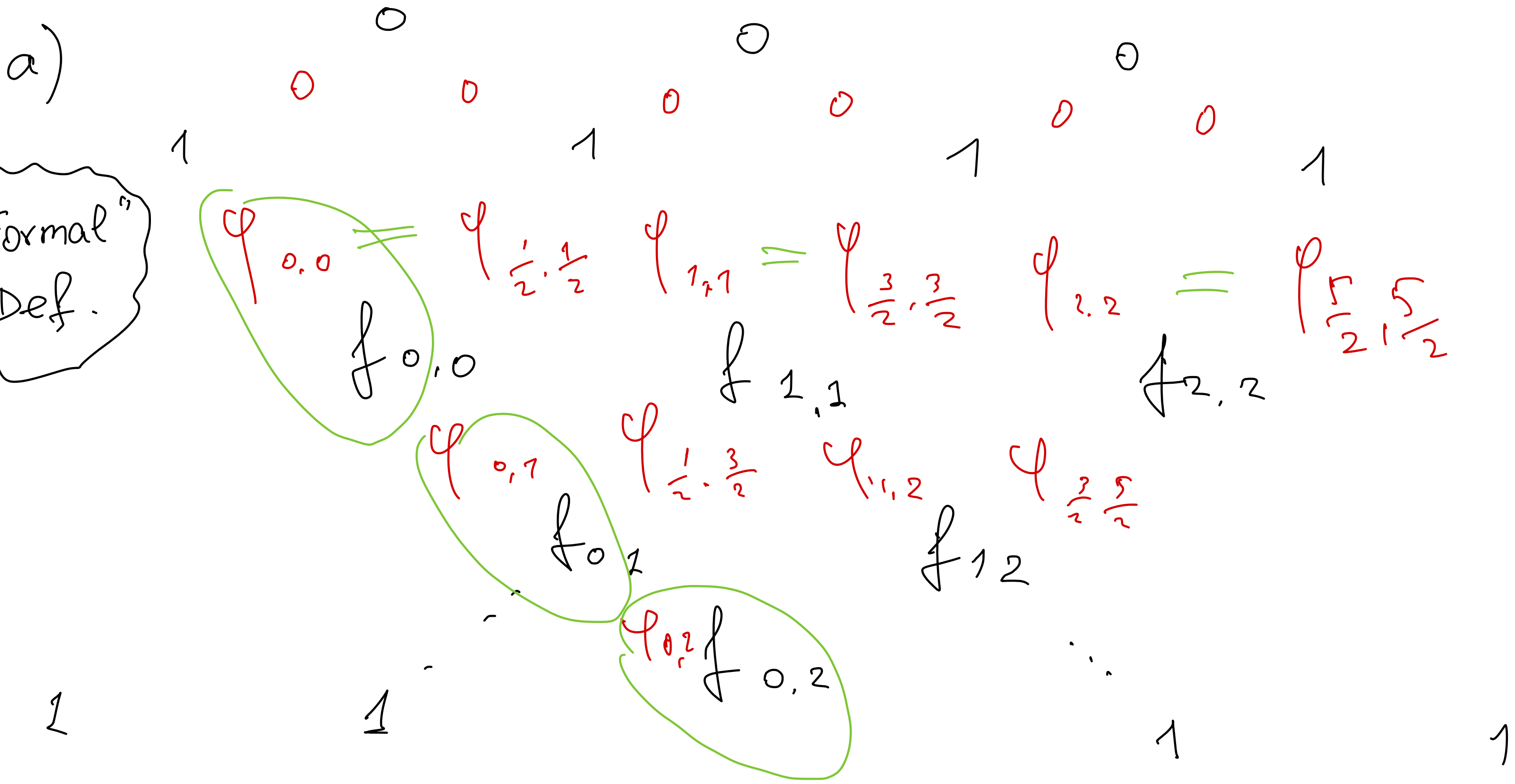
Lecture 3 Superfriezes & super continued fractions

[MG, Tab, V.O]

[Conley, V.O]

a)

"Formal" Def.



Philosophy ; "Super" = "Square root"

Think of

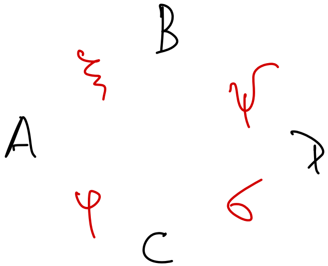
(Lie) superalgebra

$$\mathfrak{g} = \overset{\text{Lie}}{\mathfrak{g}_0} \oplus \underset{\text{"square root" of } \mathfrak{g}_0}{\mathfrak{g}_1}$$

. . . .

Superfrieze = square root of Coxeter frieze

Frieze "diamond" rule



$$\begin{cases} AD - BC = 1 + \sigma\xi \\ A\sigma - C\xi = \varphi & * \\ B\sigma - D\xi = \psi & ** \end{cases}$$

odd elements $\xi, \sigma, \varphi, \psi$

anticommute: $\xi\sigma = -\sigma\xi$

$\ldots$

$\xi^2 = 0$

$\ldots$

$*, **$ equivalent to

$$\begin{cases} B\varphi - A\psi = \xi \\ D\varphi - C\psi = \sigma \end{cases}$$

also

$$\xi\sigma = \varphi\psi$$

b) The symmetry group $SL(2) \rightsquigarrow OSp(1|2)$

Lie supergroup

$$\left(\begin{array}{cc|c} a & b & \gamma \\ c & d & \delta \\ \hline \alpha & \beta & e \end{array}\right) \qquad \begin{cases} ad - bc = 1 - \alpha\beta \\ e = 1 + \alpha\beta \\ a\delta - c\gamma = -\alpha \\ b\delta - d\gamma = -\beta \end{cases} \Rightarrow \begin{array}{l} \gamma = a\beta - b\alpha \\ \delta = c\beta - d\alpha \end{array}$$

(Manin, ...)

$$\begin{array}{ccccc} & & -a & & \\ & \gamma & & \alpha & \\ b & & & & -c \\ & -\beta & & \delta & \\ & & d & & \end{array}$$

diamond in a frieze

coefficients

$R = R_0 \oplus R_1$ $\quad\quad \mathbb{Z}_2 \; (= \mathbb{Z}/_{2\mathbb{Z}})$ - graded ring

even part $\quad \circlearrowleft \quad$ odd part $\quad\quad$ $\mathbb{Z}_2$ - commutative

α, β $\quad \alpha \cdot \beta = -\beta \cdot \alpha$

$\alpha^2 = \beta^2 = 0$

"Minimal choice" $(\mathbb{Z} \oplus \mathbb{Z}) \oplus (\mathbb{Z} \oplus \mathbb{Z})$

Notation $\quad \boxed{\mathbb{Z} + \underbrace{\xi\eta}\mathbb{Z}} \oplus (\xi\mathbb{Z} \oplus \eta\mathbb{Z})$

"Shadow" part

odd generators

c) Linear difference equations

want analog of $V_i = a_i V_{i-1} - V_{i-2}$

where

shift operator $\boxed{(TV)_i = V_{i-1}}$ $\boxed{L = T^2 - aT - Id}$

Analog of Shift operator $V + \xi W = (V_i) + \xi (W_i)$

$\mathcal{T}(V + \xi W)_i := W_i - \xi V_{i-1}$ $\boxed{\mathcal{T}^2 = -T}$

$$\mathcal{L} = \mathcal{T}^3 + \mathcal{U}\,\mathcal{T}^2 + \Pi$$

$(\beta_i'') + \xi(a_i)$ $\Pi(V + \xi W)_i = W_i + \xi V_i$

Matrix form: $A : \begin{pmatrix} V_{i-1} \\ V_i \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & a_i \end{pmatrix} \begin{pmatrix} V_{i-2} \\ V_{i-1} \end{pmatrix}$

$$A : \begin{pmatrix} V_{i-1} \\ V_i \\ W_i \end{pmatrix} = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & a_i & -B_i \\ \hline 0 & B_i & 1 \end{array}\right) \begin{pmatrix} V_{i-2} \\ V_{i-1} \\ W_{i-1} \end{pmatrix}$$

Periodicity conditions: $\begin{cases} a_{i+n} = a_i \\ B_{i+n} = -B_i \end{cases}$

$$\begin{cases} V_{i+n} = -V_i \\ W_{i+n} = W_i \end{cases}$$

$$M = \left[\begin{array}{cc|c} -1 & & \\ & -1 & \\ \hline & & 1 \end{array}\right]$$

monodromy

Superfriezes & difference equations

$$\Upsilon_i := \beta_i + \xi a_i \,, \qquad \boxed{a_i = f_{i,i} \,,\ \beta_i = \varphi_{i,i}}$$

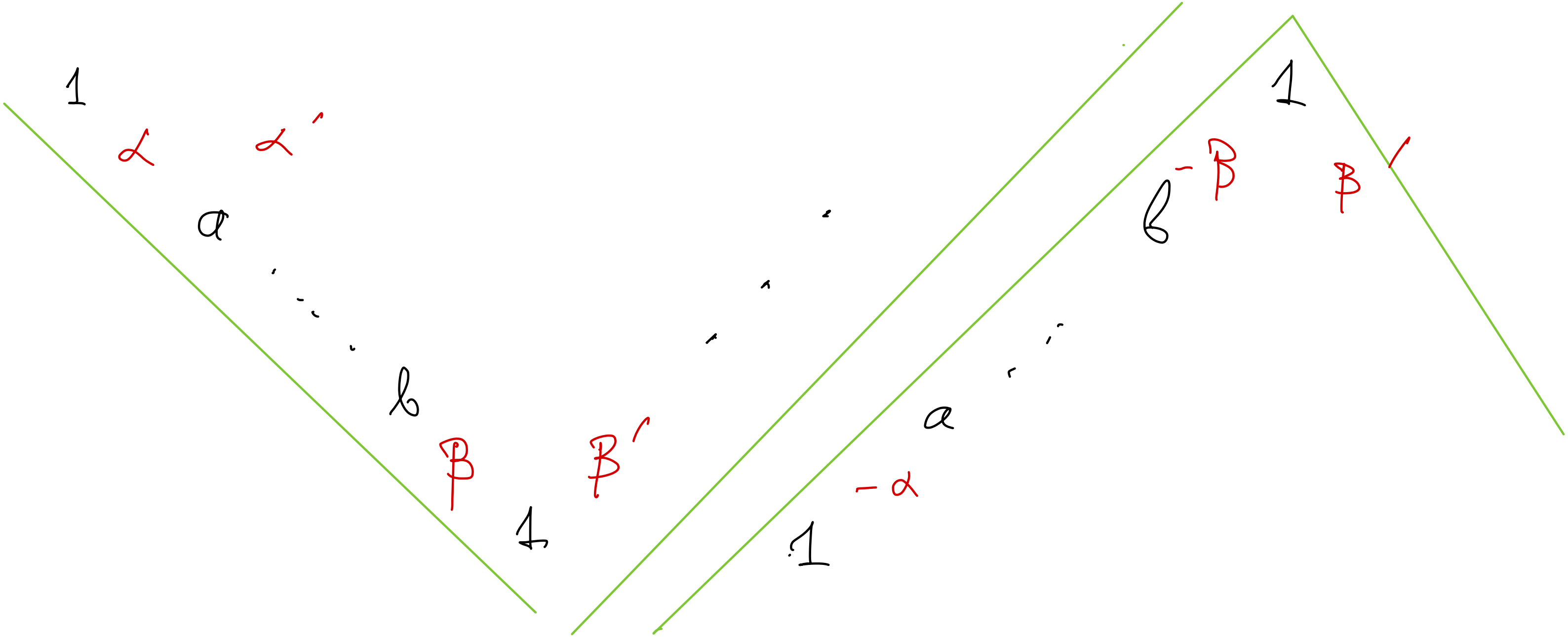
Prop $(W_i, V_i) := (\varphi_{ji}, f_{ji})$ diagonals

analog of Coxeter's observation

solution to the eq.

d) Some properties of superfriezes

Thm - Glide symmetry



e) Continued fractions

(with C. Conley)

$$R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\frac{p}{q} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 \; \ddots \; \cfrac{1}{a_n}}}$$

$$L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

initial vectors

$$\begin{pmatrix} p \\ q \end{pmatrix} = R^{a_1} L^{a_2} \dots R^{a_{2m-1}} L^{a_{2m}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad n \text{ even}$$

$$R^{a_1} L^{a_2} \dots R^{a_{2m-1}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad n \text{ odd}$$

$$\mathcal{R} = \left(\begin{array}{cc|c} 1 & 1 & \xi \\ 0 & 1 & 0 \\ \hline 0 & \xi & 1 \end{array}\right) ; \quad \mathcal{L} = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 1 & -\eta \\ \hline \eta & 0 & 1 \end{array}\right)$$

initial vectors

$$\boxed{\text{even } n} \begin{pmatrix} 1 \\ 0 \\ \xi \end{pmatrix} \quad \text{vs} \quad \begin{pmatrix} 0 \\ 1 \\ \eta \end{pmatrix} \boxed{\text{odd } n}$$

$$\mathcal{R}^{a_1} \mathcal{L}^{a_2} \mathcal{R}^{a_3} \mathcal{L}^{a_4} \cdots \mathcal{R}^{a_{2m-1}} \mathcal{L}^{a_{2m}} \begin{pmatrix} 1 \\ 0 \\ \xi \end{pmatrix}$$

or

$$\mathcal{R}^{a_1} \mathcal{L}^{a_2} \cdots \mathcal{R}^{a_{2m-1}} \begin{pmatrix} 0 \\ 1 \\ \eta \end{pmatrix}$$

f) Super continuants

Euler's continuants:

$$\det \begin{pmatrix} a_1 & 1 & & \\ -1 & a_2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & -1 & a_n \end{pmatrix} =: K(a_1 \dots a_n)$$

$$\frac{K(a_1 \dots a_n)}{K(a_2 \dots a_n)} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 \ddots + \cfrac{1}{a_n}}}$$

" algebraic regard on c.f. "

I_n super c. f. $\quad \left\{ \begin{array}{l} \mathcal{R}^{a_1}\mathcal{L}^{a_2}\cdots\mathcal{R}^{a_{2m-1}}\mathcal{L}^{a_{2m}} \\ \mathcal{R}^{a_1}\mathcal{L}^{a_2}\cdots\mathcal{R}^{a_{2m-1}} \end{array} \right.$

Thm: I_n the even part of super c. f.

shadow part!

$$\mathcal{K}(a_1 \ldots a_n) = K(a_1 \ldots a_n) + \xi\eta\, \mathcal{E}\big(K(a_1 \ldots a_n)\big)$$

$$\mathcal{E} = \frac{\partial}{\partial a_1} + \cdots + \frac{\partial}{\partial a_n}$$

(counting the degree)

(Euler operator)

Ex. $K(a_1, a_2) = a_1 a_2 + 1 \qquad \boxed{2 a_1 a_2}$

$K(a_1, a_2, a_3) = a_1 a_2 a_3 + a_1 + a_3$

$3 a_1 a_2 a_3 + a_1 + a_3 \Longleftarrow$ shadow