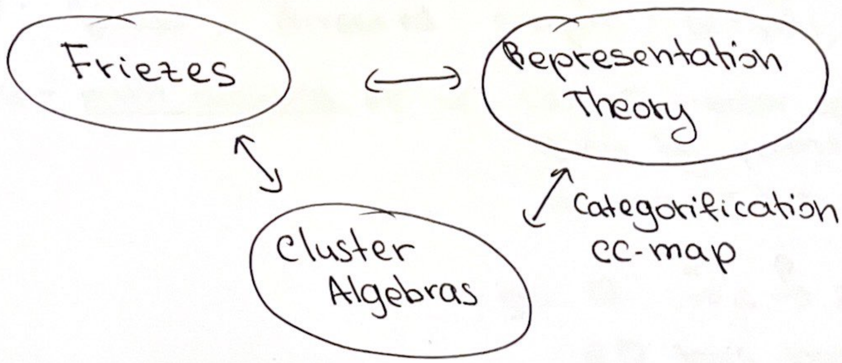
Frieze Patterns and Representation Theory



Conway-Coxeter frieze - array such that

```
 .... 0   0   0   0  ....
        1   1   1   1
width  * * * *
  =     *  *  *  *
# of rows  :  :  :
        *  *  *  *
       1   1   1   1
  ...    0   0   0   0 ...
```

- entries are in $\mathbb{Z}_{>0}$
- **diamond rule**: $\begin{matrix} & a & \\ b & & c \\ & d & \end{matrix} \rightsquigarrow \begin{vmatrix} b & a \\ d & c \end{vmatrix} = bc - ad = 1$
- **tameness**: 3×3 diamonds have determinant 0.

note first nontrivial row of the frieze is called the **quiddity sequence** and it uniquely determines the rest of the entries.

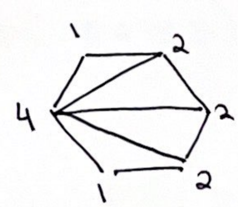
thm [Conway-Coxeter '73]

$$\left\{ \begin{matrix} \text{friezes of} \\ \text{width } n \end{matrix} \right\} \xleftrightarrow{1:1} \left\{ \begin{matrix} \text{triangulations} \\ \text{of } (n+3) \text{ gons} \end{matrix} \right\}$$

ex

```
     0  0  0  0  0
    1  1  1  1  1  1
   1  4  1  2  2  2 ...
....  3  3  1  3  3
   2  2  2  1  1  4
    1  1  1  1  1  1
     0  0  0  0  0
```

$\longleftrightarrow$



# of triangles adjacent to a vertex

Quiver Representations:

Q - quiver, directed graph $Q = (\overset{\text{vertices}}{Q_0}, \overset{\text{arrows}}{Q_1})$ ex $Q: 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$

$\mathbb{C}Q$ - path algebra of Q i.e. $\mathbb{C}$ - vector space with basis given by paths in Q and multiplication is composition of paths

ex $Q: 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$

$\mathbb{C}Q$ has basis $\{\alpha, \beta, \alpha\beta, \underbrace{e_1, e_2, e_3}_{\text{constant paths at each vertex}}\}$

$\alpha \circ \beta = \alpha\beta$

$\beta \circ \alpha = 0$

$\alpha \circ \alpha = 0$

mod $\mathbb{C}Q$ - category of finite dim $\mathbb{C}Q$-modules, it is equivalent to $\text{rep}(Q)$ - category of quiver representations.

Def A quiver representation $M = (\underset{\text{vector space}}{M_i}, \underset{\text{linear map}}{\varphi_\alpha})_{\substack{i \in Q_0 \\ \alpha \in Q_1}}$

s.t. $\alpha: i \to j$

$\varphi_\alpha : M_i \to M_j$

A morphism $f = (f_i)_{i \in Q_0} : M \to M'$ s.t.

$$\begin{tikzcd} M \arrow[d, "f"] \\ M' \end{tikzcd} \qquad \begin{tikzcd} M_i \arrow[r, "\varphi_\alpha"] \arrow[d, "f_i"'] \arrow[dr, phantom, "\circlearrowleft"] & M_j \arrow[d, "f_j"] \\ M_i' \arrow[r, "\varphi_\alpha'"'] & M_j' \end{tikzcd}$$

ex
$$\begin{tikzcd} \mathbb{C} \arrow[r, "{\begin{bmatrix}1\\0\end{bmatrix}}"] \arrow[d, "1"'] & \mathbb{C}^2 \arrow[r, "{[1\ 0]}"] \arrow[d, "{[1\ 0]}"] & \mathbb{C} \arrow[d, "0"] \\ \mathbb{C} \arrow[r, "1"'] & \mathbb{C} \arrow[r, "0"'] & 0 \end{tikzcd}$$
in $\text{rep}(1 \to 2 \to 3)$

Def

- direct sum: $M \oplus M' = \left(M_i \oplus M_i', \begin{bmatrix} \varphi_\alpha & 0 \\ 0 & \varphi_\alpha' \end{bmatrix}\right)$
- M is indecomposable if whenever $M \cong M_1 \oplus M_2$ then $M_1 = 0$ or $M_2 = 0$

ex $\mathbb{C} \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} \mathbb{C}^2 \xrightarrow{[1\ 0]} \mathbb{C} \cong (\mathbb{C} \xrightarrow{1} \mathbb{C} \xrightarrow{1} \mathbb{C}) \oplus (0 \to \mathbb{C} \to 0)$

<u>Auslander - Reiten Quiver</u> :

vertices $\leftrightsquigarrow$ ind. representations

arrows $\leftrightsquigarrow$ irreducible morphisms

ex Q: $1 \to 2 \to 3$

ind. repr.

$$\begin{array}{ll}
\mathbb{C} \to 0 \to 0 & 1 \\
0 \to \mathbb{C} \to 0 & 2 \\
0 \to 0 \to \mathbb{C} & 3 \\
\mathbb{C} \xrightarrow{1} \mathbb{C} \to 0 & \begin{smallmatrix}1\\2\end{smallmatrix} \\
0 \to \mathbb{C} \xrightarrow{1} \mathbb{C} & \begin{smallmatrix}2\\3\end{smallmatrix} \\
\mathbb{C} \xrightarrow{1} \mathbb{C} \xrightarrow{1} \mathbb{C} & \begin{smallmatrix}1\\2\\3\end{smallmatrix}
\end{array}$$

<u>AR-quiver</u> :

$$\begin{tikzcd}
& & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[dr] & & \\
& \begin{smallmatrix}2\\3\end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[dr] & \\
3 \arrow[ur] & & 2 \arrow[ur] & & 1
\end{tikzcd}$$

more generally, can consider $\mathbb{C}Q/_I$ quotient of a path algebra by ideal of relations

$\mathrm{mod}(\mathbb{C}Q/_I) \cong \mathrm{rep}(Q, I)$

representations of Q satisfying relations in I

ex Q: $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$

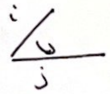
$I = \langle \alpha\beta \rangle$

$\mathrm{rep}(Q, I) = \{ M_1 \xrightarrow{\varphi_\alpha} M_2 \xrightarrow{\varphi_\beta} M_3 \mid \varphi_\beta \circ \varphi_\alpha = 0 \}$

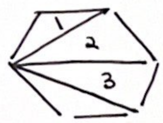
From triangulation T to Jacobian algebra $J_T = \mathbb{C}Q_T / I$

triangulation T $\rightsquigarrow$ Q_T quiver $\rightsquigarrow$ Jacobian alg J_T

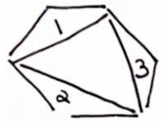
arcs — vertices



arrow $i \to j$



$\rightsquigarrow$ $1 \to 2 \to 3$ $\quad J_T = \mathbb{C}Q$



$\rightsquigarrow$

$$\begin{tikzcd} & 3 \arrow[dl, "\alpha_3"'] & \\ 1 \arrow[rr, "\alpha_1"'] & & 2 \arrow[ul, "\alpha_2"'] \end{tikzcd}$$

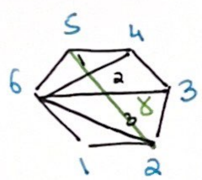
$J_T = \mathbb{C}Q / I$ ← composition of two paths in a 3-cycle

$I = \langle \alpha_1\alpha_2, \alpha_2\alpha_3, \alpha_3\alpha_1 \rangle$

Th [Caldero - Chapoton - Schiffler 2006] T - triangulation of a polygon

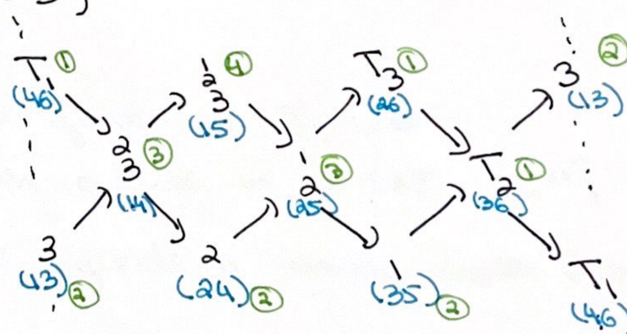
$$\left\{ \begin{matrix} \text{indecomposable} \\ J_T\text{-modules} \end{matrix} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{matrix} \text{diagonals in a} \\ \text{polygon not in } T \end{matrix} \right\}$$

ex.



$\rightsquigarrow J_T = \mathbb{C}(1 \to 2 \to 3)$

$(25) = \gamma \rightsquigarrow \mathbb{C} \xrightarrow{1} \mathbb{C} \to 0 = \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}$



AR-quiver of J_T together with T_i for every arc $i \in T$

Thm [Caldero - Chapoton 06] AR quiver of J_T yields Conway-Coxeter frieze by evaluating

$\widetilde{cc}(M) = \#$ of submodules of M

$\widetilde{cc}(T_i) = 1$

more generally

Caldero - chapoton map: $M \in \text{mod}\,\Lambda$

comes from injective presentation of M

Euler characteristic

$$CC(M) = x^{\text{ind}(M)} \sum_{\underline{e}} \chi(\underbrace{Gr_{\underline{e}}(M)}_{\text{Grassmannian of submodules of } M \text{ of dimension vector } \underline{e}})\, x^{-B_Q \underline{e}} \in \mathbb{Q}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

dimension vector

$$(B_Q)_{ij} = \#\{i \to j\} - \#\{j \to i\}$$

note in type A $\chi(Gr_{\underline{e}}(M)) = \begin{cases} 0 & \text{if } Gr_{\underline{e}}(M) = \emptyset \\ 1 & \text{o/w} \end{cases}$

ex $\mathbb{C}(1 \to 2 \to 3)$ $\quad B_Q = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$

$$CC(1) = x^{(-1,0,0)} \Big(\underbrace{x^{(0,0,0)}}_{\underline{e}=\underline{0}} + \underbrace{x^{(0,1,0)}}_{\underline{e}=(1,0,0)} \Big) = \frac{1+x_2}{x_1}$$

note $\widetilde{CC}(M) = CC(M)\big|_{\substack{x_i = 1 \\ \forall i}}$

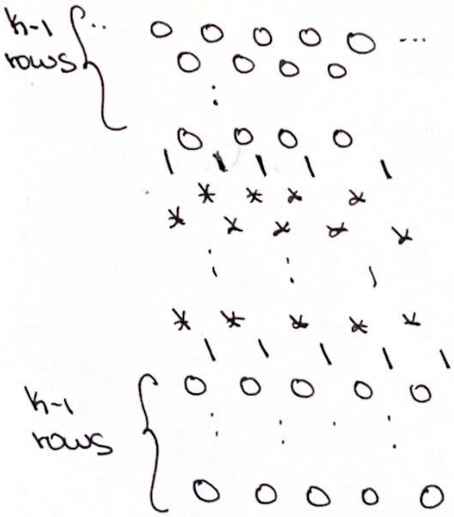
$$\widetilde{CC}(1) = \frac{1+1}{1} = 2$$

Thm Applying CC-map to AR-quiver of $\mathcal{J}_T \cup \oplus T_i$ yields a tame frieze with entries in $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ and specializing x_i's to 1 yields a Conway-Coxeter frieze

here set $CC(T_i) = x_i$

SL_k friezes and Grassmannian Cluster Algebras

An <u>SL_k-frieze</u> is an array such that:



- entries are in $\mathbb{Z}_{>0}$
- <u>diamond rule</u>  $\Rightarrow \det \boxed{*} = 1$
- <u>tameness</u>: $(k+1)\times(k+1)$ diamonds have $\det = 0$

<u>Goal</u>: find connection b/w SL_k friezes and cluster algebras/repr. th.

<u>Cluster Algebra</u>:

- Q - quiver w/o loops or 2-cycles on n vertices
- $\mathcal{A}_Q$ - cluster algebra $\subset \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, generated by a set of generators, called <u>cluster variables</u>, computed recursively via an operation called mutation, starting with an initial <u>seed</u>

$$\underbrace{(Q}_{\text{quiver}}, \underbrace{(x_1, \dots, x_n)}_{\text{cluster}})$$

<u>Grassmannians</u>

- $Gr(k,n)$ - Grassmannian of k-dim. linear subspaces of $\mathbb{C}^n$
  elements of $Gr(k,n)$ are $A = k\{\underbrace{[\quad\quad]}_{n}$ of full rank modulo row operations
- $\mathbb{C}[Gr(k,n)]$ - coordinate ring
- P_I - Plücker coordinate for $I \in \binom{[n]}{k}$, $P_I(A)$ - determinant of A on columns indexed by I

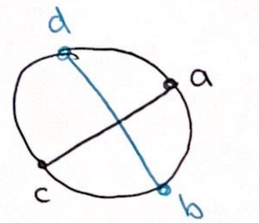
Thm $\mathbb{C}[Gr(k,n)] = \langle p_I \mid I \in \binom{[n]}{k} \rangle /$ Plücker relations

i.e. elements of $\mathbb{C}[Gr(k,n)]$ are polynomials in Plücker coordinates modulo relations

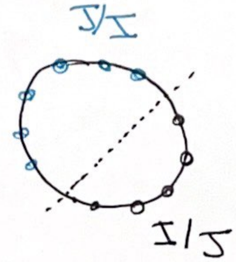
Thm [Scott 2006] $\mathbb{C}[Gr(k,n)]$ is a cluster algebra

$$\left\{ \begin{matrix} \text{Plücker} \\ \text{coordinates} \end{matrix} \right\} \subsetneq \left\{ \begin{matrix} \text{cluster} \\ \text{variables} \end{matrix} \right\}$$

Def $I, J \in \binom{[n]}{k}$ are <u>weakly separated</u> if there do not exist $a, c \in I \setminus J$, $b, d \in J \setminus I$ s.t. $a < b < c < d$ cyclically mod n



not weakly separated



weakly separated

Prop p_I, p_J are compatible cluster variables (i.e. can belong to the same cluster) iff I and J are weakly separated

$$\left\{ \begin{matrix} \text{clusters consisting} \\ \text{of Plücker coordinates} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{(maximal) pairwize weakly} \\ \text{separated sets of size} \\ k \times (n-k)+1 \end{matrix} \right\}$$

note
- p_I with $I = \{r, r+1, \dots, r+k-1\}$ is called <u>consecutive</u>, and it belongs to every cluster.
- can determine a quiver of a cluster combinatorially

Thm $\mathbb{C}[Gr(k,n)]$ is a cluster algebra of finite type (i.e. only finitely many cluster variables) iff (k,n) is

- $(2,n)$
- $(3,6)$
- $(3,7)$
- $(3,8)$
- $(n-2,n)$
- $(4,7)$
- $(5,8)$

ex $k=2$

$\mathbb{C}[Gr(2,n)]$

$\left\{\begin{array}{c}\text{cluster}\\ \text{variables}\end{array}\right\} = \left\{\begin{array}{c}\text{Plücker}\\ \text{coordinates}\\ P_{ij}\end{array}\right\} = \left\{\begin{array}{c}\text{diagonals in}\\ \text{an } n\text{-gon}\\ (ij)\end{array}\right\}$

$\{\text{clusters}\} = \left\{\begin{array}{c}\text{max. pairwise}\\ \text{weakly separated}\\ \text{sets}\end{array}\right\} = \left\{\begin{array}{c}\text{triangulations}\\ \text{of } n\text{-gons}\end{array}\right\}$

Categorification of a cluster algebra $\mathcal{A}_Q$

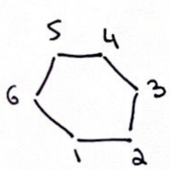
cluster category $\mathcal{C}$		cluster algebra $\mathcal{A}_Q$
ind. (rigid reachable) objects M	$\xmapsto{\text{CC-map}}$	cluster variables $x_M = CC(M)$
cluster-tilting object $T = \bigoplus_{i=1}^{n} T_i$		cluster $(x_{T_1}, \dots, x_{T_n})$
$\mathrm{End}_{\mathcal{C}} T$		quiver
mutation		mutation

ex $k=2$

Cluster Category $\mathcal{C}$
c.t.o. $T \in \mathcal{C}$ $\xrightarrow{\mathrm{Hom}(T,-)}$ Jacobian alg J_T
$J_T = \mathrm{End}_{\mathcal{C}} T$
$M \in \mathrm{mod}\, J_T$ $\longrightarrow$ Conway-Coxeter frieze
$\widetilde{CC}(M) = \#$ of submodules of M

CC-map $\dashrightarrow$ cluster algebra
$X_M \in \mathcal{A}_Q$ $\dashrightarrow$

in $k=2$ $\mathcal{C}$ = diagonals in a polygon with AR-quiver



$$\begin{tikzcd}[column sep=small]
(46) \arrow[dr] & & (15) \arrow[dr] & & (26) \arrow[dr] & & (13) \\
& (14) \arrow[ur] \arrow[dr] & & (25) \arrow[ur] \arrow[dr] & & (36) \arrow[ur] \arrow[dr] & \\
(13) \arrow[ur] & & (24) \arrow[ur] & & (35) \arrow[ur] & & (46)
\end{tikzcd}$$

c.t.o. T ⟷ triangulations

ex $T = (26) \oplus (36) \oplus (46)$

$$\mathrm{End}_e T = (26) \leftarrow (36) \leftarrow (46)$$

vertices ⟷ ind. summands of T
arrows ⟷ irred. morphisms

ex $T' = (13) \oplus (15) \oplus (35)$

$$\mathrm{End}_e T' = \begin{tikzcd} & (15) \arrow[dl] & \\ (13) \arrow[rr] & & (35) \arrow[ul] \end{tikzcd} \Big/ \text{composition of two arrows is zero}$$

<u>Back to Friezes</u>:

<u>Def</u> A <u>Plücker frieze</u> $P(k,n)$ is an array of Plückers appended with $k-1$ rows of zeros

$p_{12\ldots k-1\,n}$ $\quad$ $p_{123\ldots k}$ ~ consecutive
$p_{12\ldots k-1\,n-1}$ $\quad$ $p_{23\ldots k\,n}$ } almost consecutive
$\vdots$
$p_{12\ldots k-1\,k+2}$ $\quad \ldots$
$p_{12\ldots k-1\,k+1}$ $\quad$ $p_{23\ldots k\,k+3}$
$p_{12\ldots k}$ $\xrightarrow[\text{mod } n]{+1}$ $p_{23\ldots k\,k+2}$ $\quad$ ~ consecutive

entries are <u>almost consecutive</u> Plücker coordinates i.e.
p_I where $I = \{r, r+1, \ldots, r+k-2, s\}$

Th [Baur - Faber - Gratz - S. - Todorov]

- det of $k \times k$ diamond in $P(k,n)$ is a product of consecutive Plücker coordinates
- specializing consecutives to 1 yields an SL_k-frieze with entries in $\mathbb{C}[Gr(k,n)]$
- Specializing a cluster in $\mathbb{C}[Gr(k,n)]$ to 1's, yields an SL_k frieze over $\mathbb{Z}_{>0}$, call such friezes unitary.

  note: not every frieze is unitary only in $k=2$.
- $P(k,n)$ contains a cluster iff $k=2,3$ n-arbitrary or $k=4, n=6$
- in finite type with $k \leq \frac{n}{2}$ all SL_k friezes arize from specializing a cluster to a certain set of positive integers (uses categorification), not true in general

Remark If A is $k \times n$ matrix with entries in $\mathbb{Z}$ s.t. $P_I(A)=1$ if I is consecutive, then $P(k,n)(A)$ is an SL_k frieze with entries in $\mathbb{Z}$. Moreover every SL_k frieze arizes in this way [MGOST]

SL_k Tilings

An <u>SL_k-tiling</u> is an infinite array $\mathcal{M} = (m_{ij})_{i,j \in \mathbb{Z}}$ s.t. determinant of every $k \times k$ submatrix is 1 and determinant of every $(k+1)\times(k+1)$ submatrix is zero.

$$\mathcal{M} = \begin{matrix} & \vdots & & \\ \cdots & m_{11} \; m_{12} \; m_{13} & \cdots \\ \cdots & m_{21} \; m_{22} \; m_{23} & \cdots \\ & \vdots & & \end{matrix}$$

<u>note</u> SL_k-frieze can be extended uniquely to SL_k-tiling

$$\begin{matrix} 0 & & 0 & & 0 & \\ & 1 & & 1 & & 1 \\ & & a & & b & \\ & & & \vdots & & \end{matrix} \quad \rightsquigarrow \quad \begin{matrix} 0 & 1 & a & \\ & 0 & 1 & b & \cdots \\ & & 0 & 1 & \end{matrix}$$

<u>recall</u> [Short] SL_2-tilings $\leftrightarrow$ pairs of paths in Farey Graph

[BFGST] [CGOST] SL_k-friezes $\leftrightarrow$ Plücker friezes evaluated at a matrix A

want to generalize this to SL_k-tilings

<u>Def</u> A <u>path</u> in $\mathbb{Z}^k$ is a bi-infinite sequence $\gamma = (\gamma_i)_{i \in \mathbb{Z}}$ with $\gamma_i \in \mathbb{Z}^k$ such that $\det(\gamma_i, \gamma_{i+1}, \ldots, \gamma_{i+k-1}) = 1$ for all i.

P_k - collection of all such paths

$SL_k(\mathbb{Z})$ acts on P_k as follows $A\gamma = (A\gamma_i)_{i \in \mathbb{Z}} \in P_k$

<u>Th</u> [Peterson-S]

$$\Phi : (P_k \times P_k)/SL_k(\mathbb{Z}) \xrightarrow{\ \simeq\ } \mathbf{SL}_k$$

set of all SL_k tilings

$$(\gamma, \delta) \longmapsto \mathcal{M} = (m_{ij})_{i,j \in \mathbb{Z}}$$

$$m_{ij} = \det(\gamma_i, \gamma_{i+1}, \ldots, \gamma_{i+k-2}, \delta_j)$$

almost consecutive Plücker coordinates

<u>ex</u> $\gamma = \left(\cdots \begin{matrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 2 \end{matrix} \cdots \right) \quad \delta = \left(\cdots \begin{matrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{matrix} \cdots \right) \in P_3$

$$\Phi(\gamma, \delta) = \begin{matrix} & & \vdots & \\ \cdots & 1 & 3 & 6 & \\ & 1 & 1 & 1 & \cdots \\ & -4 & -3 & -2 & \\ & & \vdots & \end{matrix}$$

$$m_{12} = \det(\gamma_1, \gamma_2, \delta_2) = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{vmatrix} = 3$$

Cor $\quad P_k / SL_k(\mathbb{Z}) \xrightarrow{\simeq}$ infinite friezes with entries in $\mathbb{Z}$

$$\gamma \longmapsto \Phi(\tilde{\gamma}, \gamma)$$

Moreover, if γ is <u>skew-periodic</u> i.e. $\gamma_i = (-1)^{k-1}\gamma_{i+n}$ with period n

and $P_{k,n}$ - set of all skew-periodic paths with period n

Cor $\quad P_{k,n} / SL_k(\mathbb{Z}) \xrightarrow{\simeq}$ n-periodic friezes with entries in $\mathbb{Z}$

$$\gamma \longmapsto \Phi(\tilde{\gamma}, \gamma) = P(k,n)(A)$$

Plücker frieze

$k \times n$ matrix one period of γ

<u>proof idea of thm 1</u>

$\Rightarrow$ given (γ, δ) create a matrix $A = (\gamma_i, \gamma_{i+1}, \dots, \gamma_{i'}, \dots, \delta_j, \delta_{j+1}, \dots, \delta_{j'}, \dots)$ s.t. $P_I(A) = 1$ if I is consecutive

then the frieze $P(k,n)(A)$ contains a diamond $\boxed{*}$ that equals a square $\boxed{*}$ in $\Phi(\gamma, \delta)$.

This shows $\Phi(\gamma, \delta) \in SL_k$.

$\Leftarrow$ [Bergeron-Reutenauer]

$$\mathrm{Row}(i+k) = (-1)^{k-1}\mathrm{Row}(i) + \lambda_1 \mathrm{Row}(i+1) + \dots + \lambda_{k-1}\mathrm{Row}(i+k-1)$$

let $\lambda^{(i)} = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_{k-1} \end{pmatrix} \leadsto$ called <u>quiddity</u> vector for row i

Thm $\quad SL_k \xrightarrow{\simeq} SL_k(\mathbb{Z}) \times (\mathbb{Z}^{k-1})^{\mathbb{Z}} \times (\mathbb{Z}^{k-1})^{\mathbb{Z}}$

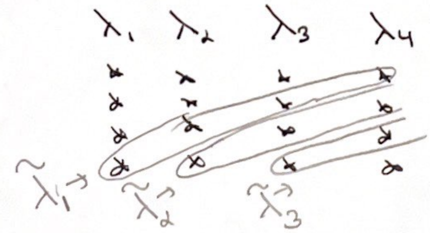
$$M \longmapsto (M_{[k],[k]}, \lambda, \mu)$$

quiddity sequences for rows and columns

Prop [PS] $M = \Phi(\gamma, \delta)$ quiddity sequence for rows equals quiddity sequence for δ, and quiddity sequence for columns equals quiddity sequence for $\tilde{\gamma}$.

$k=5$

If γ has quiddity sequence



$\tilde{\gamma}$ has quiddity sequence $\tilde{\lambda}$:

Thus given $\mathcal{M}$ can build quiddity sequences for γ and δ and then rescale them by an appropriate matrix $A \in SL_k(\mathbb{Z})$ to get the inverse of Φ

<u>Duality</u> $\mathcal{M} \in SL_k(\mathbb{Z})$, define $\mathcal{M}^*$ - dual tiling with entries m^*_{ij} - determinant of $(k-1)\times(k-1)$ submatrix of $\mathcal{M}$ with top left entry m_{ij}.

[BR] $(\mathcal{M}^*)^* = \mathcal{M}$ up to shift in indices

<u>Th</u> [PS] $(\Phi(\gamma,\delta))^* = \Phi(A\tilde{\gamma}, \tilde{\delta})$ for some $A \in SL_k(\mathbb{Z})$

$\Rightarrow (\mathcal{M}^*)^*$ coincides with $\mathcal{M}$ up to shift in indices

<u>Positivity</u> connection b/w positive friezes and alternating quiddity sequences

<u>Gale duality</u>: SL_k friezes of period n $\mathcal{F}$ $\longleftrightarrow$ SL_{n-k} friezes of period n $\mathcal{F}^G$

almos consecutive Plücker coordinates P_I, $I \in \binom{[n]}{k}$ entries of $\mathcal{F}$ $\longleftrightarrow$ semi-consecutive Plücker coordinates P_I, $I \in \binom{[n]}{k}$ entries of $\mathcal{F}^G$ = entries of quiddity sequence of $\mathcal{F}$

<u>Def</u> P_I is semi-consecutive if $I = \{i, i+1, \dots, i+k\} \setminus \{j\}$
$i < j < i+k$

<u>Thm</u> [PS] An SL_k frieze is positive iff its quiddity sequence is alternating

whenever
- $k=2$, $n \leq 9$
- $k=3,6$, $n \leq 8$
- $k=4$, $n \leq 7$
- $k=5$, $n \leq 8$ with the exception in $Gr(5,8)$ of the frieze of all 1's with quiddity sequence $\begin{pmatrix} 8 \\ \vdots \\ 8 \end{pmatrix}$

<u>Cor</u> there are 26953 positive friezes of type (5,8)
[Zhang] 26952 positive friezes of type (3,8)

<u>Questions</u>:

- investigate $Gr(4,8)$ case in the theorem above
- find geometric/comb model for quiddity sequences for positive SL_k friezes
- find a condition on the path γ s.t. $\Phi(\gamma,\gamma)$ is a positive frieze i.e. what is the analog of "clockwize" in the Farey Graph
- connection b/w paths in $\mathbb{Z}^k$ and geometry i.e. higher Farey Graph; [Felikson-Tumarkin-Karpenkov-S]
  3D Farey Graph and SL_2 tilings over Eisenstein Integers
- connection b/w SL_k tilings and cluster algebras/repr. theory

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Questions:

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