# Some determinants and relations in Heronian friezes

Anja Šneperger

# Heronian diamond

## Definition (FS)

A *Heronian diamond* is an ordered 10-tuple of complex numbers $(a, b, c, d, e, f, p, q, r, s)$ satisfying the following equations:

$$p^2 = H(b, c, e), q^2 = H(a, d, e)$$

$$r^2 = H(a, f, b), s^2 = H(c, f, d)$$

$$r + s = p + q$$

$$4ef = (p + q)^2 + (a - b + c - d)^2$$

$$e(r - s) = p(a - d) + q(b - c),$$

where

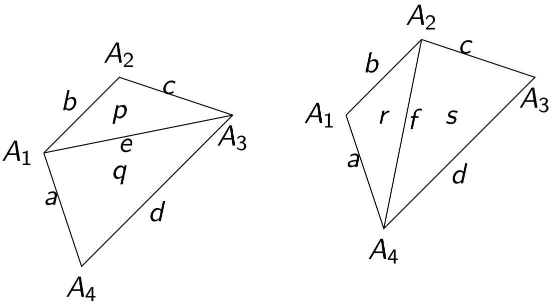
$$H(x, y, z) := -x^2 - y^2 - z^2 + 2xy + 2xz + 2yz.$$

# Heronian diamond – motivation

The measurements associated with triangulations of a plane quadrilateral satisfy the equations from the previous slide.

# Heronian diamond – motivation

The measurements associated with triangulations of a plane quadrilateral satisfy the equations from the previous slide.



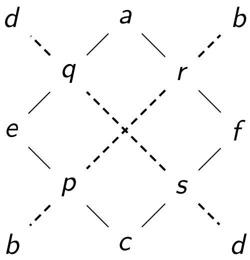

$a, b, c, d, e, f$ : **squared** distances between corresponding vertices
$p, q, r, s$ : four times signed areas of the corresponding triangles

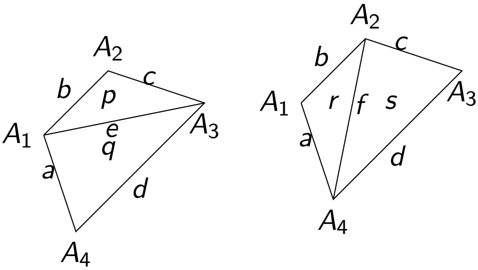
# Heronian diamond – graphical representation

Instead of listing the components of a Heronian diamond $(a, b, c, d, e, f, p, q, r, s)$ as a row of 10 numbers, we will typically arrange them in a diamond pattern:

# Heronian diamond – graphical representation

Instead of listing the components of a Heronian diamond $(a, b, c, d, e, f, p, q, r, s)$ as a row of 10 numbers, we will typically arrange them in a diamond pattern:



# Some notation...

Some notation for the future reference:

- $A_i = (x_i, y_i), A_j = (x_j, y_j), A_k = (x_k, y_k)$ points in the complex plane
- **squared distance** between $A_i$ and $A_j$:
  $\mathbf{x_{ij}} := \overline{\mathrm{A_i A_j}}^2$
- $4 \cdot$**signed area of the triangle** $\mathrm{A}_i A_j A_k$: $\mathbf{S_{ijk}}$

# Heronian friezes

How to get a **Heronian frieze**?

# Heronian friezes

How to get a **Heronian frieze**?

- $(A_1, A_2, \ldots, A_n)$ – a labeled polygon in the complex plane

# Heronian friezes

How to get a **Heronian frieze**?

- $(A_1, A_2, \ldots, A_n)$ – a labeled polygon in the complex plane
- each quadruple of vertices of the form $(A_a, A_{a+1}, A_b, A_{b+1})$ gives rise to a Heronian diamond ($a$ and $b$ are distinct elements of $\{1, 2, \ldots, n\}$, and addition is modulo $n$)

$$\begin{array}{ccccc}
x_{b(b+1)} & & x_{a(b+1)} & & x_{a(a+1)} \\
& S_{ab(b+1)} & & S_{a(a+1)(b+1)} & \\
x_{ab} & & \mathbf{a(a+1)b(b+1)} & & x_{(a+1)(b+1)} \\
& S_{a(a+1)b} & & S_{(a+1)b(b+1)} & \\
x_{a(a+1)} & & x_{(a+1)b} & & x_{b(b+1)}
\end{array}$$

# Heronian friezes

How to get a **Heronian frieze**?

- $(A_1, A_2, \dots, A_n)$ – a labeled polygon in the complex plane
- each quadruple of vertices of the form $(A_a, A_{a+1}, A_b, A_{b+1})$ gives rise to a Heronian diamond ($a$ and $b$ are distinct elements of $\{1, 2, \dots, n\}$, and addition is modulo $n$)

$$
\begin{array}{ccccc}
x_{b(b+1)} & & x_{a(b+1)} & & x_{a(a+1)} \\
 & S_{ab(b+1)} & & S_{a(a+1)(b+1)} & \\
x_{ab} & & \mathbf{a(a+1)b(b+1)} & & x_{(a+1)(b+1)} \\
 & S_{a(a+1)b} & & S_{(a+1)b(b+1)} & \\
x_{a(a+1)} & & x_{(a+1)b} & & x_{b(b+1)}
\end{array}
$$

- glue those Heronian diamonds together + impose some boundary conditions $\rightsquigarrow$ *Heronian frieze of order $n$*

# Heronian frieze – example

$$x_{45} \quad x_{45} \quad x_{51} \quad x_{51} \quad x_{12} \quad x_{12} \quad x_{23} \quad x_{23} \quad x_{34} \quad x_{34} \quad x_{45} \quad x_{45}$$

$$x_{44} \quad x_{55} \quad x_{11} \quad x_{22} \quad x_{33} \quad x_{44}$$

$$S_{434} \quad S_{454} \quad S_{545} \quad S_{515} \quad S_{151} \quad S_{121} \quad S_{212} \quad S_{232} \quad S_{323} \quad S_{343} \quad S_{434} \quad S_{454}$$

$$x_{43} \quad \mathbf{4534} \quad x_{54} \quad \mathbf{5145} \quad x_{15} \quad \mathbf{1251} \quad x_{21} \quad \mathbf{2312} \quad x_{32} \quad \mathbf{3423} \quad x_{43} \quad \mathbf{4534} \quad x_{54}$$

$$S_{453} \quad S_{534} \quad S_{514} \quad S_{145} \quad S_{125} \quad S_{251} \quad S_{231} \quad S_{312} \quad S_{342} \quad S_{423} \quad S_{453} \quad S_{534} \quad S_{514}$$

$$x_{53} \quad \mathbf{5134} \quad x_{14} \quad \mathbf{1245} \quad x_{25} \quad \mathbf{2351} \quad x_{31} \quad \mathbf{3412} \quad x_{42} \quad \mathbf{4523} \quad x_{53} \quad \mathbf{5134} \quad x_{14}$$

$$S_{513} \quad S_{134} \quad S_{124} \quad S_{245} \quad S_{235} \quad S_{351} \quad S_{341} \quad S_{412} \quad S_{452} \quad S_{523} \quad S_{513} \quad S_{134}$$

$$x_{13} \quad \mathbf{1234} \quad x_{24} \quad \mathbf{2345} \quad x_{35} \quad \mathbf{3451} \quad x_{41} \quad \mathbf{4512} \quad x_{52} \quad \mathbf{5123} \quad x_{13} \quad \mathbf{1234}$$

$$S_{123} \quad S_{123} \quad S_{234} \quad S_{234} \quad S_{345} \quad S_{345} \quad S_{451} \quad S_{451} \quad S_{512} \quad S_{512} \quad S_{123} \quad S_{123}$$

$$x_{12} \quad \mathbf{1223} \quad x_{23} \quad \mathbf{2334} \quad x_{34} \quad \mathbf{3445} \quad x_{45} \quad \mathbf{4551} \quad x_{51} \quad \mathbf{5112} \quad x_{12} \quad \mathbf{1223} \quad x_{23}$$

$$S_{112} \quad S_{122} \quad S_{223} \quad S_{233} \quad S_{334} \quad S_{344} \quad S_{445} \quad S_{455} \quad S_{551} \quad S_{511} \quad S_{112} \quad S_{122} \quad S_{223}$$

$$x_{11} \quad x_{22} \quad x_{33} \quad x_{44} \quad x_{55} \quad x_{11} \quad x_{22}$$

$$x_{12} \quad x_{12} \quad x_{23} \quad x_{23} \quad x_{34} \quad x_{34} \quad x_{45} \quad x_{45} \quad x_{51} \quad x_{51} \quad x_{12} \quad x_{12} \quad x_{23}$$

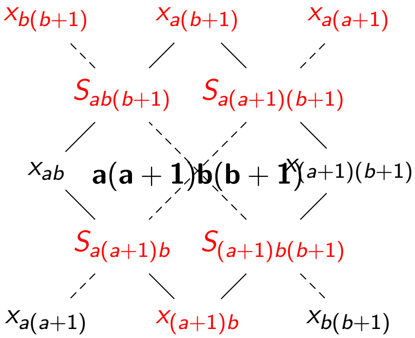
Figure: Fragment of a polygonal Heronian frieze of order 5

# Vanishing determinants

**Theorem (Š)**

*Let $P = (A_1, A_2, \ldots, A_n)$ be a cyclic n-gon with anticlocwise ordering of vertices. Then, for every diamond of the corresponding Heronian frieze the following determinant vanishes:*

$$\begin{vmatrix} x_{b(b+1)} & S_{ab(b+1)} & S_{(a+1)b(b+1)} \\ x_{a(a+1)} & S_{a(a+1)(b+1)} & S_{a(a+1)b} \\ 0 & x_{a(b+1)} & x_{(a+1)b} \end{vmatrix} = 0.$$


$x_{b(b+1)}$
$x_{a(b+1)}$
$x_{a(a+1)}$
$S_{ab(b+1)}$
$S_{a(a+1)(b+1)}$
$x_{ab}$
$\mathbf{a(a+1)b(b+1)}$
$x_{(a+1)(b+1)}$
$S_{a(a+1)b}$
$S_{(a+1)b(b+1)}$
$x_{a(a+1)}$
$x_{(a+1)b}$
$x_{b(b+1)}$

# Adjacent diamonds

## Theorem (Š)

*Let $P = (A_1, A_2, ..., A_{m+1})$ be a cyclic $(m+1)-$ gon, with anticlockwise ordering of the vertices, where $m \geq 2$ is an integer. Then*

$$S_{1,m,m+1} \prod_{i=2}^{m-1} x_{i,m+1} = \sum_{j=1}^{m-1} S_{j,j+1,m+1} \frac{\prod_{k=1}^{m} x_{k,m+1}}{x_{j,m+1} x_{j+1,m+1}},$$

*where the entries $S_{***}$ and $x_{**}$ are the entries of the corresponding Heronian frieze.*

# Adjacent diamonds

## Corollary

*$P = (A_1, A_2, \ldots, A_n)$ – a cyclic n-gon, $F$ – the corresponding Heronian frieze;*
*if $m \in \mathbb{Z}$, $2 \le m \le n-1$, and $q, q+1, \ldots, q+m-2, r, r+1$ are distinct, then*

$$S_{q,r,r+1} \prod_{k=q+1}^{q+m-2} x_{k,r+1} =$$

$$S_{q,q+1,r+1} x_{q+2,r+1} x_{q+3,r+1} \cdots x_{q+m-2,r+1} x_{r,r+1}$$
$$+ S_{q+1,q+2,r+1} x_{q,r+1} x_{q+3,r+1} \cdots x_{q+m-2,r+1} x_{r,r+1}$$
$$\vdots$$
$$+ S_{q+m-2,r,r+1} x_{q,r+1} x_{q+1,r+1} \cdots x_{q+m-4,r+1} x_{q+m-3,r+1}.$$

# Example

Let $P = (A_1, A_2, ..., A_{10})$ be a cyclic 10-gon, with anticlockwise ordering of vertices, and $m = 5$.

# Example

Let $P = (A_1, A_2, ..., A_{10})$ be a cyclic 10-gon, with anticlockwise ordering of vertices, and $m = 5$.
Consider the vertices $A_3, A_4, A_5, A_6, A_9, A_{10}$.

# Example

Let $P = (A_1, A_2, ..., A_{10})$ be a cyclic 10-gon, with anticlockwise ordering of vertices, and $m = 5$.
Consider the vertices $A_3, A_4, A_5, A_6, A_9, A_{10}$. We get

$$S_{3,9,10}x_{4,10}x_{5,10}x_{6,10} = S_{3,4,10}x_{5,10}x_{6,10}x_{9,10} + S_{4,5,10}x_{3,10}x_{6,10}x_{9,10} + S_{5,6,10}x_{3,10}x_{4,10}x_{9,10} + S_{6,9,10}x_{3,10}x_{4,10}x_{5,10}.$$

# Main Theorem

## Theorem (Š)

*Let $P = (A_1, A_2, \dots, A_n)$ be a cyclic n-gon with vertices ordered anticlockwise, and $n > 4$ an even number. Then*

$$\sum_{m=1}^{n} (-1)^{m+1} x(m) S(m) = 0,$$

*where*

$$x(m) = x_{12} x_{34}^{c} \prod_{\substack{l=m \\ l \text{ even}}}^{n-2} x_{l,l+1}^{\frac{l-2}{2}} \prod_{\substack{l=5 \\ l \text{ odd}}}^{m-1} x_{l,l+1}^{\frac{l-3}{2}},$$

$$S(m) = \left( \prod_{k=1}^{m-1} \left( \prod_{\substack{l=k+1 \\ l \text{ even}}}^{m-2} S_{k,l,l+1} \prod_{\substack{l=m+1 \\ l \text{ odd}}}^{n-1} S_{k,l,l+1} \right) \right) \left( \prod_{k=m}^{n-3} \prod_{\substack{l=k+2 \\ l \text{ odd}}}^{n-1} S_{k+1,l,l+1} \right),$$

*for even m.*

# Example

Take $n = 6$. We get

$$x_{12}x_{45}S_{234}S_{256}S_{356}S_{456} - x_{12}x_{45}S_{134}S_{156}S_{356}S_{456} + x_{12}x_{45}S_{124}S_{156}S_{256}S_{456} - x_{12}x_{45}S_{123}S_{156}S_{256}S_{356} + x_{45}x_{56}S_{123}S_{124}S_{126}S_{346} - x_{12}x_{56}S_{123}S_{145}S_{245}S_{345} = 0.$$

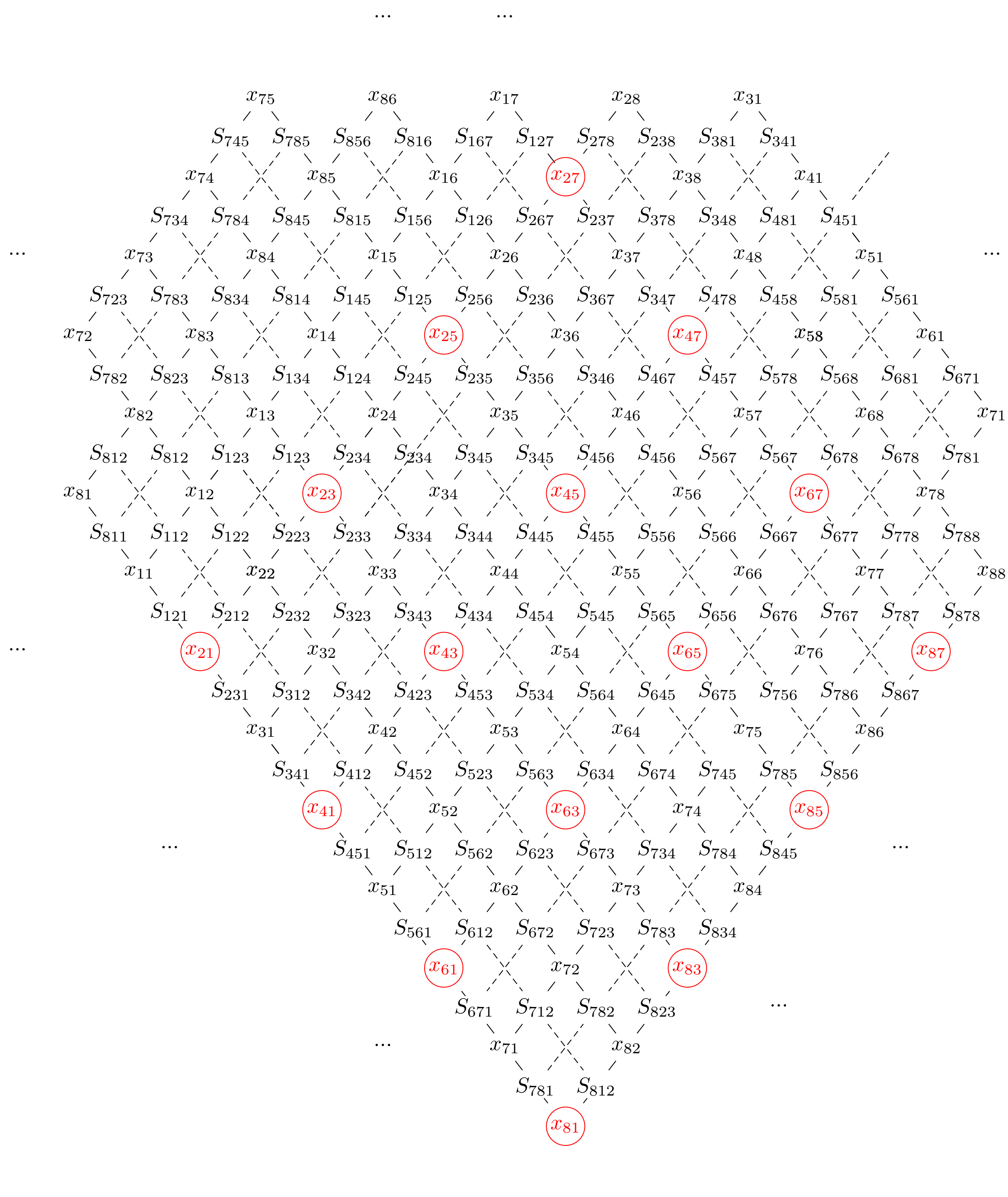


Figure 12: