Exclusion sets for Gaussian Integer Continued Fractions

(Note to accompany a poster presented at CIRM, 13th May 2025)

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1 PRELIMINARIES

This note gives results about the convergence of the infinite negative Gaussian integer continued fraction (GICF)

$$\gamma = [b_0, b_1, \ldots, b_k, \ldots] = b_0 - \cfrac{1}{b_1 - \cfrac{1}{b_2 - \cdots}}$$

with $b_i \in \mathbb{Z}[i]$, and $b_k \neq 0$ for $k > 0$. Truncating the continued fraction at b_k, we obtain the k'th convergent, the Gaussian rational $\dfrac{p_k}{q_k}$, with $p_k, q_k \in \mathbb{Z}[i]$, and $q_k \neq 0$ if $k > 0$. We use the recurrences given by

$$\begin{pmatrix} p_{k+1} & p_k \\ q_{k+1} & q_k \end{pmatrix} = \begin{pmatrix} p_k & p_{k-1} \\ q_k & q_k - 1 \end{pmatrix} \begin{pmatrix} b_k & 1 \\ -1 & 0 \end{pmatrix}; \quad \frac{p_0}{q_0} = \frac{1}{0}, \ \frac{p_1}{q_1} = \frac{b_0}{1}.$$

The continued fraction γ converges if the infinite sequence $\left(\dfrac{p_k}{q_k}\right)$ converges in $\mathbb{C}$.

As $\mathbb{C}$ is complete, γ converges if and only if

$$\left|\frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}}\right| = \left|\frac{p_k q_{k-1} - p_{k-1} q_k}{q_{k-1} q_k}\right| = \frac{1}{|q_{k-1}||q_k|} \longrightarrow 0 \text{ as } \ k \to \ \infty,$$

that is if and only if $|q_{k-1}||q_k| \to \infty$ as $k \to \infty$.

It is straightforward to prove that the GICF $(i, i, i, i, \ldots)$ converges to $\frac{1}{2}(1+\sqrt{5})i$, and that the GICF $(1, -1, 1, -1, \ldots)$ converges to $\frac{1}{2}(1+\sqrt{5})$. It is also straightforward to prove that the GICF $(1+i, 1-i, 1+i, 1-i, \ldots)$ does not converge. We have not been able to find a necessary and sufficient condition for convergence which only depends on the coefficients. Instead, we give sufficent conditions in the form of exclusion sets. An exclusion set $\mathcal{E}$ is a sets of strings of coefficients such that, if the list of coefficients of a GICF does not contain, infinitely often, any strings in $\mathcal{E}$, then it converges.

2 A FIRST EXCLUSION SET

The following theorem is well known. We give a proof for completeness.

Theorem 2.1. *The GICF $[b_0, b_1, \ldots, b_k, \ldots]$ converges if $|b_i| \geq 2$ for all $i > 0$.*

Proof. Aiming for a proof by induction, we note that $|q_1| > |q_0|$, and assume that $|q_i| > |q_{i-1}|$ for all $i \leq k$. Then

$$|q_{k+1}| = |b_k q_k - q_{k-1}| \geq ||b_k||q_k| - |q_{k-1}|| > |q_k| \, ||b_k| - 1| \, .$$

□

So a GICF converges if it has no coefficients with modulus stictly less than 2 occuring infinitely often, which gives us an exclusion set.

Corollary 2.2. *The set $\{(0), \pm(1), \pm(i), \pm(1 \pm i)\}$ is an exclusion set for GICFs.*

3 A better exclusion set

The exclusion set given in Corollary 2.2 excludes continued fractions such as $(i, i, i, \ldots)$ which are known to converge. We attempt to find a more satisfactory result.

Theorem 3.1. *The GICF $[b_0, b_1, \ldots, b_k, \ldots]$ converges if $|b_{k-1}b_k - 1| \geq 3$ for all $k > 0$.*

Proof. Aiming for a proof by induction, note that $|q_2| = |b_1| > |q_0|$, and assume that $|q_i| > |q_{i-2}|$ for all $i \leq k$. From $q_{k+1} = b_k q_k - q_{k-1}$, we obtain $b_k q_k = q_{k+1} + q_{k-1}$, so we have

$$|b_k||q_{k-2}| < |b_k|||q_k| \leq |q_{k+1} + |q_{k-1}|,$$

and also, putting $g_k = b_{k-1}b_k$,

$$\begin{aligned} |q_{k+1}| &= |b_k(b_{k-1}q_k - q_{k-2}) - q_{k-1}| \\ &\geq ||g_k - 1||q_{k-1}| - |b_k||q_{k-2}|| \\ &> |g_k - 1||q_{k-1}| - |q_{k-1}| - |q_{k+1}.| \end{aligned}$$

As $|g_k - 1| \geq 3$, this implies $|q_{k+1} > 2|q_{k-1}| - |q_{k+1}|$. It follows that $|q_{k+1}| > |q_{k-1}|$, therefore $|q_{k-1}||q_k| < |q_k||q_{k+1}|$ for all k, and the continued fraction converges.

□

So a GICF converges unless its list if coefficients is such that the product g_i of two successive coefficients is not such that $|g_i - 1| < 3$ infinitely often. So an exclusion set consists of those pairs whose product is a Gaussian integer situated inside the circle centred at 1 with radius 3, as shown in Figure 3.1.

Corollary 3.2. *An exclusion set for GICFs consists of those pairs whose product is in the set*

$$\{(0), \pm(1), \pm(i), \pm 2i, \pm(1 \pm i), \pm(1 \pm 2i), 2, 2 \pm i, 2 \pm 2i, 3, 3 \pm i, 3 \pm 2i\}.$$

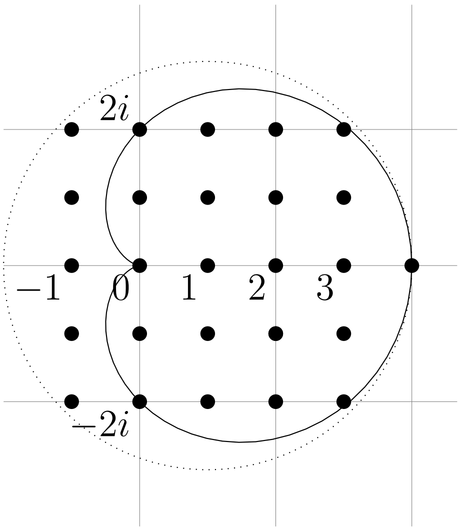

Figure 3.1: The cardioid $r = 2(1 + \cos\theta)$ and the circle centered at 1 with radius 3 (dotted).

4 A more satisfactory exclusion set

The exclusion set given in Corollary 3.2 excludes fewer continued fractions than that in Corollary 2.2. However it still excludes continued fractions such as $(i, i, i, \ldots)$ which are known to converge. We continue to attempt to find a more satisfactory result.

Theorem 4.1. *The GICF* $[b_0, b_1, \ldots, b_k, \ldots]$ *converges if* $r_k > 2(1 + \cos\theta_k)$ *for all* $k > 0$*, where*

$$r_k e^{i\theta_k} = g_k = b_{k-1} b_k.$$

Proof.

$$\text{For } k > 1, \quad \text{let} \quad z_k = \frac{q_{k-2}}{q_k}, \quad w_k = \frac{1}{z_{k+1}} = \frac{q_{k+1}}{q_{k-1}}, \quad \text{and} \quad a_k = \frac{q_{k-1}}{q_k}.$$

The assumption in the theorem implies that g_k is outside the cardioid $r_k = 2(1 + \cos\theta_k)$, as shown in Figure 3.1. The Gaussian integers inside the circle with centre 1 and radius 2 are inside the cardioid, so we can assume that g_k is not such that $|g_k - 1| < 2$, and, in particular, that $|g_2 - 1| \geq 2$. Then we have $|z_2| = 0 < 1$ and, as $|q_3| = |b_2 b_1 - 1| = |g_2 - 1| \geq 2$, $\quad z_3 = \frac{q_1}{q_3} < 1$. Aiming for a proof by induction, we now show that, if $|z_i| < 1$ for all $i \leq k$, then $|w_k| > 1$ and so $|z_{k+1}| < 1$ for all k.

From $q_k = b_{k-1} q_{k-1} - q_{k-2}$, $\quad 1 = a_k b_{k-1} - z_k$ so that $a_k = \dfrac{1 + z_k}{b_{k-1}}$.

From $q_{k+1} = b_k q_k - q_{k-1}$, $\quad w_k = \dfrac{b_k}{a_k} - 1 = \dfrac{g_k}{1 + z_k} - 1.$

Therefore, putting $f_k(z) = \dfrac{g_k}{1 + z} - 1$, we have $f_k(z_k) = w_k$.

For ease of legibility, we omit the suffix k in what follows. Let Δ be the disk $\{z : |z| < 1\}$. Define $f_1(z) = 1 + z,\ f_2(z) = \frac{1}{z},\ f_3(z) = rz,\ f_4(z) = ze^{i\theta}$ and $\quad f_5(z) = z - 1$, so that

$$f = f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1(z).$$

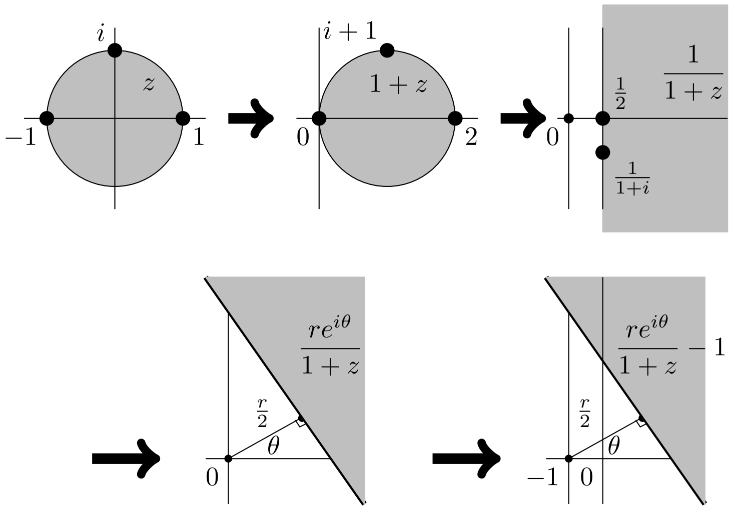

Figure 4.1: The transformation of the unit circle $|z < 1|$ by the function $f : f(z) = \frac{g_k}{1+z} - 1$

As shown in Figure 4.1, $f_1(\Delta) = \{z : |z-1| < 1\}$; $f_2 \circ f_1(\Delta) = \{z = x+iy, x, y \in \mathbb{R} : x > 1/2\}$ and $f_3 \circ f_2 \circ f_1(\Delta) = \{z = x+iy, x, y \in \mathbb{R} : x > r/2\}$.

Let l be the line $x = r/2$. The function f_4 rotates it through an angle θ about the origin, giving a line $f_4(l)$ which crosses the real axis at $\frac{r}{2\cos\theta}$, and the imaginary axis at $\frac{r}{2\sin\theta}i$. Then f_5 translates the line $f_4(l)$ through -1, so that $f(\Delta)$ is a line which cuts the real axis at $\frac{r}{2\cos\theta} - 1$. If d is the distance from the origin to the line $f(\Delta)$, we have

$$\frac{d}{r/2} = \frac{\frac{r}{2\cos\theta}}{\frac{r}{2\cos\theta} - 1}, \quad \text{so that} \quad d = \frac{r}{2} - \cos\theta.$$

Then $|w| > 1$ if $d > 1$, and therefore if $r > 2(1 + \cos\theta)$ the continued fraction converges. $\square$