Exclusion sets for Gaussian Integer Continued Fractions

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Gaussian Integer Continued Fractions

Given an infinite sequence $(b_0, b_1, \ldots, b_k, \ldots)$, where coefficients b_k are members of $\mathbf{Z}[i]$, and $b_k \neq 0$ for $k > 0$, an infinite Gaussian integer continued fractions is

$$(b_0, b_1, \ldots, b_k, \ldots) = b_0 - \cfrac{1}{b_1 - \cfrac{1}{b_2 - \cdots}}.$$

Truncating the fraction at b_{k-1} gives the convergent $v_k = \dfrac{p_k}{q_k}$.

The sequence of convergents of the continued fraction $(i, i, i, i, \ldots)$ is

$$(i, 2i, \tfrac{3}{2}i, \tfrac{5}{2}i, \tfrac{8}{5}i, \tfrac{13}{8}i, \ldots), \quad \text{which converges (to} \tfrac{1}{2}(1+\sqrt{5})i).$$

The continued fraction $(0, i, 1+i, 1-i, 1+i, 1-i, \ldots)$ has convergents

$$0, i, 1+i, 1, 0, i, \ldots.$$

These form the path in the complex plane shown in Figure 1. It is a repeating loop, and the continued fraction does not converge.

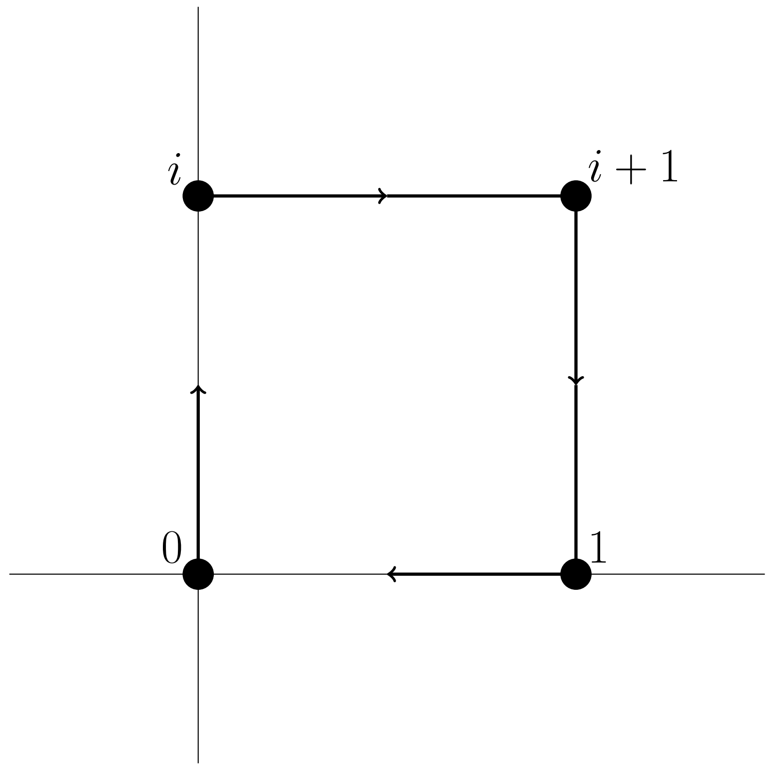

Figure 1. The path of the continued fraction $(0, i, 1+i, 1-i, 1+i, 1-i, \ldots)$

Which continued fractions converge?

Can we tell, just by considering the coefficients, whether a continued fraction converges?

It does if the distance between its convergents tends to zero, that is if $\dfrac{1}{|q_{k-1}||q_k|} \to \infty$ as $k \to \infty$. This is the case if $|q_{k-1}| < |q_k|$, that is when

$$|b_k| \geq 2 \quad \text{for all } k.$$

So the fraction converges if it does not contain, infinitely often, any coefficients in the set

$$\{\pm 1, \pm i, 1+i, 1-i, -1+i, -1-i\}.$$

Exclusion sets

An exclusion set for a continued fraction is a set of strings of coefficients such that, if the list of coefficients of the continued fraction does not contain, infinitely often, any of the strings in the exclusion set, then it converges.

Our first, one element, exclusion set for Gaussian integer continued fractions is the set

$$\{\pm 1, \pm i, 1+i, 1-i, -1+i, -1-i\}.$$

This unfortunately excludes the continued fraction $(i, i, i, \ldots)$, which does converge. We can improve this result.

A continued fraction also converges if $|q_{k+1}| > |q_{k-1}|$ for all k. This is true if $|b_{k-1}b_k - 1| \geq 3$. So it converges unless it contains, infinitely often, pairs whose product is in the open circle with centre 1 and radius 3. As shown by the dotted line in Figure 2, this circle contains many Gaussian integers, and we obtain a two element exclusion set. Representing the equivalence class of pairs (b_{k-1}, b_k) such that $b_{k-1}b_k = g$ by the Gaussian integer g, we write this as the set of pairs whose product g is in the set

$$\{-1, 1, 2, 3, -1 \pm i, \pm i, 1 \pm i, 2 \pm i, 3 \pm i, -1 \pm 2i, \pm 2i, 1 \pm 2i, 2 \pm 2i, 3 \pm 2i\}.$$

This set is rather large and still excludes the continued fraction $(i, i, i, \ldots)$, as $b_{k-1}b_k = ii = -1$. It does not contain 4. In the next column, we show a way of obtaining a better exclusion set.

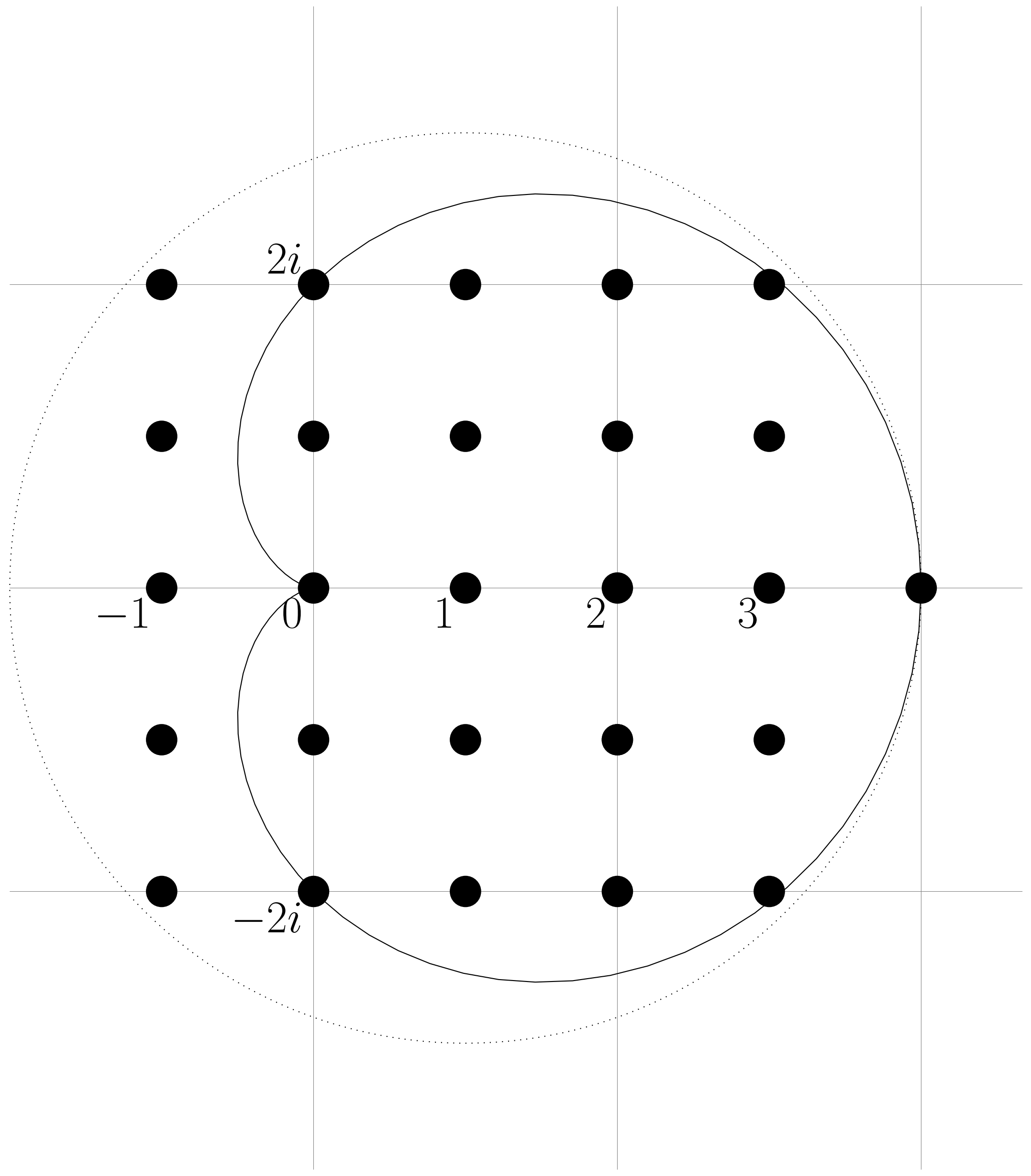

Figure 2. The cardioid $r = 2(1 + \cos\theta)$ and the circle centered at 1 with radius 3 (dotted).

$$\text{Putting}\, z = \frac{q_{k-2}}{q_k} \quad \text{and } f(z) = \frac{q_{k+1}}{q_{k-1}} \text{ gives } \quad f(z) = \frac{b_{k-1}b_k}{1+z} - 1.$$

So, if $|z| < 1$ implies $|f(z)| > 1$ for all k, the continued fraction converges. For which values of $b_{k-1}b_k$ is this true? We put $b_{k-1}b_k = re^{i\pi\theta}$.

If $|z| < 1$, z lies inside the unit circle; the function f transforms the unit circle as shown in Figure 3.

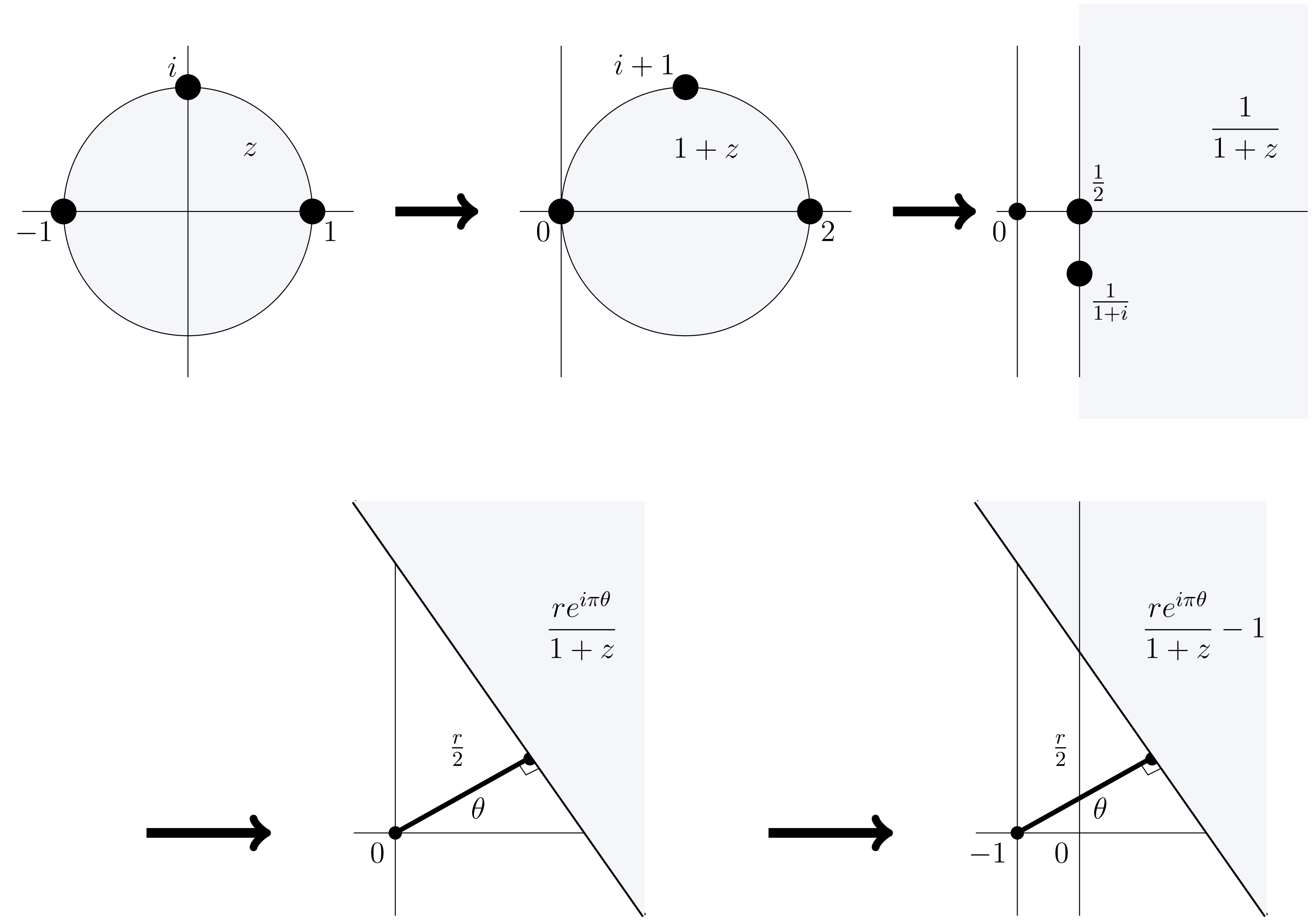

Figure 3. The transformation of the unit circle $|z < 1|$ by the function $f : f(z) = \frac{\beta_k}{1+z} - 1$

The distance from 0 to the shaded region in the final diagram is $1 + \cos\theta$, so $f(z)$ is outside the unit circle, and the continued fraction converges, if

$$r > 2(1 + \cos\theta).$$

The curve $r = 2(1 + \cos\theta)$ is a cardioid, as shown in Figure 2. The continued fraction will not converge if g is inside or in the border of this cardioid. We do not exclude $g = 4$, as it satisfies $g = b_{k-1}b_k - 1| \geq 3$. So we have the following result, which does not exclude $(i, i, i, i \ldots)$, though still excludes several converging continued fractions such as $(1, 1+i, 1, 1+i, \ldots)$.

Theorem

A Gaussian integer continued fraction converges if its list of coefficients does not contain, infinitely often, a pair (b_{k-1}, b_k) such that the product $b_{k-1}b_k$ is in the set

$$\{1, 2, 3, \pm i, 1 \pm i, 2 \pm i, 3 \pm i, \pm 2i, 1 \pm 2i, 2 \pm 2i, 3 \pm 2i\}.$$