Counting tame SL_3- and some SL_4-frieze patterns over finite fields

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Frieze patterns in algebra, combinatorics and geometry

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SL_k-frieze patterns

1. SL_k-frieze patterns are arrays of elements of some field, where each $k \times k$-diamond has determinant 1.
2. The first $k-1$ rows only contain zeroes and the k-th row ones. Same for the last $k-1$ rows and the row above that.
3. The number of nontrivial rows is the width w of the SL_k-frieze pattern.
4. The SL_k-frieze pattern is called tame if every $(k+1) \times (k+1)$-diamond has determinant zero.
5. Tame SL_k-frieze patterns are periodic with period $n = w + k + 1$.
6. There is an embedding of tame SL_k-friezes with period n into the Grassmannian $\mathscr{G}(k, n)$ such that the entries correspond to certain Plücker coordinates.

Consider the fundamental region of a tame SL_k-frieze pattern of width w with entries in some field K, where $n = w + k + 1$:

$$\begin{array}{ccccccccccccc}
\Delta_w(1) & & \Delta_w(2) & & \dots & \Delta_w(n) & & & & & \\
& \Delta_{w-1}(2) & & & \Delta_{w-1}(3) & \dots & \Delta_{w-1}(1) & & & & \\
& \ddots & & & \ddots & & \ddots & & & & \\
& & \Delta_2(w-1) & & & \Delta_2(w) & \dots & \Delta_2(w-2) & & & \\
& & & \Delta_1(w) & & & \Delta_1(w+1) & & \dots & \Delta_1(w-1).
\end{array}$$

The entries fulfill certain difference equations:

$$\Delta_j(i) = \sum_{t=0}^{k-2} (-1)^t a_{k-1-t,i+t} \Delta_{j-t-1}(i+t+1) + (-1)^{k-1} \Delta_{j-k}(i+k)$$

with coefficients $(a_{j,i})_{j \in [k-1], i \in [n]}$.

Definition 1

$(a_{j,i})$ are called the generalized quiddity elements of the SL_k-frieze pattern, collectively they are called the quiddity matrix. $\mathrm{Quid}(k, n)$ is the set of all quiddity matrices of tame SL_k-frieze patterns of width $w = n - k - 1$.

Lemma 1

There is a bijection between the set of tame SL_k-frieze patterns of width w and $\mathrm{Quid}(k, w + k + 1)$.

The sets $C_k(n)$

Definition 2

$$C_k(n) := \{(v_1, \ldots, v_n) \in (\mathbb{P}^{k-1}(K))^n \mid \{v_i, v_{i+1} \ldots, v_{i+k-1}\} \text{ independent for } i \in [n]\},$$

where the indices of the v_i are modulo n and

$$\mathcal{M}_k(n) := C_k(n)/\operatorname{PGL}(k, K).$$

The sets $C_k^*(n)$

Let $(v_1, \ldots, v_n) \in C_k(n)$ and let $(V_1, \ldots, V_n) \in K^k$ be some lift. Let

$$d_i := \det(V_i, V_{i+1}, \ldots, V_{i+k-1}).$$

Definition 3

Let $k, n \in \mathbb{N}$ with $k \geqslant 2$ and $n \geqslant k$. Let $g := gcd(k, n)$. If $g > 1$ we define

$$C_k^*(n) := \{v \in C_k(n) \mid d_i d_{i+g} \ldots d_{i+n-g} = d_{i+1} d_{i+g+1} \ldots d_{i+n-g+1} \text{ for } i \in [g-1]\}.$$

and

$$\mathcal{M}_k^*(n) := C_k^*(n) / \operatorname{PGL}(k, K).$$

Difference equations for $C_k^{(*)}(n)$

Let $v \in C_k^{(*)}(n)$. The vectors $\{V_{i+1}, \dots, V_{i+k}\}$ are a basis of K^k, therefore V_i can be expressed in this basis. Moreover, it can be shown that there is a lift $(V_1, \dots, V_n)$ of v in which all these expansions have the form

$$V_i = a_{k-1,i} V_{i+1} - a_{k-2,i+1} V_{i+2} + \dots + (-1)^{k-2} a_{1,i+k-2} V_{i+k-1} + (-1)^{k-1} V_{i+k} \quad (1)$$

with coefficients $a_{j,i} \in K$. Here the second index of $a_{j,i}$ is considered modulo n and $V_{n+i} := (-1)^{k-1} V_i$,

Definition 4

Let $v \in C_k^{(*)}(n)$. We write

$$\mathrm{Coeff}(v) = \{(a_{j,i})_{j,i} \mid (a_{j,i}) \text{ fulfills (1) for a suitable lift of } v\} \subset K^{(k-1)\times n}$$

for the set of coefficient matrices that can arise from v for different choices of lifts with constant consecutive determinants.

Lemma 2

We have

$$\mathrm{Quid}(k,n) = \bigcup_{v \in C_k^{(*)}(n)} \mathrm{Coeff}(v).$$

Lemma 3

Let $v, w \in C_k^{(*)}(n)$ with classes $[v], [w] \in \mathcal{M}_k^{(*)}(n)$.

- (i) If $[v] = [w]$, then $\mathrm{Coeff}(v) = \mathrm{Coeff}(w)$.
- (ii) If $[v] \neq [w]$, then $\mathrm{Coeff}(v) \cap \mathrm{Coeff}(w) = \varnothing$.

The number of friezes

Theorem 1

Let $k \geqslant 2$ and $w \geqslant 1$. Let $n := w + k + 1$. Let $g := \gcd(k, n)$. Let $K = \mathbb{F}_q$ be a finite field with q elements.

1. *If $g = 1$, then the number of tame SL_k-frieze patterns of width w is*

$$\frac{|C_k(n)|}{|\mathrm{PGL}(k, K)|}.$$

2. *If $g > 1$, then the number of tame SL_k-frieze patterns of width w is*

$$\frac{|C_k^*(n)|(q-1)^{g-1}}{|\mathrm{PGL}(k, K)|}.$$

The case $k = 3$ and $\gcd(3, n) = 1$

We define set $C_3^{+-}(n)$, $C_3^{-+}(n)$ and $C_3^{--}(n)$:

	$\{v_1, v_2, v_3\}$	$\{v_2, v_3, v_4\}$	$\dots$	$\{v_{n-2}, v_{n-1}, v_n\}$	$\{v_{n-1}, v_n, v_1\}$	$\{v_n, v_1, v_2\}$
$C_3(n)$	indep.	indep.	$\dots$	indep.	indep.	indep.
$C_3^{+-}(n)$	indep.	indep.	$\dots$	indep.	indep.	dep.
$C_3^{-+}(n)$	indep.	indep.	$\dots$	indep.	dep.	indep.
$C_3^{--}(n)$	indep.	indep.	$\dots$	indep.	dep.	dep.

For $C_3^{--}(n)$ we additionally require $v_1 \neq v_n$ (for the other sets this follows from the other conditions). A small c denotes the cardinality of these sets.

Lemma 4

Let K be finite with $|K| = q$. The following recursions hold for $n \geqslant 4$:

$$\begin{aligned} c_3(n) &= (q-1)^2 c_3(n-1) + 2q(q-1)c_3^{+-}(n-1) + q^2 c_3^{--}(n-1), \\ c_3^{+-}(n) &= (q-1)c_3(n-1) + qc_3^{+-}(n-1), \\ c_3^{--}(n) &= (q-1)c_3^{+-}(n-1) + qc_3(n-2). \end{aligned}$$

Theorem 2

Let $n \geqslant 3$ and $|K| = q < \infty$. Then we have

$$c_3(n) = \begin{cases} q^{2n} - q^{n+2} - q^{n+1} + q^3, & n \not\equiv 0 \mod 3, \\ q^{2n} + 2q^{n+2} + 2q^{n+1} + q^3, & n \equiv 0 \mod 3. \end{cases}$$

The case $k = 3$ and $\gcd(3, n) = 1$ [Cont.]

Corollary 1

Let $|K| = q < \infty$. Let $w \geqslant 1$ and $n = w + 4$. If $\gcd(3, n) = 1$, then the number of tame SL_3-frieze patterns of width w is

$$\frac{q^{2n} - q^{n+2} - q^{n+1} + q^3}{(q^2+q+1)(q+1)q^3(q-1)^2} = \frac{(q^{n-1}-1)(q^{n-2}-1)}{(q^2+q+1)(q+1)(q-1)^2}.$$

Theorem 3

Let $|K| = q < \infty$ *and* $n \in 3\mathbb{N}_{>0}$. *Then*

$$c_3^*(n) = \frac{q^{2n} + 3f(q)q^{\frac{4}{3}n} + (q^5 - 3q^4 - 2q^3 + 5q^2 + 9q + 6)q^n + 3f(q)q^{\frac{2}{3}n+1} + q^3}{(q-1)^2}$$

with

$$f(q) = q^3 - q^2 - q - 2.$$

The case $k = 3$ and $3 \mid n$ [Cont.]

Corollary 2

Let $|K| = q < \infty$. Let $w \geqslant 1$ and $n = w + 4$. If $\gcd(3, n) \neq 1$, then the number of tame SL_3-frieze patterns of width w is

$$\frac{q^{2n} + 3f(q)q^{\frac{4}{3}n} + (q^5 - 3q^4 - 2q^3 + 5q^2 + 9q + 6)q^n + 3f(q)q^{\frac{2}{3}n+1} + q^3}{(q^2 + q + 1)(q + 1)q^3(q - 1)^2}$$

with

$$f(q) = q^3 - q^2 - q - 2.$$

The case $k = 4$ and n odd

Theorem 4

Let K be a finite field with q elements and $n \geqslant 4$. Then

$$c_4(n) = \begin{cases} q^{3n} - (q^3 + q^2 + q)q^{2n} + (q^5 + q^4 + q^3)q^n - q^6, & \textit{if } \gcd(n,4) = 1, \\ q^{3n} - (q^3 + q^2 + q)q^{2n} + 2(q^4 + q^2)q^{\frac{3}{2}n} & \\ -(q^5 + q^4 + q^3)q^n + q^6, & \textit{if } \gcd(n,4) = 2, \\ q^{3n} + 3(q^3 + q^2 + q)q^{2n} + 2(q^4 + q^2)q^{\frac{3}{2}n} & \\ +3(q^5 + q^4 + q^3)q^n + q^6, & \textit{if } \gcd(n,4) = 4. \end{cases}$$

Corollary 3

Let $|K| = q < \infty$. Let $w \geqslant 1$ and $n = w + 5$. If n is odd, then the number of tame SL_4-frieze patterns of width w is

$$\frac{q^{3n} - (q^3 + q^2 + q)q^{2n} + (q^5 + q^4 + q^3)q^n - q^6}{(q^3 + q^2 + q + 1)(q^2 + q + 1)(q + 1)q^6(q - 1)^3}$$
$$= \frac{(q^{n-1} - 1)(q^{n-2} - 1)(q^{n-3} - 1)}{(q^3 + q^2 + q + 1)(q^2 + q + 1)(q + 1)(q - 1)^3}$$

Some numbers

The number of tame SL_3-friezes of width w over $K = \mathbb{F}_q$ for small values of q and w:

w \ q	2	3	4	5	7
1	5	10	17	26	50
2	29	145	433	1001	3529
3	93	847	4433	16401	120443
4	381	7651	70993	410151	5902051
5	1597	72775	1172305	10443901	291371347

The number of tame SL_4-friezes of width w over $K = \mathbb{F}_q$ for small values of q and w:

w \ q	2	3	4	5	7
2	93	847	4433	16401	120443
4	6477	627382	18245201	256754526	14176726502

Conjecture for higher k

The results in the cases $k = 2$, $k = 3$ and $k = 4$ suggest that the general formula might be as follows if $\gcd(k, n) = 1$:

Conjecture 1

Let $k \geqslant 5$ and $w \geqslant 1$ such that $\gcd(k, w + k + 1) = 1$. Then there are

$$\frac{(q^n - q)(q^n - q^2)\dots(q^n - q^{k-1})}{|\mathrm{PGL}(k, K)|}$$

tame SL_k-frieze patterns of width w over $K = \mathbb{F}_q$.