Spherical friezes

Katie Waddle
University of Michigan

Plan for the talk

1. Coxeter-Conway friezes: geometric interpretations

2. Euclidean distance geometry: Heronian friezes

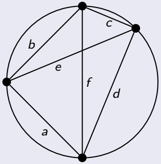
3. Spherical distance geometry: Spherical Heronian friezes

K. Waddle, *Spherical friezes*, arXiv:2501.03587, 2025.

A combinatorial interpretation

Theorem [Ptolemy, ≈ 150 CE]

Consider a cyclic quadrilateral with consecutive sides a, b, c, d and diagonals e, f. Then $ac + bd = ef$.



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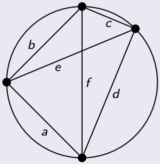

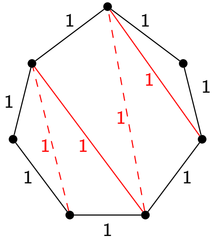

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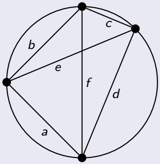

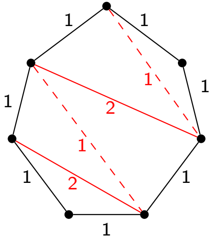

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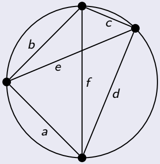

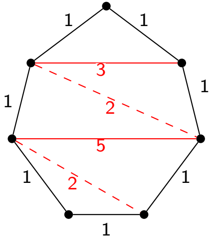

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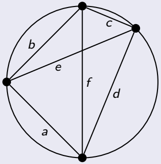

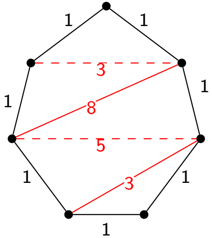

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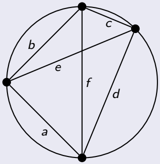

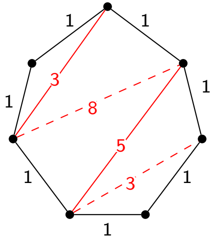

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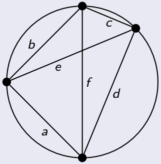

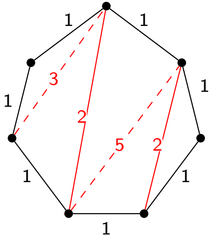

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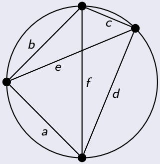

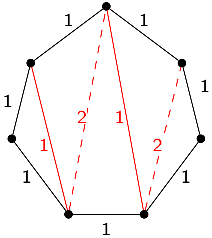

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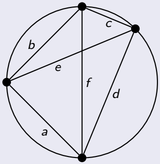

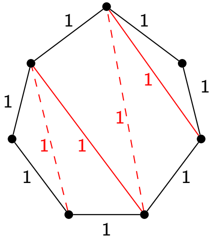

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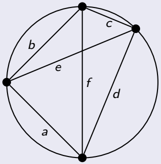

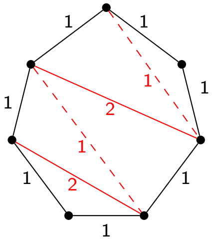

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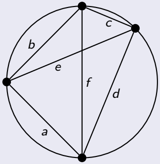

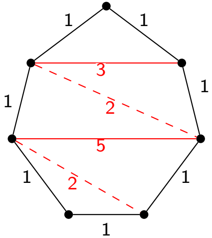

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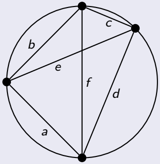

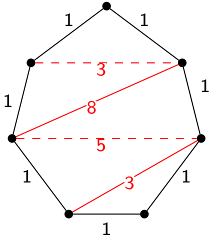

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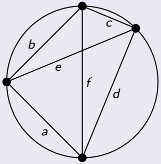

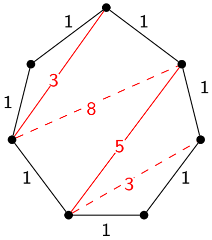

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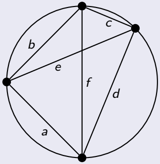

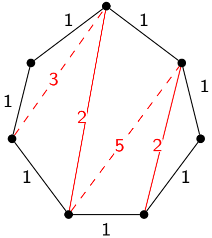

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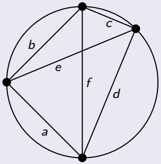

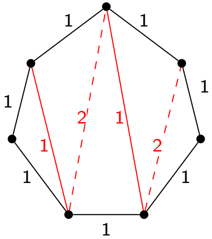

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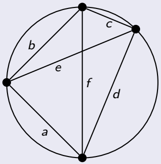

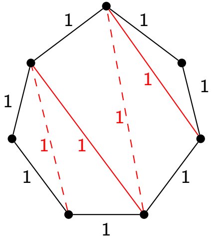

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Geometric interpretations

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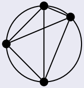
Let d_{ij} be the distance from A_i to A_j. Then



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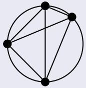

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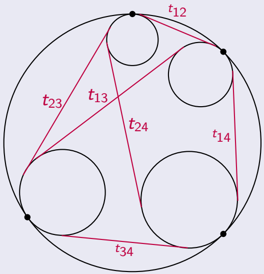

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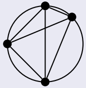

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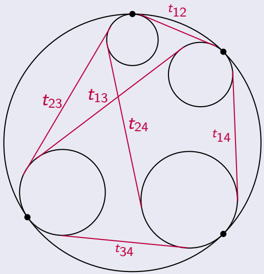

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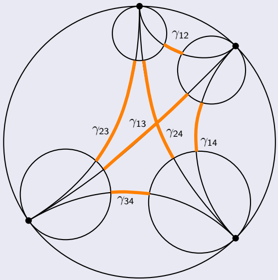

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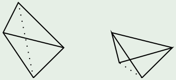

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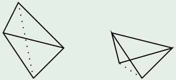

Question

How can we supplement the data corresponding to a triangulation to enable unique reconstruction of an n-point configuration, up to oriented isometry?

The Cayley-Menger determinant

Definition

Given points $A_1, \ldots, A_n$ in a Euclidean space $\mathbb{R}^d$, let x_{ij} denote the squared distance between A_i and A_j. The associated Cayley-Menger determinant is defined by:

$$M_n^0(x_{12}, x_{13}, \ldots, x_{n-1,n}) = \det \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & x_{12} & \cdots & x_{1,n-1} & x_{1,n} \\ 1 & x_{12} & 0 & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 1 & x_{1,n-1} & \cdots & \cdots & 0 & x_{n-1,n} \\ 1 & x_{1,n} & \cdots & \cdots & x_{n,n-1} & 0 \end{bmatrix}.$$

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Theorem

Consider a simplex $A_1 \cdots A_{d+1}$ in $\mathbb{R}^d$, with volume V. Let x_{ij} denote the squared distance from A_i to A_j. Then

$$V^2 = \frac{(-1)^{d+1}}{(d!)^2 \cdot 2^d} M_{d+1}^0(x_{12}, x_{13}, \ldots, x_{d,d+1}).$$

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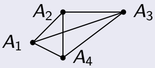
Corollary

Given $A_1, A_2, A_3, A_4 \in \mathbb{R}^2$, we have

$$M_4^0(x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}) = 0.$$



Heron's formula

We will use the notation

$$\begin{aligned} H(a,b,c) &= -M_3^0(a,b,c) \\ &= -\det \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & a & b \\ 1 & a & 0 & c \\ 1 & b & c & 0 \end{bmatrix} \\ &= -a^2 - b^2 - c^2 + 2ab + 2ac + 2bc. \end{aligned}$$

Theorem [Heron, $\approx$ 250 CE]

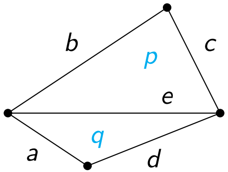
Let $A_1A_2A_3$ be a triangle with squared side lengths a, b, c. Then

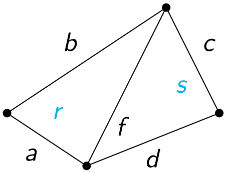
$$\underbrace{(4 \cdot \text{area of } A_1A_2A_3)}_{S}{}^2 = H(a,b,c).$$

The converse is also true if we work over $\mathbb{C}$.

Solution of the reconstruction problem on the Euclidean plane [Fomin-Setiabrata, 2021]

p, q, r, s are normalized signed areas





Solution of the reconstruction problem on the Euclidean plane [Fomin-Setiabrata, 2021]

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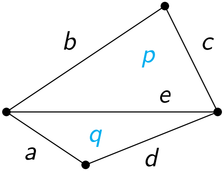

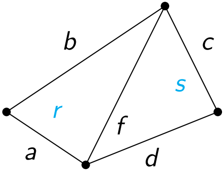

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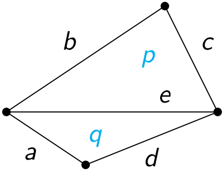

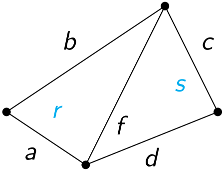

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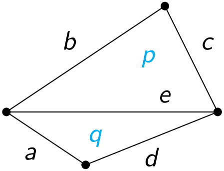

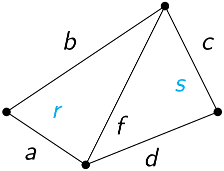

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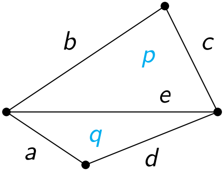

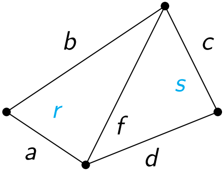

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Distance geometry on a sphere

Let $\mathbf{S} \subset \mathbb{R}^3$ be a sphere with radius R centered at the origin. Set $K = \frac{1}{R^2}$.

Problem

Identify a collection of $O(n)$ measurements that uniquely determines a labeled set of n points on $\mathbf{S}$, considered up to oriented isometry. Write the corresponding formulas explicitly.

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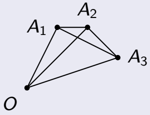
Definition

$$S^K(A_1, A_2, A_3) = \frac{12}{R} V(OA_1A_2A_3)$$



Spherical Cayley-Menger determinant [T. Tao, 2019]

As before, $K = \frac{1}{R^2}$. We will use the notation

$$\begin{aligned} H^K(a,b,c) &= \frac{K}{2}\det\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & a & c & \frac{1}{K} \\ 1 & a & 0 & b & \frac{1}{K} \\ 1 & c & b & 0 & \frac{1}{K} \\ 1 & \frac{1}{K} & \frac{1}{K} & \frac{1}{K} & 0 \end{bmatrix} \\ &= -a^2 - b^2 - c^2 + 2ab + 2ac + 2bc - Kabc. \end{aligned}$$

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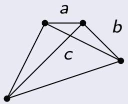
$$= -a^2 - b^2 - c^2 + 2ab + 2ac + 2bc - Kabc.$$

Spherical Heron's formula

$$(S^K)^2 = H^K(a,b,c).$$



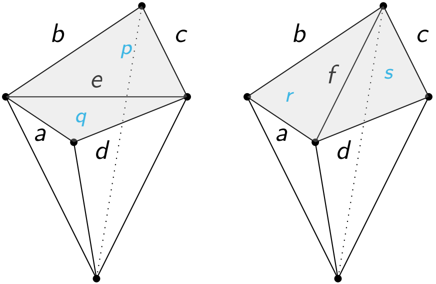
The converse holds if we work over $\mathbb{C}$.

Reconstruction problem on the sphere

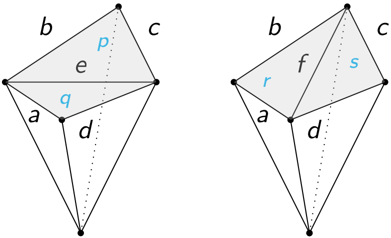
Problem

Given measurement data corresponding to a triangulation of a polygon (squared side distances and S^K measurements), compute all remaining distance measurements.

p, q, r, s are S^K measurements (normalized signed volumes)



Solution on the sphere [W., 2025]



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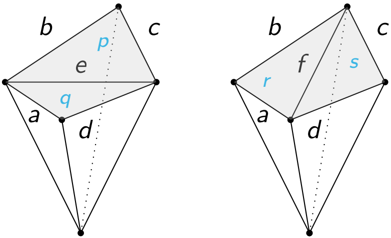

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- $p + q = r + s + \frac{K}{2}(ap + bq - er)$
- $p + q = r + s + \frac{K}{2}(fp - cr - bs)$
- $p + q = r + s + \frac{K}{2}(dp + cq - es)$
- $p + q = r + s + \frac{K}{2}(fq - dr - as)$

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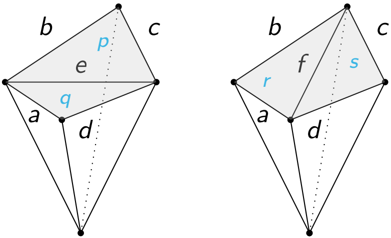

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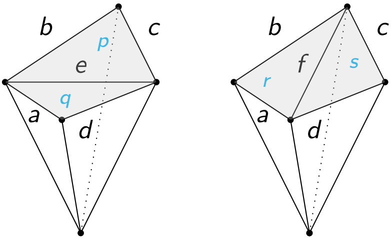

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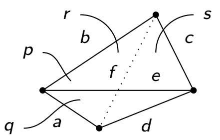
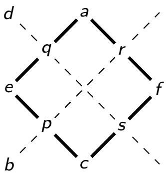
Spherical Heronian diamond



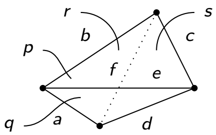


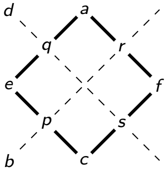
Definition

A *spherical Heronian diamond* is a 10-tuple $(a, b, c, d, e, f, p, q, r, s)$ satisfying the equations

$$\begin{aligned}
&(1) \qquad p^2 = H^K(b, c, e) \qquad\qquad q^2 = H^K(a, d, e)\\
&(2) \qquad r^2 = H^K(a, f, b) \qquad\qquad s^2 = H^K(c, f, d)\\
&(3) \qquad p + q = r + s + \tfrac{K}{2}(ap + bq - er)\\
&(4) \qquad e(r - s) = p(a - d) + q(b - c)\\
&(5) \qquad 4ef = (p + q)^2 + (a - b + c - d)^2 - Ke(a - b)(c - d)
\end{aligned}$$

Spherical Heronian diamond: propagation rule





Proposition [W., 2025]

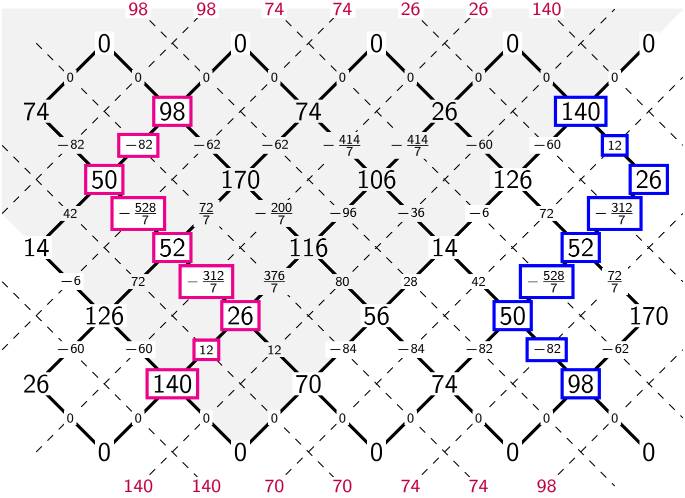
Suppose that a, b, c, d, e, p, q satisfy spherical Heronian relations (1). Assuming $e \notin \left\{0, \frac{4}{K}\right\}$, there exist unique f, r, s such that $(a, b, c, d, e, f, p, q, r, s)$ is a spherical Heronian diamond. Namely,

$$f = \frac{(p+q)^2 + (a-b+c-d)^2 - Ke(a-b)(c-d)}{4e\left(1 - \frac{Ke}{4}\right)},$$

$$r = \frac{p\left(e+a-d-\frac{Kae}{2}\right) + q\left(e-c+b-\frac{Kbe}{2}\right)}{2e\left(1-\frac{Ke}{4}\right)},$$

$$s = \frac{p\left(e-a+d-\frac{Kde}{2}\right) + q\left(e+c-b-\frac{Kce}{2}\right)}{2e\left(1-\frac{Ke}{4}\right)}.$$

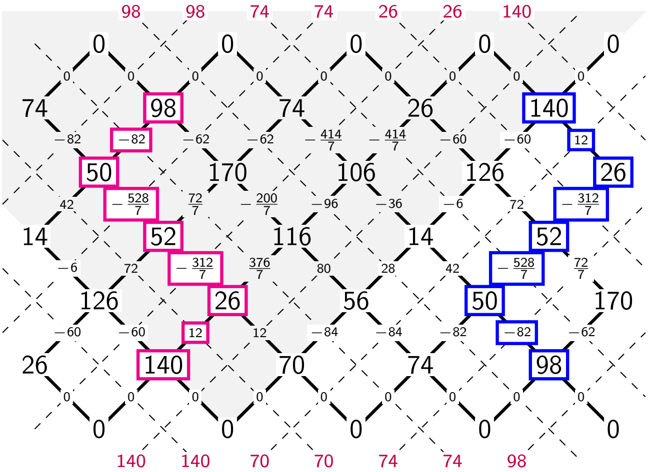
Spherical Heronian friezes



$$K = \frac{1}{49}$$

Main results

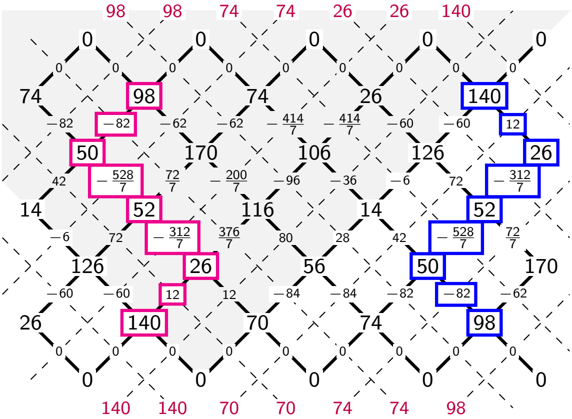
The entries along a traversing path determine the entire frieze.



Main results

Theorem [W., 2025]

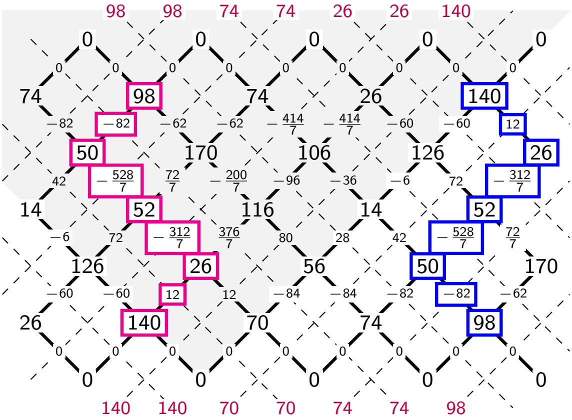
A sufficiently generic spherical Heronian frieze is periodic, has glide symmetry, and exhibits a form of the Laurent phenomenon.



Main results

Theorem [W., 2025]

Every generic spherical Heronian frieze comes from a polygon on the complexified sphere. This polygon is unique.



Thank you!

Metropolitan Museum of Art 4000 BCE

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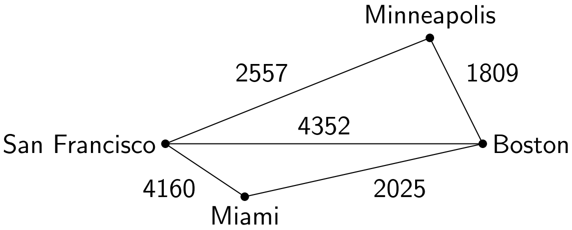
- T. Tao, The spherical Cayley-Menger determinant and the radius of the Earth, 2019.
- K. Waddle, *Spherical friezes*, 2025.

What next?

One application of spherical Heronian friezes is to measuring and computing distances on a globe.



Question

What if we have a point (or points) on a different sphere, like a satellite?