Laurent-Shmaurent, integer friezes, and polygon folding

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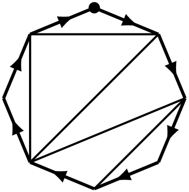
Frieze patterns in algebra, combinatorics and geometry
CIRM

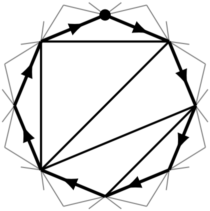
n-gon with rooted, directed boundary, dissected into $\triangle$

homomorphism into $\mathcal{F}$ (the Farey graph), sending the boundary to a non-self-intersecting $\circlearrowright$ path, modulo $\mathrm{SL}_2(\mathbb{Z})$

positive integer frieze of width* n



$\longleftrightarrow$



$\longleftrightarrow$

0 0 0 0 0 0 0 0
1 1 1 1 1 1 1
3 3 1 2 4 1
8 2 1 7 3
5 1 3 5
2 2 2
3 1
1

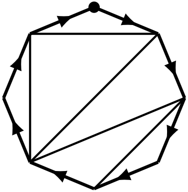
[Conway & Coxeter; Morier-Genoud, Ovsienko & Tabachnikov]

A humble goal: extend this correspondence to tame [not-necessarily-positive] integer friezes.

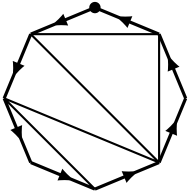
n-gon
dissected into
$\triangle$

homomorphism into $\mathcal{F}$,
non-self-intersecting,
$\circlearrowright$ for odd n,
$\circlearrowright$ or $\circlearrowleft$ for even n

nonzero frieze



$\longleftarrow$



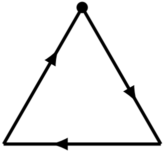
$\longleftrightarrow$

$$\begin{array}{ccccccccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
& 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
& & -3 & & -3 & & -1 & & -2 & & -4 & & -1 & & \\
& & & 8 & & 2 & & 1 & & 7 & & 3 & & & \\
& & & & -5 & & -1 & & -3 & & -5 & & & & \\
& & & & & 2 & & 2 & & 2 & & & & & \\
& & & & & & -3 & & -1 & & & & & & \\
& & & & & & & 1 & & & & & & &
\end{array}$$

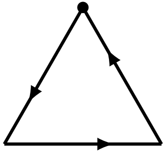
n-gon dissected into $\triangle$

homomorphism into $\mathcal{F}$, non-self-intersecting, ↻ or ↺

nonzero frieze possibly ending with -1s



$\longleftarrow$

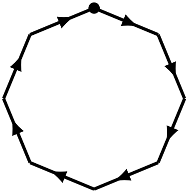


$\longleftrightarrow$

$$\begin{array}{cccccc} 0 & & 0 & & 0 & \\ & 1 & & 1 & & 1 \\ & & -1 & & -1 & & -1 \end{array}$$

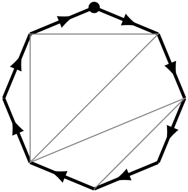
We want a frieze that ends with a row of 1s.

n-gon



$\longleftarrow$

any homomorphism
into $\mathcal{F}$



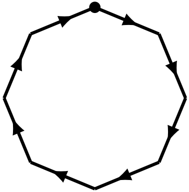
$\longleftrightarrow$

tame frieze
possibly ending with -1s

$$\begin{array}{ccccccccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
& 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
& & 3 & & 3 & & 1 & & 2 & & 4 & & 1 & & \\
& & & 8 & & 2 & & 1 & & 7 & & 3 & & & \\
& & & & 5 & & 1 & & 3 & & 5 & & & & \\
& & & & & 2 & & 2 & & 2 & & & & & \\
& & & & & & 3 & & 1 & & & & & & \\
& & & & & & & 1 & & & & & & &
\end{array}$$

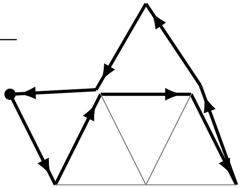
[Short]

n-gon



$\longleftarrow$

any homomorphism into $\mathcal{F}$



$\longleftrightarrow$

tame frieze
possibly ending with -1s

$0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0$

$1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1$

$-1 \quad 2 \quad 2 \quad 0 \quad -3 \quad -1$

$3 \quad 3 \quad -1 \quad -1 \quad 2$

$4 \quad -2 \quad 1 \quad 1$

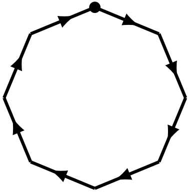
$-3 \quad 3 \quad 0$

$-4 \quad -1$

1

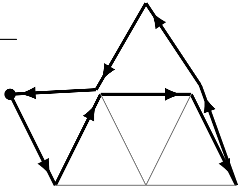
[Short]

n-gon



$\longleftarrow$

homomorphism into $\mathcal{F}$, regular



$\longleftrightarrow$

tame frieze

$$
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & \\
-1 & 2 & 2 & 0 & -3 & -1 & & \\
3 & 3 & -1 & -1 & 2 & & & \\
4 & -2 & 1 & 1 & & & & \\
-3 & 3 & 0 & & & & & \\
-4 & -1 & & & & & & \\
1 & & & & & & &
\end{array}
$$

Lemma

Every closed path in $\mathcal{F}$ of length > 4 has a pair of nonconsecutive but adjacent vertices.

Complete freedom from diagonals, as in Short's theorem, is not necessary.

Theorem

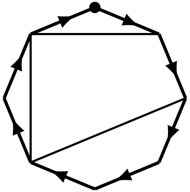
Any closed path in $\mathcal{F}$ is the image of the sides of a polygon dissected into triangles and quadrilaterals under some graph homomorphism into $\mathcal{F}$.

[Banaian et al.]: dissections into $\triangle$ & $\square$ and friezes over $\mathbb{Z}[\sqrt{2}]$
[Conley & Ovsienko]: $3d$-dissections of polygons and positive quiddities
[Cuntz & Holm]: labelled dissections into $\triangle$ & $\square$ and friezes over $\mathbb{Z}$

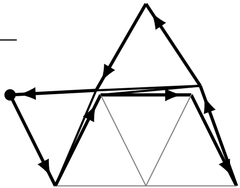
n-gon dissected into $\triangle$ and $\square$

homomorphism into $\mathcal{F}$, regular

tame frieze



$\longleftarrow$



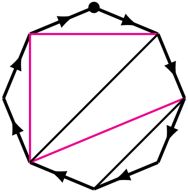
$\longleftrightarrow$

$$
\begin{array}{ccccccccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
& 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
& & -1 & & 2 & & 2 & & 0 & & -3 & & -1 & & \\
& & & 3 & & 3 & & -1 & & -1 & & 2 & & & \\
& & & & 4 & & -2 & & 1 & & 1 & & & & \\
& & & & & -3 & & 3 & & 0 & & & & & \\
& & & & & & -4 & & -1 & & & & & & \\
& & & & & & & 1 & & & & & & &
\end{array}
$$

We'd like to have a unique homomorphism for each polygon, like for positive friezes; possibly with some extra data.

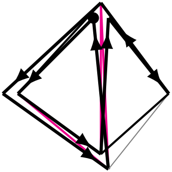
Forget about quadrilaterals for now.
Label diagonals with the corresponding frieze entries: 1s and -1s.
(The sides are always 1.)

n-gon
dissected into $\triangle$,
diagonals labelled
with 1s & -1s



$\longleftrightarrow$

homomorphism into $\mathcal{F}$,
respects labels,
regular



$\longrightarrow$

tame frieze

$0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0$

$1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1$

$-1 \quad -1 \quad 1 \quad 0 \quad -2 \quad -1$

$0 \quad -2 \quad -1 \quad -1 \quad 1$

$1 \quad 1 \quad 1 \quad 1$

$0 \quad 0 \quad 0$

$-1 \quad -1$

1

Polygon folding

What happens to a path in $\mathcal{F}$
when the label of a diagonal is flipped from 1 to -1?

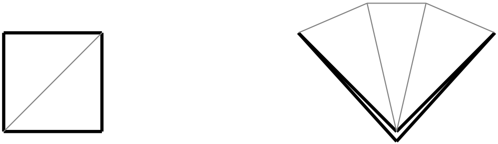
(See the demonstration!)

Well-defined for self-intersecting paths too.

Now we can interpret “regular” as “resulting from a folding sequence”.

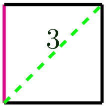
What is the image of a quadrilateral in $\mathcal{F}$ under a homomorphism?

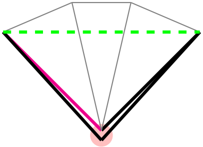
Either a quadrilateral with a diagonal or a self-intersecting path of length 4.



Self-intersection happens exactly when the quadrilateral is *odd*:
bounded by 1 or 3 diagonals labelled -1.

The extra data needed to specify the image of an odd quadrilateral is: which pair of opposite vertices of the n-gon is mapped to distinct vertices in $\mathcal{F}$ (if any), and how many triangles in $\mathcal{F}$ between them.





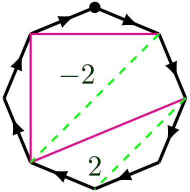
We "bend" the quadrilateral (and the entire path) at the self-intersection.

Theorem

Consider an n-gon dissected into triangles and quadrilaterals, with edges labelled with 1s and -1s such that each quadrilateral is odd, and then diagonals labelled by any integers are inserted into quadrilaterals. For each such object, there exists exactly one tame integer frieze with elements corresponding to the diagonals equal to the diagonals' labels, and all tame integer friezes arise this way.

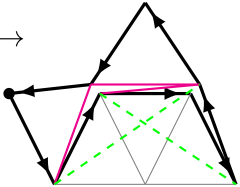
Cf. [Cuntz & Holm, 2019].

n-gon with data as in the theorem



$\longleftrightarrow$

homomorphism into $\mathcal{F}$ via folding and bending



$\longleftrightarrow$

tame frieze

$$\begin{array}{cccccccccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
& 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
& & -1 & & 2 & & 2 & & 0 & & -3 & & -1 & & \\
& & & 3 & & 3 & & -1 & & -1 & & 2 & & & \\
& & & & 4 & & -2 & & 1 & & 1 & & & & \\
& & & & & -3 & & 3 & & 0 & & & & & \\
& & & & & & -4 & & -1 & & & & & & \\
& & & & & & & 1 & & & & & & &
\end{array}$$

Note: there is no 1-to-1 correspondence between polygons on the left and friezes on the right; same frieze can arise from different dissections.

Cluster-algebraic perspective

The elements of a tame frieze of width n are given by the specialisation of cluster variables of the cluster algebra of type A_{n-3}.

The cluster variables are Laurent polynomials in the initial cluster variables, so setting the initial cluster vars to 1s and/or -1s guarantees integer outputs.

This explains labelling with 1s and -1s but not "bending" associated with odd quadrilaterals.

Laurent-Shmaurent

Theorem (Berenstein, Fomin, Zelevinsky)

A cluster algebra associated with a totally mutable acyclic seed (in particular, A_{n-3}) is generated by x_is and x_i's.

A flip of a diagonal corresponds to the mutation of a cluster variable:

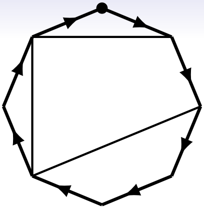
$$x' = \frac{ac + bd}{x}.$$

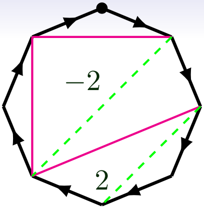
Note that if the quadrilateral is odd, and the labels of its sides are substituted for a, b, c, d, then $x' = 0$,
and therefore all remaining cluster variables become *polynomial* in x!

The Shmaurent phenomenon

Theorem

Suppose an n-gon is dissected into triangles and some of its diagonals are labelled by 1s and -1s.
Let $\mathcal{A}$ be the algebra obtained from the corresponding cluster algebra of type A_{n-3} by substituting the label of each labelled diagonal for the corresponding initial cluster variable.
Consider a diagonal that is unlabelled itself but is inside an odd quadrilateral. Then the elements of $\mathcal{A}$ are **polynomial** *in the initial cluster variable corresponding to that diagonal.*





$x_1x_4 - x_4 + 1$

- Our group: `SL2tilings.github.io`
- Visual Cluster Algebras: `VisualCA.net`
- My mathematical blog: `mathstodon.xyz/@PSL2Z`

I am graduating in 2026.

Thank you!
Questions, please.