A positive dichotomy between frieze patterns and Diophantine geometry

Robin Zhang

Massachusetts Institute of Technology

Frises en algèbre, combinatoire et géométrie
Frieze patterns in algebra, combinatorics and geometry

15 May 2025

Outline

1. Counting friezes
2. Between friezes and Diophantine geometry
3. Applications of frieze enumeration to Diophantine geometry
4. Generalizations

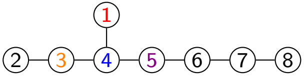
Positive integral Dynkin friezes

Talk convention:
A "frieze pattern" will be an array of positive integers with an SL_2 unimodular rule.

Positive integral Dynkin friezes

Talk convention:
A “frieze pattern” will be an array of positive integers with an SL_2 unimodular rule.

Dynkin diagram for E_8



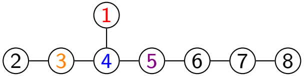
Positive integral Dynkin friezes

Talk convention:
A "frieze pattern" will be an array of positive integers with an SL_2 unimodular rule.

Example of a frieze of type E_8

...	1		1		1		1		1		1	...
...		4		3		3		4		4		...
...	15		11		8		11		15		11	...
...	7	41	6	29	5	29	6	41	7	41	6	...
...	16		18		21		18		16		18	...
...		7		13		13		7		7		...
...	3		5		8		5		3		5	...
...		2		3		3		2		2		...
...	1		1		1		1		1		1	...

Dynkin diagram for E_8



All diamonds $\begin{smallmatrix} & b & \\ a & & d \\ & c & \end{smallmatrix}$ and $\begin{smallmatrix} & f & \\ e & g & i \\ & h & \end{smallmatrix}$ satisfy $ad - bc = 1$ and $ei - fgh = 1$.

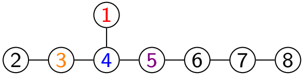
Positive integral Dynkin friezes

Talk convention:
A "frieze pattern" will be an array of positive integers with an SL_2 unimodular rule.

Example of a frieze of type E_8

...	1		1		1		1		1		1	...
...		4		3		3		4		4		...
...	15		11		8		11		15		11	...
...	7	41	6	29	5	29	6	41	7	41	6	...
...	16		18		21		18		16		18	...
...		7		13		13		7		7		...
...	3		5		8		5		3		5	...
...		2		3		3		2		2		...
...	1		1		1		1		1		1	...

Dynkin diagram for E_8



Question: Are there finitely many friezes for each Dynkin diagram?

All diamonds $\begin{smallmatrix} & b & \\ a & & d \\ & c & \end{smallmatrix}$ and $\begin{smallmatrix} & f & \\ e & g & i \\ & h & \end{smallmatrix}$ satisfy $ad - bc = 1$ and $ei - fgh = 1$.

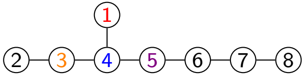
Positive integral Dynkin friezes

Talk convention:
A "frieze pattern" will be an array of positive integers with an SL_2 unimodular rule.

Example of a frieze of type E_8

...	1		1		1		1		1		1	...
...		4		3		3		4		4		...
...	15		11		8		11		15		11	...
...	7	41	6	29	5	29	6	41	7	41	6	...
...	16		18		21		18		16		18	...
...		7		13		13		7		7		...
...	3		5		8		5		3		5	...
...		2		3		3		2		2		...
...	1		1		1		1		1		1	...

Dynkin diagram for E_8



Question: Are there finitely many friezes for each Dynkin diagram?

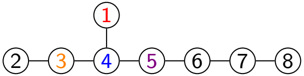
Answer: Yes, if and only if it is a finite Dynkin type!

All diamonds $\begin{smallmatrix} & b & \\ a & & d \\ & c & \end{smallmatrix}$ and $\begin{smallmatrix} & f & \\ e & g & i \\ & h & \end{smallmatrix}$ satisfy $ad - bc = 1$ and $ei - fgh = 1$.

Positive integral Dynkin friezes

Talk convention:
A "frieze pattern" will be an array of positive integers with an SL_2 unimodular rule.

Dynkin diagram for E_8



Theorem (Morier-Genoud 2012)

If Δ is an infinite type, then there are infinitely many friezes of type Δ.

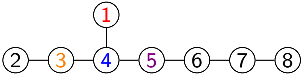
Question: Are there finitely many friezes for each Dynkin diagram?

Answer: Yes, if and only if it is a finite Dynkin type!

Positive integral Dynkin friezes

Talk convention:
A "frieze pattern" will be an array of positive integers with an SL_2 unimodular rule.
(or a homomorphism $\mathcal{A} \to \mathbb{Z}$ sending cluster variables to $\mathbb{N}$)

Dynkin diagram for E_8



Theorem (Morier-Genoud 2012)

If Δ is an infinite type, then there are infinitely many friezes of type Δ.

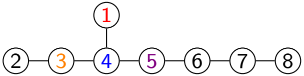
Question: Are there finitely many friezes for each Dynkin diagram?

Answer: Yes, if and only if it is a finite Dynkin type!

Positive integral Dynkin friezes

Talk convention:
A "frieze pattern" will be an array of positive integers with an SL_2 unimodular rule.
(or a homomorphism $\mathcal{A} \to \mathbb{Z}$ sending cluster variables to $\mathbb{N}$)

Dynkin diagram for E_8



Theorem (Morier-Genoud 2012)

If Δ is an infinite type, then there are infinitely many friezes of type Δ.

Idea: Fomin–Zelevinsky classification of cluster algebras + unitary friezes

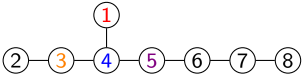
Question: Are there finitely many friezes for each Dynkin diagram?

Answer: Yes, if and only if it is a finite Dynkin type!

Positive integral Dynkin friezes

Talk convention:
A "frieze pattern" will be an array of positive integers with an SL_2 unimodular rule.
(or a homomorphism $\mathcal{A} \to \mathbb{Z}$ sending cluster variables to $\mathbb{N}$)

Dynkin diagram for E_8



Theorem (Morier-Genoud 2012)

If Δ is an infinite type, then there are infinitely many friezes of type Δ.

Idea: Fomin–Zelevinsky classification of cluster algebras + unitary friezes

Question: Are there finitely many friezes for each Dynkin diagram?

Theorem (Gunawan–Muller 2022, Muller 2023)

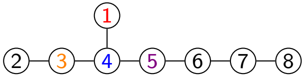
If Δ is a finite type, then there are finitely many friezes of type Δ.

Answer: Yes, if and only if it is a finite Dynkin type!

Positive integral Dynkin friezes

Talk convention:
A "frieze pattern" will be an array of positive integers with an SL_2 unimodular rule.
(or a homomorphism $\mathcal{A} \to \mathbb{Z}$ sending cluster variables to $\mathbb{N}$)

Dynkin diagram for E_8



Theorem (Morier-Genoud 2012)

If Δ is an infinite type, then there are infinitely many friezes of type Δ.

Idea: Fomin–Zelevinsky classification of cluster algebras + unitary friezes

Question: Are there finitely many friezes for each Dynkin diagram?

Theorem (Gunawan–Muller 2022, Muller 2023)

If Δ is a finite type, then there are finitely many friezes of type Δ.

Answer: Yes, if and only if it is a finite Dynkin type!

Gunawan–Muller: Non-effective proof using the geometry of cluster algebras

Muller: Effective proof using average logarithms of rows

Enumeration of Dynkin friezes

Question: How many friezes are there for each Dynkin type?

Enumeration of Dynkin friezes

Question: How many friezes are there for each Dynkin type?

Answer:

Type	Number of friezes	Proof
A_n	$\frac{1}{n+2}\binom{2n+2}{n+1}$	Conway–Coxeter 1973
B_n	$\sum_{m=1}^{\lfloor\sqrt{n+1}\rfloor}\binom{2n-m^2+1}{n}$	Fontaine–Plamondon 2016
C_n	$\binom{2n}{n}$	Fontaine–Plamondon 2016
D_n	$\sum_{m=1}^{n} d(m)\binom{2n-m-1}{n-m}$	Morier-Genoud–Ovsienko–Tabachnikov 2012 Fontaine–Plamondon 2016
E_6	868	Cuntz–Plamondon 2021
E_7	?	
E_8	?	
F_4	112	Cuntz–Plamondon 2021
G_2	9	Fontaine–Plamondon 2016
Infinite	∞	Morier-Genoud 2012

Enumeration of Dynkin friezes

Question: How many friezes are there for each Dynkin type?

Answer:

Type	Number of friezes	Proof
A_n	$\frac{1}{n+2}\binom{2n+2}{n+1}$	Conway–Coxeter 1973
B_n	$\sum_{m=1}^{\lfloor\sqrt{n+1}\rfloor}\binom{2n-m^2+1}{n}$	Fontaine–Plamondon 2016
C_n	$\binom{2n}{n}$	Fontaine–Plamondon 2016
D_n	$\sum_{m=1}^{n} d(m)\binom{2n-m-1}{n-m}$	Morier-Genoud–Ovsienko–Tabachnikov 2012 Fontaine–Plamondon 2016
E_6	868	Cuntz–Plamondon 2021
E_7	?	
E_8	?	
F_4	112	Cuntz–Plamondon 2021
G_2	9	Fontaine–Plamondon 2016
Infinite	∞	Morier-Genoud 2012

Conjecture (Fontaine–Plamondon 2016)

There are exactly 4400 and 26952 friezes of type E_7 and E_8.

Enumeration of Dynkin friezes

Question: How many friezes are there for each Dynkin type?

Answer:

Type	Number of friezes	Proof
A_n	$\frac{1}{n+2}\binom{2n+2}{n+1}$	Conway–Coxeter 1973
B_n	$\sum_{m=1}^{\lfloor\sqrt{n+1}\rfloor}\binom{2n-m^2+1}{n}$	Fontaine–Plamondon 2016
C_n	$\binom{2n}{n}$	Fontaine–Plamondon 2016
D_n	$\sum_{m=1}^{n} d(m)\binom{2n-m-1}{n-m}$	Morier-Genoud–Ovsienko–Tabachnikov 2012 Fontaine–Plamondon 2016
E_6	868	Cuntz–Plamondon 2021
E_7	?	
E_8	?	
F_4	112	Cuntz–Plamondon 2021
G_2	9	Fontaine–Plamondon 2016
Infinite	∞	Morier-Genoud 2012

Conjecture (Fontaine–Plamondon 2016)

There are exactly 4400 and 26952 friezes of type E_7 and E_8.

Muller 2023: effective bound essentially reduces E_8 problem to searching across $\approx 10^{50}$-many 16-tuples (last rows of E_8-friezes)

Enumeration of Dynkin friezes

Question: How many friezes are there for each Dynkin type?

Conjecture (Fontaine–Plamondon 2016)

There are exactly 4400 and 26952 friezes of type E_7 and E_8.

Muller 2023: effective bound essentially reduces E_8 problem to searching across $\approx 10^{50}$-many 16-tuples (last rows of E_8-friezes)

Theorem (Z. 2025)

There are exactly 4400 and 26952 friezes of type E_7 and E_8.

Answer:

Type	Number of friezes	Proof
A_n	$\frac{1}{n+2}\binom{2n+2}{n+1}$	Conway–Coxeter 1973
B_n	$\sum_{m=1}^{\lfloor\sqrt{n+1}\rfloor}\binom{2n-m^2+1}{n}$	Fontaine–Plamondon 2016
C_n	$\binom{2n}{n}$	Fontaine–Plamondon 2016
D_n	$\sum_{m=1}^{n} d(m)\binom{2n-m-1}{n-m}$	Morier-Genoud–Ovsienko–Tabachnikov 2012 Fontaine–Plamondon 2016
E_6	868	Cuntz–Plamondon 2021
E_7	4400	Z. 2025
E_8	26952	Z. 2025
F_4	112	Cuntz–Plamondon 2021
G_2	9	Fontaine–Plamondon 2016
Infinite	∞	Morier-Genoud 2012

Enumeration of Dynkin friezes

Question: How many friezes are there for each Dynkin type?

Conjecture (Fontaine–Plamondon 2016)

There are exactly 4400 and 26952 friezes of type E_7 and E_8.

Muller 2023: effective bound essentially reduces E_8 problem to searching across $\approx 10^{50}$-many 16-tuples (last rows of E_8-friezes)

Theorem (Z. 2025)

There are exactly 4400 and 26952 friezes of type E_7 and E_8.

Idea: Transfer the problem to Diophantine geometry

Answer:

Type	Number of friezes	Proof
A_n	$\frac{1}{n+2}\binom{2n+2}{n+1}$	Conway–Coxeter 1973
B_n	$\sum_{m=1}^{\lfloor\sqrt{n+1}\rfloor}\binom{2n-m^2+1}{n}$	Fontaine–Plamondon 2016
C_n	$\binom{2n}{n}$	Fontaine–Plamondon 2016
D_n	$\sum_{m=1}^{n} d(m)\binom{2n-m-1}{n-m}$	Morier-Genoud–Ovsienko–Tabachnikov 2012 Fontaine–Plamondon 2016
E_6	868	Cuntz–Plamondon 2021
E_7	4400	Z. 2025
E_8	26952	Z. 2025
F_4	112	Cuntz–Plamondon 2021
G_2	9	Fontaine–Plamondon 2016
Infinite	∞	Morier-Genoud 2012

Enumeration of Dynkin friezes

Question: How many friezes are there for each Dynkin type?

Answer:

Type	Number of friezes	Proof
A_n	$\frac{1}{n+2}\binom{2n+2}{n+1}$	Conway–Coxeter 1973
B_n	$\sum_{m=1}^{\lfloor\sqrt{n+1}\rfloor}\binom{2n-m^2+1}{n}$	Fontaine–Plamondon 2016
C_n	$\binom{2n}{n}$	Fontaine–Plamondon 2016
D_n	$\sum_{m=1}^{n} d(m)\binom{2n-m-1}{n-m}$	Morier-Genoud–Ovsienko–Tabachnikov 2012 Fontaine–Plamondon 2016
E_6	868	Cuntz–Plamondon 2021
E_7	4400	Z. 2025
E_8	26952	Z. 2025
F_4	112	Cuntz–Plamondon 2021
G_2	9	Fontaine–Plamondon 2016
Infinite	∞	Morier-Genoud 2012

Conjecture (Fontaine–Plamondon 2016)

There are exactly 4400 and 26952 friezes of type E_7 and E_8.

Muller 2023: effective bound essentially reduces E_8 problem to searching across $\approx 10^{50}$-many 16-tuples (last rows of E_8-friezes)

Theorem (Z. 2025)

There are exactly 4400 and 26952 friezes of type E_7 and E_8.

Idea: Transfer the problem to Diophantine geometry
Use an affine variety X_Δ given by de Saint Germain–Huang–Lu 2023

Enumeration of Dynkin friezes

Question: How many friezes are there for each Dynkin type?

Conjecture (Fontaine–Plamondon 2016)

There are exactly 4400 and 26952 friezes of type E_7 and E_8.

Muller 2023: effective bound essentially reduces E_8 problem to searching across $\approx 10^{50}$-many 16-tuples (last rows of E_8-friezes)

Theorem (Z. 2025)

There are exactly 4400 and 26952 friezes of type E_7 and E_8.

Idea: Transfer the problem to Diophantine geometry
Use an affine variety X_Δ given by de Saint Germain–Huang–Lu 2023
Modified Muller's bound: search $X_{E_8}(\mathbb{N})$ with coordinates $< 10^{41}$

Answer:

Type	Number of friezes	Proof
A_n	$\frac{1}{n+2}\binom{2n+2}{n+1}$	Conway–Coxeter 1973
B_n	$\sum_{m=1}^{\lfloor\sqrt{n+1}\rfloor}\binom{2n-m^2+1}{n}$	Fontaine–Plamondon 2016
C_n	$\binom{2n}{n}$	Fontaine–Plamondon 2016
D_n	$\sum_{m=1}^{n} d(m)\binom{2n-m-1}{n-m}$	Morier-Genoud–Ovsienko–Tabachnikov 2012 Fontaine–Plamondon 2016
E_6	868	Cuntz–Plamondon 2021
E_7	4400	Z. 2025
E_8	26952	Z. 2025
F_4	112	Cuntz–Plamondon 2021
G_2	9	Fontaine–Plamondon 2016
Infinite	∞	Morier-Genoud 2012

Enumeration of Dynkin friezes

Question: How many friezes are there for each Dynkin type?

Conjecture (Fontaine–Plamondon 2016)

There are exactly 4400 and 26952 friezes of type E_7 and E_8.

Muller 2023: effective bound essentially reduces E_8 problem to searching across $\approx 10^{50}$-many 16-tuples (last rows of E_8-friezes)

Theorem (Z. 2025)

There are exactly 4400 and 26952 friezes of type E_7 and E_8.

Idea: Transfer the problem to Diophantine geometry

Use an affine variety X_Δ given by de Saint Germain–Huang–Lu 2023

Modified Muller's bound: search $X_{E_8}(\mathbb{N})$ with coordinates $< 10^{41}$

Reduce further using $\mathbb{Z}/m\mathbb{Z}$-orbits in $X_{E_8}(\mathbb{N})$

Answer:

Type	Number of friezes	Proof
A_n	$\frac{1}{n+2}\binom{2n+2}{n+1}$	Conway–Coxeter 1973
B_n	$\sum_{m=1}^{\lfloor\sqrt{n+1}\rfloor}\binom{2n-m^2+1}{n}$	Fontaine–Plamondon 2016
C_n	$\binom{2n}{n}$	Fontaine–Plamondon 2016
D_n	$\sum_{m=1}^{n} d(m)\binom{2n-m-1}{n-m}$	Morier-Genoud–Ovsienko–Tabachnikov 2012 Fontaine–Plamondon 2016
E_6	868	Cuntz–Plamondon 2021
E_7	4400	Z. 2025
E_8	26952	Z. 2025
F_4	112	Cuntz–Plamondon 2021
G_2	9	Fontaine–Plamondon 2016
Infinite	∞	Morier-Genoud 2012

Enumeration of Dynkin friezes

Question: How many friezes are there for each Dynkin type?

Conjecture (Fontaine–Plamondon 2016)

There are exactly 4400 and 26952 friezes of type E_7 and E_8.

Muller 2023: effective bound essentially reduces E_8 problem to searching across $\approx 10^{50}$-many 16-tuples (last rows of E_8-friezes)

Theorem (Z. 2025)

There are exactly 4400 and 26952 friezes of type E_7 and E_8.

Idea: Transfer the problem to Diophantine geometry

Use an affine variety X_Δ given by de Saint Germain–Huang–Lu 2023

Modified Muller's bound: search $X_{E_8}(\mathbb{N})$ with coordinates $< 10^{41}$

Reduce further using $\mathbb{Z}/m\mathbb{Z}$-orbits in $X_{E_8}(\mathbb{N})$

Search for $X_{E_8}(\mathbb{N})$ with one coordinate $< 10^{10}$

Answer:

Type	Number of friezes	Proof
A_n	$\frac{1}{n+2}\binom{2n+2}{n+1}$	Conway–Coxeter 1973
B_n	$\sum_{m=1}^{\lfloor\sqrt{n+1}\rfloor}\binom{2n-m^2+1}{n}$	Fontaine–Plamondon 2016
C_n	$\binom{2n}{n}$	Fontaine–Plamondon 2016
D_n	$\sum_{m=1}^{n} d(m)\binom{2n-m-1}{n-m}$	Morier-Genoud–Ovsienko–Tabachnikov 2012 Fontaine–Plamondon 2016
E_6	868	Cuntz–Plamondon 2021
E_7	4400	Z. 2025
E_8	26952	Z. 2025
F_4	112	Cuntz–Plamondon 2021
G_2	9	Fontaine–Plamondon 2016
Infinite	∞	Morier-Genoud 2012

Enumeration of Dynkin friezes

Question: How many friezes are there for each Dynkin type?

Conjecture (Fontaine–Plamondon 2016)

There are exactly 4400 and 26952 friezes of type E_7 and E_8.

Muller 2023: effective bound essentially reduces E_8 problem to searching across $\approx 10^{50}$-many 16-tuples (last rows of E_8-friezes)

Theorem (Z. 2025)

There are exactly 4400 and 26952 friezes of type E_7 and E_8.

Idea: Transfer the problem to Diophantine geometry

Use an affine variety X_Δ given by de Saint Germain–Huang–Lu 2023

Modified Muller's bound: search $X_{E_8}(\mathbb{N})$ with coordinates $< 10^{41}$

Reduce further using $\mathbb{Z}/m\mathbb{Z}$-orbits in $X_{E_8}(\mathbb{N})$

Search for $X_{E_8}(\mathbb{N})$ with one coordinate $< 10^{10}$

Answer:

Type	Number of friezes	Proof
A_n	$\frac{1}{n+2}\binom{2n+2}{n+1}$	Conway–Coxeter 1973
B_n	$\sum_{m=1}^{\lfloor\sqrt{n+1}\rfloor}\binom{2n-m^2+1}{n}$	Fontaine–Plamondon 2016
C_n	$\binom{2n}{n}$	Fontaine–Plamondon 2016
D_n	$\sum_{m=1}^{n} d(m)\binom{2n-m-1}{n-m}$	Morier-Genoud–Ovsienko–Tabachnikov 2012 Fontaine–Plamondon 2016
E_6	868	Cuntz–Plamondon 2021
E_7	4400	Z. 2025
E_8	26952	Z. 2025
F_4	112	Cuntz–Plamondon 2021
G_2	9	Fontaine–Plamondon 2016
Infinite	∞	Morier-Genoud 2012

Enumeration of Dynkin friezes

Question: How many friezes are there for each Dynkin type?

Conjecture (Fontaine–Plamondon 2016)

There are exactly 4400 and 26952 friezes of type E_7 and E_8.

Muller 2023: effective bound essentially reduces E_8 problem to searching across $\approx 10^{50}$-many 16-tuples (last rows of E_8-friezes)

Theorem (Z. 2025)

There are exactly 4400 and 26952 friezes of type E_7 and E_8.

Idea: Transfer the problem to Diophantine geometry
- Use an affine variety X_Δ given by de Saint Germain–Huang–Lu 2023
- Modified Muller's bound: search $X_{E_8}(\mathbb{N})$ with coordinates $< 10^{41}$
- Reduce further using $\mathbb{Z}/m\mathbb{Z}$-orbits in $X_{E_8}(\mathbb{N})$
- Search for $X_{E_8}(\mathbb{N})$ with one coordinate $< 10^{10}$

Corollary: new proof of frieze counts for all Dynkin types of rank $n < 8$

Answer:

Type	Number of friezes	Proof
A_n	$\frac{1}{n+2}\binom{2n+2}{n+1}$	Conway–Coxeter 1973
B_n	$\sum_{m=1}^{\lfloor\sqrt{n+1}\rfloor}\binom{2n-m^2+1}{n}$	Fontaine–Plamondon 2016
C_n	$\binom{2n}{n}$	Fontaine–Plamondon 2016
D_n	$\sum_{m=1}^{n} d(m)\binom{2n-m-1}{n-m}$	Morier-Genoud–Ovsienko–Tabachnikov 2012 Fontaine–Plamondon 2016
E_6	868	Cuntz–Plamondon 2021
E_7	4400	Z. 2025
E_8	26952	Z. 2025
F_4	112	Cuntz–Plamondon 2021
G_2	9	Fontaine–Plamondon 2016
Infinite	∞	Morier-Genoud 2012

Cluster varieties

(Fomin–Zelevinsky) classification of cluster algebras $\mathcal{A}$	$\longleftrightarrow$	(Killing–Cartan) classification of simple Lie groups G	$\longleftrightarrow$	classification of generalized Cartan matrices $C = (c_{i,j})$

Cluster varieties

(Fomin–Zelevinsky) classification of cluster algebras $\mathcal{A}$	$\longleftrightarrow$	(Killing–Cartan) classification of simple Lie groups G	$\longleftrightarrow$	classification of generalized Cartan matrices $C = (c_{i,j})$

Definition (Cluster varieties)

An affine variety is of cluster algebra type $\mathcal{A}$ if it is isomorphic to $\mathrm{Spec}\,(\mathcal{A})$ over $\mathbb{C}$.

Cluster varieties

(Fomin–Zelevinsky) classification of cluster algebras $\mathcal{A}$ $\longleftrightarrow$ (Killing–Cartan) classification of simple Lie groups G $\longleftrightarrow$ classification of generalized Cartan matrices $C = (c_{i,j})$

Definition (Cluster varieties)

An affine variety is of cluster algebra type $\mathcal{A}$ if it is isomorphic to $\mathrm{Spec}\,(\mathcal{A})$ over $\mathbb{C}$.

Definition (Lower bound frieze polynomials)

For each $i \in \{1, \ldots, n\}$, consider the polynomial in $\mathbb{Z}[x_1, \ldots, x_n, y_1, \ldots, y_n]$

$$f_{C,i} := x_i y_i - \prod_{j=1}^{i-1} x_j^{-c_{j,i}} - \prod_{j=i+1}^{n} x_j^{-c_{j,i}}.$$

Define the affine variety $X_C := \{f_{C,i} = 0\}$ to be their vanishing locus.

Cluster varieties

(Fomin–Zelevinsky) classification of cluster algebras $\mathcal{A}$	$\longleftrightarrow$	(Killing–Cartan) classification of simple Lie groups G	$\longleftrightarrow$	classification of generalized Cartan matrices $C = (c_{i,j})$

Definition (Lower bound frieze polynomials)

For each $i \in \{1, \ldots, n\}$, consider the polynomial in $\mathbb{Z}[x_1, \ldots, x_n, y_1, \ldots, y_n]$

$$f_{C,i} := x_i y_i - \prod_{j=1}^{i-1} x_j^{-c_{j,i}} - \prod_{j=i+1}^{n} x_j^{-c_{j,i}}.$$

Define the affine variety $X_C := \{f_{C,i} = 0\}$ to be their vanishing locus.

Definition (Cluster varieties)

An affine variety is of cluster algebra type $\mathcal{A}$ if it is isomorphic to $\mathrm{Spec}\,(\mathcal{A})$ over $\mathbb{C}$.

Theorem (de Saint Germain–Huang–Lu 2023)

Let Δ be a Dynkin type with generalized Cartan matrix C and cluster algebra $\mathcal{A}_C$ with trivial coefficients.
There is an isomorphism $\mathrm{Spec}\,(\mathcal{A}_C) \cong X_C$ over $\mathbb{C}$, and there is a bijection

$$\{\text{friezes of type } \Delta\} \longleftrightarrow X_C(\mathbb{N})$$

Rational points, integral points, and positive integral points

The fundamental result in Diophantine geometry about *rational* points (conjectured by Poincaré 1901 and Mordell 1922):

Theorem (Mordell 1922, Weil 1929, Faltings 1983)

Let X be a smooth curve of genus g.

- If $g = 0$, then $\#X(\mathbb{Q}) = 0$ or ∞.
- If $g = 1$, then $X(\mathbb{Q})$ is a finitely-generated abelian group.
- If $g \geq 2$, then $X(\mathbb{Q})$ is finite.

Rational points, integral points, and positive integral points

The fundamental result in Diophantine geometry about *rational* points (conjectured by Poincaré 1901 and Mordell 1922):

Theorem (Mordell 1922, Weil 1929, Faltings 1983)

Let X be a smooth curve of genus g.

- If $g = 0$, then $\#X(\mathbb{Q}) = 0$ or ∞.
- If $g = 1$, then $X(\mathbb{Q})$ is a finitely-generated abelian group.
- If $g \geq 2$, then $X(\mathbb{Q})$ is finite.

The fundamental result in Diophantine geometry about *integral* points:

Theorem (Siegel 1929)

Let X be a smooth curve of genus g.

- If $g \geq 1$, then $X(\mathbb{Z})$ is finite.

Rational points, integral points, and positive integral points

The fundamental result in Diophantine geometry about *rational* points (conjectured by Poincaré 1901 and Mordell 1922):

Theorem (Mordell 1922, Weil 1929, Faltings 1983)

Let X be a smooth curve of genus g.

- If $g = 0$, then $\#X(\mathbb{Q}) = 0$ or ∞.
- If $g = 1$, then $X(\mathbb{Q})$ is a finitely-generated abelian group.
- If $g \geq 2$, then $X(\mathbb{Q})$ is finite.

The fundamental result in Diophantine geometry about *integral* points:

Theorem (Siegel 1929)

Let X be a smooth curve of genus g.

- If $g \geq 1$, then $X(\mathbb{Z})$ is finite.

The classification of cluster algebras & enumeration of friezes yields a result about *positive integral* points:

Theorem (Z. 2025)

Let X be an affine variety of cluster algebra type Δ with smallest principal minor t_Δ.

- If $t_\Delta \leq 0$, then $\#X(\mathbb{N}) = \infty$.
- If $t_\Delta \geq 1$, then $X(\mathbb{N})$ is finite and precisely given by the frieze counts.

The Mordell–Schinzel program

Theorem (Mordell 1952)

For any $G \in \mathbb{Z}[x, y]$, there are infinitely many integer solutions (x, y, z) to the equation

$$xyz = G(x, y).$$

The Mordell–Schinzel program

Theorem (Mordell 1952)

For any $G \in \mathbb{Z}[x, y]$, there are infinitely many integer solutions (x, y, z) to the equation

$$xyz = G(x, y).$$

- Proved by Mordell 1952 for $G(x, y) = x^3 + y^3 + 1$

The Mordell–Schinzel program

Theorem (Mordell 1952)

For any $G \in \mathbb{Z}[x, y]$, there are infinitely many integer solutions (x, y, z) to the equation

$$xyz = G(x, y).$$

- Proved by Mordell 1952 for $G(x, y) = x^3 + y^3 + 1$
- Quadratic counter-examples by Jacobsthal 1939, Barnes 1953, Mills 1954, Schinzel 2015

The Mordell–Schinzel program

Theorem (Mordell 1952)

For any $G \in \mathbb{Z}[x, y]$, there are infinitely many integer solutions (x, y, z) to the equation

$$xyz = G(x, y).$$

- Proved by Mordell 1952 for $G(x, y) = x^3 + y^3 + 1$
- Quadratic counter-examples by Jacobsthal 1939, Barnes 1953, Mills 1954, Schinzel 2015
- Some cases salvaged by Schinzel 2015, Schinzel 2018, and Kollár–Li 2024

The Mordell–Schinzel program

"Theorem" (Mordell 1952)

For any $G \in \mathbb{Z}[x, y]$, there are infinitely many integer solutions (x, y, z) to the equation

$$xyz = G(x, y).$$

- Proved by Mordell 1952 for $G(x, y) = x^3 + y^3 + 1$
- Quadratic counter-examples by Jacobsthal 1939, Barnes 1953, Mills 1954, Schinzel 2015
- Some cases salvaged by Schinzel 2015, Schinzel 2018, and Kollár–Li 2024

The Mordell–Schinzel program

"Theorem" (Mordell 1952)

For any $G \in \mathbb{Z}[x, y]$, there are infinitely many integer solutions (x, y, z) to the equation

$$xyz = G(x, y).$$

- Proved by Mordell 1952 for $G(x, y) = x^3 + y^3 + 1$
- Quadratic counter-examples by Jacobsthal 1939, Barnes 1953, Mills 1954, Schinzel 2015
- Some cases salvaged by Schinzel 2015, Schinzel 2018, and Kollár–Li 2024

Theorem (Z. 2025)

If $ab \geq 4$, then there are infinitely many *positive* integer solutions (x, y, z) to the equation

$$xyz = (x^a + 1)^b + y.$$

Furthermore if $ab = 1$, 2 or 3, then the number of positive integer solutions is 5, 6, or 9 respectively.

If $abcd \geq 3$, then there are infinitely many positive integer solutions (x, y, z, w) to the equation

$$xyzw = (x^a + 1)^b y + (x^c + 1)^d z.$$

Generalizations

Question: What about other number/finite/function fields, rings of integers, etc.?

Generalizations

Question: What about other number/finite/function fields, rings of integers, etc.?

Example over finite fields:

Theorem (Morier-Genoud 2021)

$$\#\{\text{Friezes of type } A_n \text{ over } \mathbb{F}_q\} = \begin{cases} \frac{q^{n+2}-1}{q^2-1} & \text{if } n \text{ is even} \\ \frac{(q^{\frac{n+3}{2}}-1)(q^{\frac{n+1}{2}}-1)}{q^2-1} & \text{if } n \text{ is odd and } \mathrm{char}(\mathbb{F}_q) > 2 \\ \frac{(q^{\frac{n+3}{2}}-1)(q^{\frac{n+1}{2}}-1)}{q^2-1} + q^{\frac{n+1}{2}} & \text{if } n \text{ is odd and } \mathrm{char}(\mathbb{F}_q) = 2 \end{cases}$$

Generalizations

Question: What about other number/finite/function fields, rings of integers, etc.?

Example over finite fields:

Theorem (Morier-Genoud 2021)

$$\#\{\text{Friezes of type } A_n \text{ over } \mathbb{F}_q\} = \begin{cases} \frac{q^{n+2}-1}{q^2-1} & \text{if } n \text{ is even} \\ \frac{(q^{\frac{n+3}{2}}-1)(q^{\frac{n+1}{2}}-1)}{q^2-1} & \text{if } n \text{ is odd and } \mathrm{char}(\mathbb{F}_q) > 2 \\ \frac{(q^{\frac{n+3}{2}}-1)(q^{\frac{n+1}{2}}-1)}{q^2-1} + q^{\frac{n+1}{2}} & \text{if } n \text{ is odd and } \mathrm{char}(\mathbb{F}_q) = 2 \end{cases}$$

- Short–van Son–Zabolotskii 2025: another proof using Farey complexes

Generalizations

Question: What about other number/finite/function fields, rings of integers, etc.?

Example over finite fields:

Theorem (Morier-Genoud 2021)

$$\#\{\text{Friezes of type } A_n \text{ over } \mathbb{F}_q\} = \begin{cases} \frac{q^{n+2}-1}{q^2-1} & \text{if } n \text{ is even} \\ \frac{(q^{\frac{n+3}{2}}-1)(q^{\frac{n+1}{2}}-1)}{q^2-1} & \text{if } n \text{ is odd and } \mathrm{char}(\mathbb{F}_q) > 2 \\ \frac{(q^{\frac{n+3}{2}}-1)(q^{\frac{n+1}{2}}-1)}{q^2-1} + q^{\frac{n+1}{2}} & \text{if } n \text{ is odd and } \mathrm{char}(\mathbb{F}_q) = 2 \end{cases}$$

- Short–van Son–Zabolotskii 2025: another proof using Farey complexes
- Z. in-progress: another proof using $\#X_{A_n}(\mathbb{F}_q)$ counts

Generalizations

Question: What about other number/finite/function fields, rings of integers, etc.?

Example over finite fields:

Theorem (Morier-Genoud 2021)

$$\#\{\text{Friezes of type } A_n \text{ over } \mathbb{F}_q\} = \begin{cases} \frac{q^{n+2}-1}{q^2-1} & \text{if } n \text{ is even} \\ \frac{(q^{\frac{n+3}{2}}-1)(q^{\frac{n+1}{2}}-1)}{q^2-1} & \text{if } n \text{ is odd and } \mathrm{char}(\mathbb{F}_q) > 2 \\ \frac{(q^{\frac{n+3}{2}}-1)(q^{\frac{n+1}{2}}-1)}{q^2-1} + q^{\frac{n+1}{2}} & \text{if } n \text{ is odd and } \mathrm{char}(\mathbb{F}_q) = 2 \end{cases}$$

- Short–van Son–Zabolotskii 2025: another proof using Farey complexes
- Z. in-progress: another proof using $\#X_{A_n}(\mathbb{F}_q)$ counts

Question: Can this generalize to SL_k friezes?

Generalizations

Question: What about other number/finite/function fields, rings of integers, etc.?

Example over finite fields:

Theorem (Morier-Genoud 2021)

$$\#\{\text{Friezes of type } A_n \text{ over } \mathbb{F}_q\} = \begin{cases} \frac{q^{n+2}-1}{q^2-1} & \text{if } n \text{ is even} \\ \frac{(q^{\frac{n+3}{2}}-1)(q^{\frac{n+1}{2}}-1)}{q^2-1} & \text{if } n \text{ is odd and } \operatorname{char}(\mathbb{F}_q) > 2 \\ \frac{(q^{\frac{n+3}{2}}-1)(q^{\frac{n+1}{2}}-1)}{q^2-1} + q^{\frac{n+1}{2}} & \text{if } n \text{ is odd and } \operatorname{char}(\mathbb{F}_q) = 2 \end{cases}$$

- Short–van Son–Zabolotskii 2025: another proof using Farey complexes
- Z. in-progress: another proof using $\#X_{A_n}(\mathbb{F}_q)$ counts

Question: Can this generalize to SL_k friezes?

Question: Are there other interesting varieties containing / contained in cluster varieties?

Generalizations

Question: What about other number/finite/function fields, rings of integers, etc.?

Example over finite fields:

Theorem (Morier-Genoud 2021)

$$\#\{\text{Friezes of type } A_n \text{ over } \mathbb{F}_q\} = \begin{cases} \frac{q^{n+2}-1}{q^2-1} & \text{if } n \text{ is even} \\ \frac{(q^{\frac{n+3}{2}}-1)(q^{\frac{n+1}{2}}-1)}{q^2-1} & \text{if } n \text{ is odd and } \mathrm{char}(\mathbb{F}_q) > 2 \\ \frac{(q^{\frac{n+3}{2}}-1)(q^{\frac{n+1}{2}}-1)}{q^2-1} + q^{\frac{n+1}{2}} & \text{if } n \text{ is odd and } \mathrm{char}(\mathbb{F}_q) = 2 \end{cases}$$

- Short–van Son–Zabolotskii 2025: another proof using Farey complexes
- Z. in-progress: another proof using $\#X_{A_n}(\mathbb{F}_q)$ counts

Question: Can this generalize to SL_k friezes?

Question: Are there other interesting varieties containing / contained in cluster varieties?

Example with more general cluster algebras:

- de Saint Germain–Z. in-progress: more Mordell–Schinzel equations