Continued Fractions and SL_2-tilings

Schedule

Monday 25th		Tuesday 26th		Wednesday 27th		Thursday 28th	
		Rietsch	9:30–10:20	Yıldırım	9:30–10:20	Blackman	9:30–10:20
		Break		Break		Break	
		Plamondon	11:00–11:50	Pedon	11:00–11:50	Parker	11:00–11:50
Registration	13:15	Van Son	12:00–12:40	Evans	12:00–12:40	Karpenkov	12:00–12:50
LMS Opening	13:45						
Morier-Genoud	14:00–14:50	Banaian	14:30–15:20	Veselov	14:30–15:20		
Break		Break		Break			
Short	15:30–16:20	Baur	16:00–16:50	Çanakçı	16:00–16:50		
Pressland	16:30–17:20	Zabolotskii	17:00–17:40	de Saint Germain	17:00–17:40		
Reception	17:30	Dinner	18:45				

Titles and Abstracts

Esther Banaian (Aarhus)
Continued Fractions and Orbifold Markov numbers

It is known that Markov numbers can be viewed as specializations of cluster variables in the cluster algebra from a once-punctured torus. This connection has inspired formulas for Markov numbers involving continued fractions and these formulas in turn can be used to better understand Markov numbers. We consider similar formulas for solutions to several variants of the Markov equation coming from triangulated orbifolds. This is partially based on joint work with Archan Sen.

Karin Baur (Leeds)
Infinite friezes of affine type D

Cluster and module categories of affine type D give rise to a triple of infinite periodic friezes of integers: the friezes are obtained by specialising the Caldero-Chapoton map. These numbers in turn correspond to the submodule count for certain modules in these categories. We show the following remarkable property of this triple of friezes: their entries all grow in the same way, i.e. their growth coefficients are equal. To prove this, we use the surface model for the categories, where modules correspond to (generalised) arcs in a twice punctured disk.

This is joint work with L. Bittmann, E. Gunawan, G. Todorov and E. Yildirim.

John Blackman (Liverpool)
Cutting Sequences, Hecke Congruence Subgroups, and the p-adic Littlewood Conjecture

One of the main themes of Diophantine approximation is the study of how well real numbers can be approximated by rational numbers. Classically, a real number is defined to be well-approximable if the Markov constant is 0, i.e. $M(x) := \liminf\{q||qx||\} = 0$. Otherwise, the number is badly approximable, with larger values of $M(x)$ indicating worse rates of approximation. As a slight twist on this notion of approximability, the p-adic Littlewood Conjecture asks if – given a prime p and a badly approximable number x – one can always find a subsequence of xp^k such that the Markov constant of this sequence tends to 0, i.e. if $\liminf M(xp^k) = 0$.

In this talk, I will outline a brief history of the p-adic Littlewood Conjecture and discuss how hyperbolic geometry can be used to help understand the problem further. In particular, I will discuss how one can represent integer multiplication of continued fractions by replacing one triangulation of the hyperbolic plane with an alternative triangulation. Finally, I will give a reformulation of pLC using infinite loops – a family of objects that arise from this setting.

İlke Çanakçı (VU Amsterdam)
"Super" cluster algebras of type A reinterpreted

Abstract: One can explicitly compute the generators of a surface cluster algebra either combinatorially, through dimer covers of snake graphs, or homologically, through the CC-map applied to indecomposable modules over the appropriate algebra. Recent work by Musiker, Ovenhouse and Zhang used Penner and Zeitlin's decorated super Teichmüller theory to define a super version of the cluster algebra of type A and gave a combinatorial formula to compute the even generators. We extend this theory by giving a homological way of explicitly computing these generators by defining a super CC-map for type A.

Antoine de Saint Germain (The University of Hong Kong)
Y-frieze patterns

In this talk, we will begin by introducing a variant of Coxeter's frieze patterns called Y-frieze patterns. We will show that Y-frieze patterns share many properties with frieze patterns: glide symmetry, finiteness, existence of "unitary" friezes, etc.. Using the connection with cluster algebras, we will show that frieze patterns and Y-frieze patterns fit naturally in Fock and Goncharov's notion of a cluster ensemble. This will allow us to formulate a conjecture about the exact number of Y-frieze patterns. This talk is based on arxiv:2311.03073.

Sam Evans (Loughborough)
Arithmetic and geometry of Markov polynomials

Markov polynomials are the Laurent-polynomial solutions of the Markov equation

$$X^2 + Y^2 + Z^2 = aXYZ,$$

which are the results of the cluster mutations applied to the initial triple (x, y, z).

They were first discussed by Propp, who proved that their coefficients are non-negative integers. We present more results and formulate new conjectures about the coefficients of Markov polynomials.

The talk is based on the ongoing joint work with Alexander Veselov and Brian Winn.

Oleg Karpenkov (Liverpool)
Continued Fraction approach to Gauss-Reduction theory

Jordan Normal Forms serve as excellent representatives of conjugacy classes of matrices over closed fields. Once we knows normal forms, we can compute functions of matrices, their main invariant, etc. The situation is more complicated if we search for normal forms for conjugacy classes over fields that are not closed and especially over rings. In this paper we study PGL(2,Z)-conjugacy classes of GL(2,Z) matrices. For the ring of integers Jordan approach has various limitations and in fact it is not effective. The normal forms of conjugacy classes of GL(2,Z) matrices are provided by alternative theory, which is known as Gauss Reduction Theory. We introduce a new techniques to compute reduced forms In Gauss Reduction Theory in terms of the elements of certain continued fractions. Current approach is based on recent progress in geometry of numbers. The proposed technique provides an explicit computation of periods of continued fractions for the slopes of eigenvectors.

Sophie Morier-Genoud (Reims)
Combinatorial aspects of continued fractions, SL(2,Z)-tilings, and their q-analogs

The talk will start from the elementary fact that positive rational numbers can be expanded as finite continued fractions with positive integer coefficients. I will present combinatorial models where the positive integer coefficients can be interpreted, meaning that ''they count something". Introducing a formal parameter q, I will then make a deformations of the objects and refine the countings. This will bring notions of q-rationals, q-continued fractions, q-SL(2,Z), and even more. I will present some properties of these q-analogs. The results are parts of collaborations with Valentin Ovsienko and with Ludivine Leclere.

John Parker (Durham)
Diophantine approximation and Margulis regions for screw-parabolic maps in four dimensions

There is a celebrated result of Margulis about discrete subgroups of isometries of Riemannian manifolds with non-positive curvature. The result says that there is a constant only depending on the manifold so that, for each point on the manifold, the group generated by those isometries with displacement less than this constant is nilpotent. For many points this group is trivial, but the sets where it is infinite are called Margulis regions. In this talk, I will discuss the Margulis regions associated to screw parabolic maps with infinite order rotational part in real hyperbolic 4-space and, if there is time, in complex hyperbolic 2-space. A particularly striking aspect of these results is the way they depend on the continued fraction expansion associated to the rotational part of the map, and the use of results from Diophantine approximation.

Emmanuel Pedon (Reims)
Continued fractions and Hankel determinants for q-metallic numbers.

The 'metallic numbers', or 'metallic means', are the real numbers whose regular continued fraction expansion has the form $n + 1/(n + 1/(n + 1/(n + ...$ for some positive integer n; the golden ratio (corresponding to $n = 1$) is certainly the most famous example. My talk is devoted to their 'q-deformation' in the sense of S. Morier-Genoud and V. Ovsienko.

I will present several continued fractions which characterize q-metallic numbers and show, when $n = 1$ or $n = 2$, that their Hankel determinants have amazing properties: they consist of -1, 0 and 1 only; they are periodic; and they form Somos or Gale-Robinson recurrences. I will explain why these properties make q-metallic numbers resembling Catalan and Motzkin numbers.

The talk is based on joint work with Valentin Ovsienko.

Pierre-Guy Plamondon (Versailles)
Friezes, reduction, and categorification

Friezes of integers have a now well-known interpretation in terms of cluster algebras. I will discuss how the reduction of friezes of Dynkin type ADE (as defined by Baur, Faber, Gratz, Serhiyenko and Todorov) can be "reversed". This will be done by interpretation reduction in terms of the categorification of cluster algebras. This is an application of a joint work with Bernhard Keller and Fan Qin on multiplication formulas for cluster algebras.

Matthew Pressland (Glasgow)
The geometry and representation theory of frieze patterns

Frieze patterns were introduced and developed by Coxeter and Conway in the 80s, who presented them as a kind of combinatorial game, whereby the player attempts to fill a grid with positive integers obeying certain rules. It turns out, however, that frieze patterns appear naturally in a range of mathematical problems from a number of different areas. For example, in geometry, a frieze pattern represents a positive integer valued point in a decorated Teichmüller space, or in the totally positive Grassmannian. In representation theory, the entries in a frieze pattern count submodules of quiver representations. In this talk, I will survey some of these different interpretations of frieze patterns, and the connections between them.

Konstanze Rietsch: (King's College London)
Generalising Euler's Tonnetz

This talk is connected to music theory. We give a definition of a what we call a 'tonnetz' on a triangulated surface, which is inspired by the famous tonnetz of Euler (1739). In Euler's tonnetz the vertices of a regular 'A_2 triangulation' of the plane are labelled with notes, or pitch-classes (integers module 12), leading to a geometric arrangement of all major and minor triads. In our generalisation we allow much more general labellings of triangulated surfaces. In particular, edge labellings turn out to lead to a rich set of examples. We give various constructions of such objects, including a set of examples related to crystallographic reflection groups that live on triangulations of tori. We also construct a tonnetz on a sphere that precisely encodes all major ninth chords.

Ian Short (Open University)
Continued fractions, SL_2-tilings, and the Farey complex

This talk explores interactions between continued fractions, frieze patterns, and SL_2-tilings using geometric and numerical properties of the Farey complex embedded in the upper half-plane. This perspective was championed by Morier-Genoud, Ovsienko, and Tabachnikov who, in 2015, used the geometry of the Farey complex to interpret Conway and Coxeter's classification of frieze patterns by triangulated polygons. Since then this approach has been adopted by many others. Here we will explain how integer continued fractions and tame integer SL_2-tilings can be represented by paths in the Farey complex, illuminating old results and inspiring new results. We also review the same story for real continued fractions and SL_2-tilings. To finish, we will attempt to tame wild tilings.

Matty Van Son (Open University)
Lifting SL_2 tilings

Following the work of Morier-Genoud, Ovsienko, and Tabachnikov, in 2019 Short established a correspondence between tame SL_2-tilings over $\mathbb{Z}$ and paths in the Farey complex. We introduce Farey complexes for all commutative rings R, and use these to obtain an analogous correspondence for tame SL_2-tilings over R. We discuss different types of friezes in this setting.

There is a natural projection from SL_2-tilings over $\mathbb{Z}$ to SL_2-tilings over $\mathbb{Z}_n$. We show that any tame SL_2-tiling over $\mathbb{Z}_n$ can be lifted to an SL_2-tiling over $\mathbb{Z}$. The situation for friezes is more complicated; we present topological conditions for when a tame frieze over $\mathbb{Z}_n$ can be lifted to a frieze over $\mathbb{Z}$ of the same width. Time permitting we will discuss the situation for lifting friezes over general rings R.

This talk is based on joint work with Ian Short and Andrei Zabolotskii.

Alexander Veselov (Loughborough)
Braids and q-rationals

I will talk about a link between the new theory of q-deformed rational numbers and the classical Burau representation of the braid group B_3. This will be applied to the open problem of classification of faithful complex specializations of this representation.

The talk is based on a joint work with Sophie Morier-Genoud and Valentin Ovsienko.

Emine Yıldırım (Leeds)
Matrix computation of cluster expansions

This is a joint work with Ezgi Kantarcı Oğuz. Sophie Morier-Genoud and Valentin Ovsienko define the q-deformed rational numbers and q-deformed continued fractions in their celebrated paper titled "q-deformed rationals and q-continued fractions". These notions are certain quantized versions of ordinary rational numbers and continued fractions. Their work inspired us to discover a new way of writing cluster expansions via matrices. In this talk, we will see the beautiful combinatorial nature of newly discovered q-deformed continued fractions and their application to the setting of cluster algebras.

Andrei Zabolotskii (Open University)
Gallery of Farey graphs, and Wild SL_2-tilings

The Farey graphs over a given ring R and a group of units U, which were introduced in the talks by Ian Short and Matty van Son, provide combinatorial models and hierarchical classification for SL_2-tilings over R. These graphs did not come out of nowhere, but are actually known to arise in different contexts such as tessellations of hyperbolic space. We discuss some notable examples in the first part of the talk.

In the second part of the talk, we present Farey surfaces which provide combinatorial models for wild integer SL_2-tilings and discuss examples of wild tilings with notable properties.