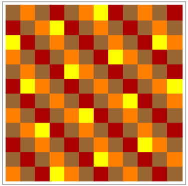
Gallery of Farey graphs, and Wild SL_2-tilings

Andrei Zabolotskii



26 March 2024

With Ian Short and Matty van Son
EPSRC EP/W002817/1 & EP/W524098/1

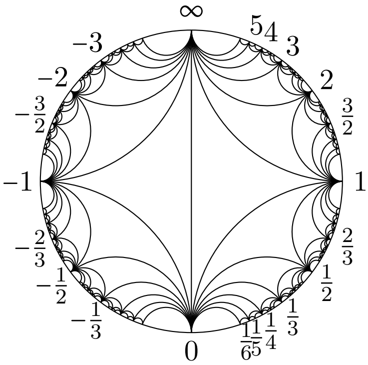
The Open University

Gallery of Farey graphs

The Farey complex over integers is a graph that has

vertices: equiv. classes $\left\{\frac{a}{b} : a, b \in \mathbb{Z},\ a, b \text{ coprime}\right\}/\sim$, with $\frac{a}{b} \sim \frac{-a}{-b}$;

edges: from $\frac{a}{b}$ to $\frac{c}{d}$ whenever $ad - bc = \pm 1$.



see arXiv:2312.12953 and talks by Ian and Matty

Farey complexes

For ring R and group of units $U \subset R^\times$, Farey complex $\mathscr{F}_{R,U}$ has
vertices: equiv. classes $\left\{\frac{a}{b} :\ a, b \in R,\ aR + bR = R\right\}/\sim$, $\quad \frac{a}{b} \sim \frac{ua}{ub}\ \ \forall u \in U$;
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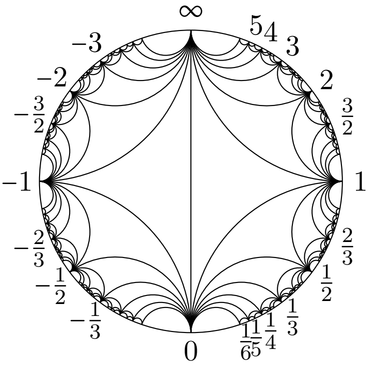

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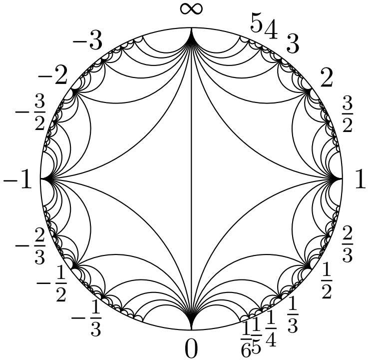
For ring R and group of units $U \subset R^\times$, Farey complex $\mathscr{F}_{R,U}$ has vertices: equiv. classes $\left\{\frac{a}{b} : \ a, b \in R, \ aR + bR = R\right\} / \sim, \quad \frac{a}{b} \sim \frac{ua}{ub} \ \forall u \in U$;
edges: from $\frac{a}{b}$ to $\frac{c}{d}$ whenever $ad - bc \in U$.

Define $\mathscr{F}_R = \mathscr{F}_{R,\{\pm 1\}}$.

Example: the Farey graph over integers $\mathscr{F}_{\mathbb{Z}}$



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$$\mathsf{SL}_2(R)\Big\backslash\left\{\begin{matrix}\text{pairs of bi-infinite}\\ \text{paths in } \mathscr{F}_{R,U}\end{matrix}\right\} \quad\longleftrightarrow\quad (U\times U)\Big\backslash\left\{\begin{matrix}\text{tame } \mathsf{SL}_2\text{-tilings}\\ \text{over } R\end{matrix}\right\}$$

$$\mathsf{SL}_2(R)\Big\backslash\left\{\begin{matrix}\text{paths in } \mathscr{F}_{R,U} \text{ from}\\ \text{any } \frac{a}{b} \text{ to } \frac{ua}{ub},\ u\in R^\times\end{matrix}\right\} \quad\longleftrightarrow\quad U\Big\backslash\left\{\begin{matrix}\text{tame friezes over } R\\ \text{with first row of 1s}\end{matrix}\right\}$$

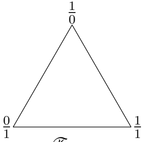
Gallery of Farey graphs

- $\mathscr{F}_{\mathbb{Z}/N\mathbb{Z}}$ and uniqueness of lifting
- $\mathscr{F}_{\mathbb{F}_4}$, $\mathscr{F}_{K,K^\times}$, and symmetries of Farey complex
- $\mathscr{F}_{\mathcal{O}_d,\mathcal{O}_d^\times}$ and tessellations of hyperbolic space

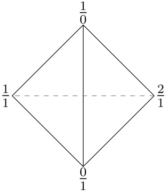
by Jorge Royan, commons.wikimedia.org/wiki/File:Oxford_-_Pitt_Rivers_Museum_-_0269.jpg, CC BY-SA 3.0

Gallery of Farey graphs: $\mathscr{F}_{\mathbb{Z}/N\mathbb{Z}}$ and uniqueness of lifting

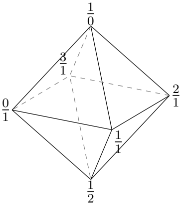
$\mathscr{F}_{\mathbb{Z}/N\mathbb{Z}}$ and uniqueness of lifting: Farey complexes for integers modulo N



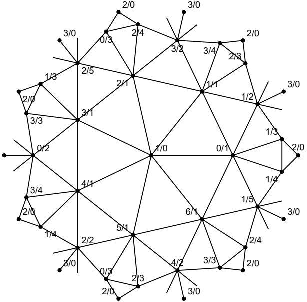
$\mathscr{F}_{\mathbb{Z}/2\mathbb{Z}}$



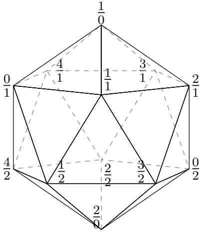
$\mathscr{F}_{\mathbb{Z}/3\mathbb{Z}}$



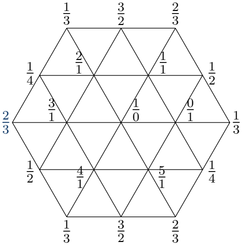
$\mathscr{F}_{\mathbb{Z}/4\mathbb{Z}}$



$\mathscr{F}_{\mathbb{Z}/7\mathbb{Z}}$ (genus 3)



$\mathscr{F}_{\mathbb{Z}/5\mathbb{Z}}$



$\mathscr{F}_{\mathbb{Z}/6\mathbb{Z}}$ (torus)

genus$(N) = 1 + \frac{1}{24}N^2(N-6)\prod_{p|N}\left(1 - 1/p^2\right)$

$\mathscr{F}_{\mathbb{Z}/N\mathbb{Z}}$ and uniqueness of lifting

A path in $\mathscr{F}_{\mathbb{Z}/N\mathbb{Z}}$ lifts to a closed path in $\mathscr{F}_{\mathbb{Z}}$ (and defines an integer frieze) when it is closed and strongly contractible.

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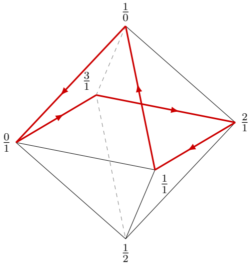

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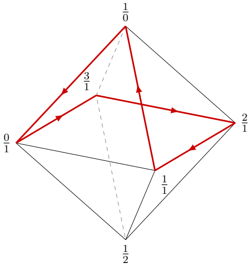

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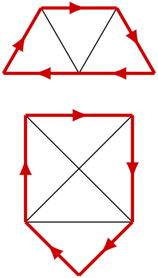

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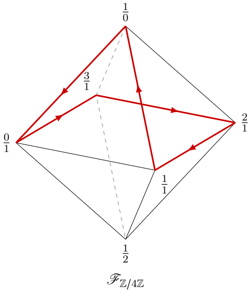

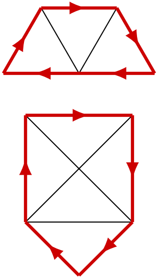

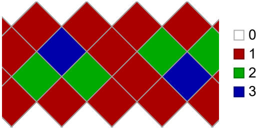

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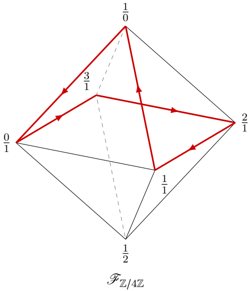
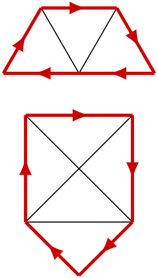
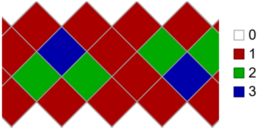
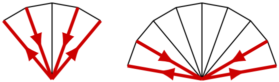






The lifting of a path can be unique up to $\mathrm{SL}_2(\mathbb{Z})$ only if it has no self-intersections.



$\mathscr{F}_{\mathbb{Z}/N\mathbb{Z}}$ and uniqueness of lifting

Can a non-self-intersecting path in $\mathscr{F}_{\mathbb{Z}/N\mathbb{Z}}$ be lifted to different paths in $\mathscr{F}_{\mathbb{Z}}$, up to the action of $\mathrm{SL}_2(\mathbb{Z})$ transformations?

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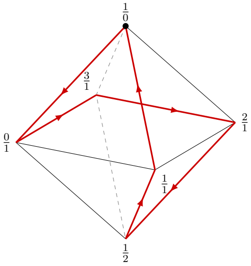
It has to bound two disks with no internal vertices
$=$ be a Hamiltonian cycle on a sphere.



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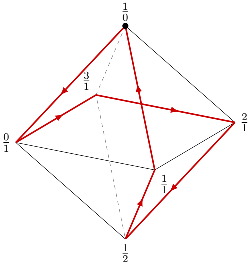

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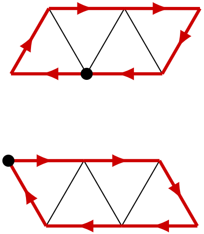

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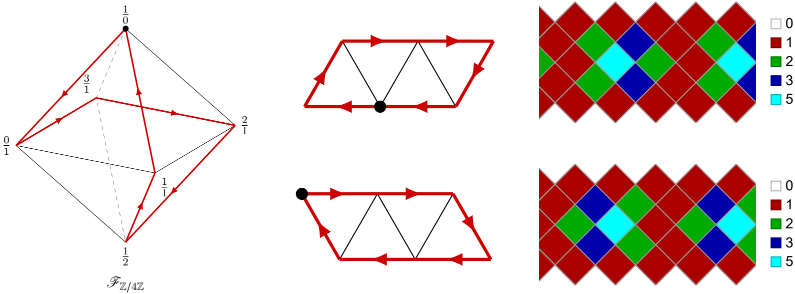

These two are related by reflection $+$ basepoint shift.

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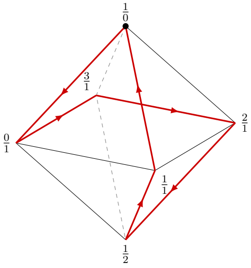

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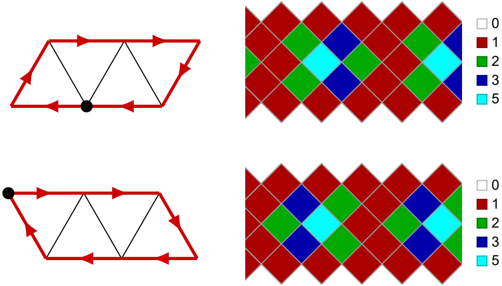

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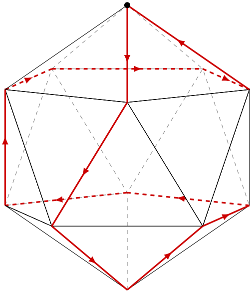
Can a path be lifted to two *really* different paths?

$\mathscr{F}_{\mathbb{Z}/N\mathbb{Z}}$ and uniqueness of lifting



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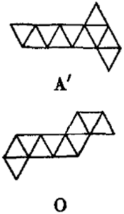

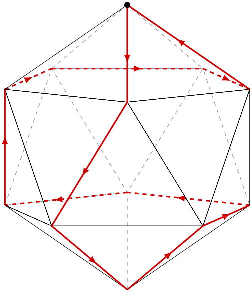

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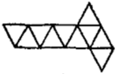
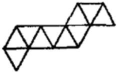


A′



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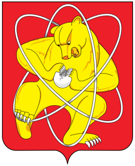

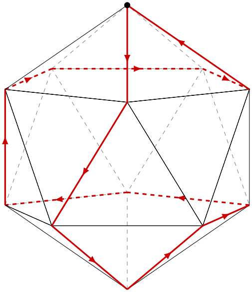

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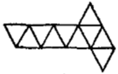

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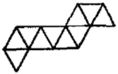

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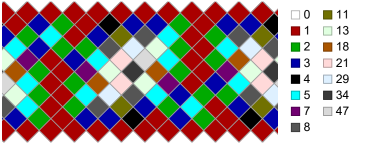

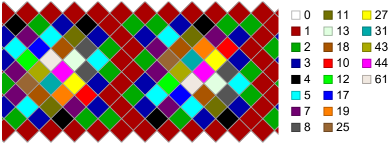

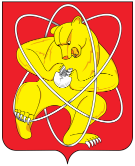

$\mathscr{F}_{\mathbb{Z}/N\mathbb{Z}}$ and uniqueness of lifting

There are 17 Hamiltonian cycles on an icosahedron,
9 of which separate the icosahedron into two differently triangulated disks.
(Sainte-Laguë, 1937)

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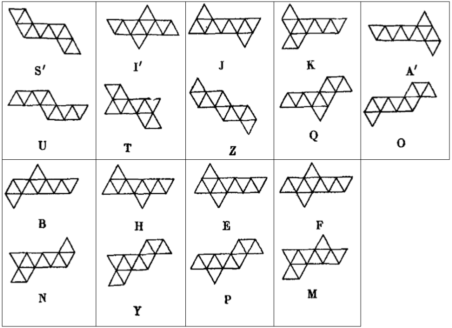

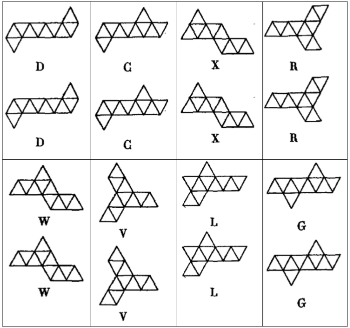

$\mathscr{F}_{\mathbb{Z}/N\mathbb{Z}}$ and uniqueness of lifting: conclusion

Theorem. Suppose γ is a path in $\mathscr{F}_{\mathbb{Z}/N\mathbb{Z}}$. Consider its lift(s) to closed path(s) in $\mathscr{F}_{\mathbb{Z}}$, up to the action of $\mathrm{SL}_2(\mathbb{Z})$.

- If $N = 5$ and γ is one of the 9 distinguished Hamiltonian cycles, then γ has 2 inequivalent* lifts;
- else, if $N = 2, 3, 4, 5$ and γ is a Hamiltonian cycle, then γ has 2 equivalent lifts;
- else, if γ is non-self-intersecting, then γ has at most 1 lift;
- otherwise, γ has either no or infinitely many lifts.

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*under reflection with direction change and a basepoint shift

Gallery of Farey graphs: $\mathscr{F}_{\mathbb{F}_4}$, $\mathscr{F}_{K,K^\times}$, and symmetries of Farey complexes

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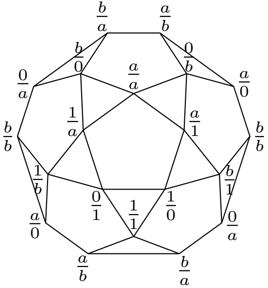
Example: let $R = \mathbb{F}_4 = \{0, 1, a, b\}$ be a field, $U = \{1\}$.



$\mathscr{F}_{\mathbb{F}_4}$ is embedded into projective plane

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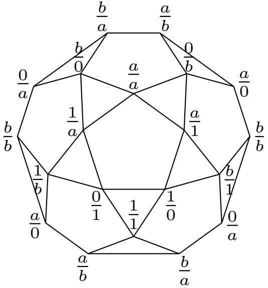
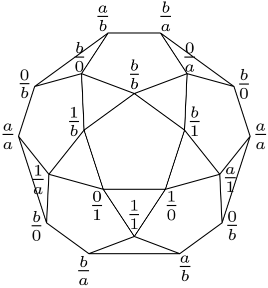
Example: let $R = \mathbb{F}_4 = \{0, 1, a, b\}$ be a field, $U = \{1\}$. Frobenius automorphism $a \leftrightarrow b$ acts on $\mathscr{F}_{\mathbb{F}_4}$:



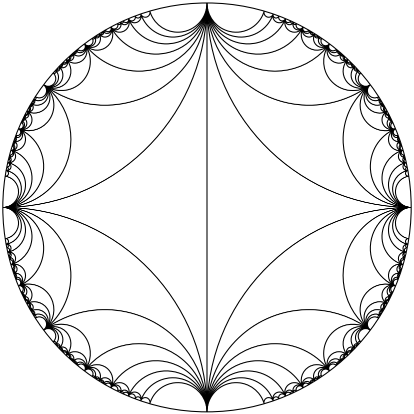


$\mathscr{F}_{\mathbb{F}_4}$ is embedded into projective plane in two different ways

- Any $\mathscr{F}_{R,U}$ has symmetries represented by the special linear group $\mathrm{SL}_2(R)$. SL_2-tilings are invariant w.r.t. its action.
- Moreover, any $\mathscr{F}_{R,U}$ has symmetries coming from the general semilinear group $\Gamma\mathrm{L}_2(R) = \mathrm{GL}_2(R) \rtimes \mathrm{Aut}(R)$.
- Some Farey complexes have more symmetries.
  - For a field K, $\mathscr{F}_{K,K^\times}$ is the projective line over K.
  - It is a complete graph, its symmetry group (collineation group) is the symmetric group.

Gallery of Farey graphs: $\mathscr{F}_{\mathcal{O}_d,\mathcal{O}_d^\times}$ and tessellations of hyperbolic space

Farey graph over integers is a hyperbolic tessellation



Tessellation of the hyperbolic plane by ideal triangles

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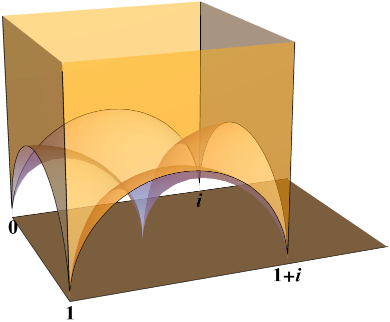

(see e.g. papers by M. Hockman for the relation to continued fractions)

Farey graph over Eisenstein integers

$\mathscr{F}_{R,U}$ with $R = \mathbb{Z}[\sigma]$ with $\sigma = e^{i\pi/3}$, $U = \mathbb{Z}[\sigma]^{\times} = \{\pm 1, \pm\sigma, \pm\sigma^2\}$.

(A. Felikson, O. Karpenkov, Kh. Serhiyenko, P. Tumarkin, arXiv:2306.17118)

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Again, edges drawn in $\mathbb{H}^3$ as the orbit of the edge $1/0$ to $0/1$ under $\mathrm{SL}_2(\mathbb{Z}[\sigma])$, or $\mathrm{PSL}_2(\mathbb{Z}[\sigma])$.

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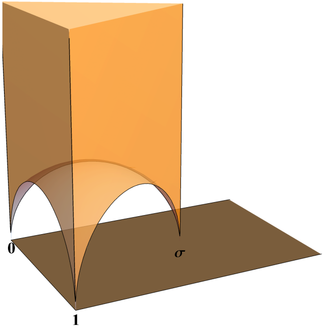

Symmetry group is $\mathrm{P\Gamma L}_2(\mathbb{Z}[\sigma]) = \mathrm{PGL}_2(\mathbb{Z}[\sigma]) \rtimes \langle \text{c. c.} \rangle$.

(A. Felikson, O. Karpenkov, Kh. Serhiyenko, P. Tumarkin, arXiv:2306.17118)

More generally, let R be the ring of integers of the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$, denoted $\mathcal{O}_d$, and $U = \mathcal{O}_d^\times$.

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Let $d = 1, 2, 3, 7, 11$:

- $\mathcal{O}_1 = \mathbb{Z}[i]$,
- $\mathcal{O}_3 = \mathbb{Z}[\sigma]$,
- $\mathcal{O}_2 = \mathbb{Z}[i\sqrt{2}]$,
- $\mathcal{O}_7 = \mathbb{Z}[(1 + i\sqrt{7})/2]$,
- $\mathcal{O}_{11} = \mathbb{Z}[(1 + i\sqrt{11})/2]$.

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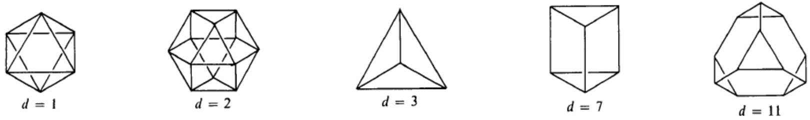
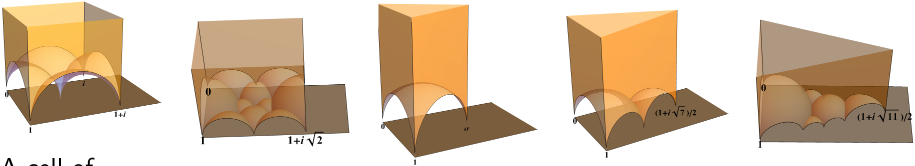
- $\mathcal{O}_{11} = \mathbb{Z}[(1 + i\sqrt{11})/2]$.

The Farey graph $\mathscr{F}_{\mathcal{O}_d, \mathcal{O}_d^\times}$ is again the orbit of $1/0$—$0/1$ under the Bianchi group $\mathrm{PSL}_2(\mathcal{O}_d)$.

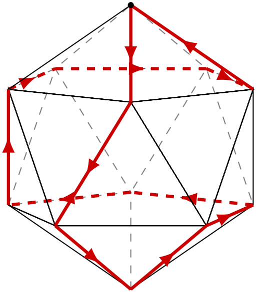
$\mathscr{F}_{\mathcal{O}_d,\mathcal{O}_d^\times}$ are again tessellations of $\mathbb{H}^3$ by ideal polyhedra.

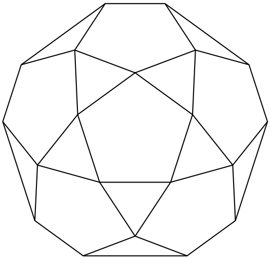


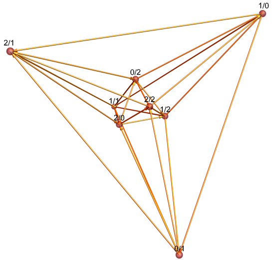


A cell of $\mathscr{F}_{\mathcal{O}_1,\langle i\rangle}$ $\mathscr{F}_{\mathcal{O}_2,\{\pm1\}}$ $\mathscr{F}_{\mathcal{O}_3,\langle\sigma\rangle}$ $\mathscr{F}_{\mathcal{O}_7,\{\pm1\}}$ $\mathscr{F}_{\mathcal{O}_{11},\{\pm1\}}$

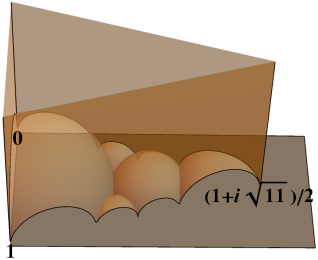
For $d = 1, 2, 3, 7, 11$, see Hatcher, J. London Math. Soc. (1983).
For other d, tessellations also exist but their edges
are not the same as $\mathscr{F}_{\mathcal{O}_d,\mathcal{O}_d^\times}$.
See Vulakh, Cremona, and Yasaki for different generalisations.



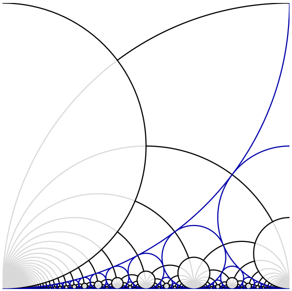




1/0
2/1
0/2
1/1
2/2
1/2
2/0
0/1



0
1
$(1+i\sqrt{11})/2$



Wild SL_2-tilings

Farey surfaces: combinatorial model for wild integer SL_2-tilings

An SL_2-tiling is *tame* if every 3×3 block has determinant 0. Otherwise, it is *wild*.

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$$
\begin{array}{ccccccccc}
 & & & & \vdots & & & & \\
 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & \\
 & -1 & 0 & 1 & \breve{0} & -1 & \breve{0} & 1 & \\
\cdots & 0 & -1 & 0 & 1 & 9 & -1 & 0 & \cdots \\
 & 1 & 0 & -1 & \breve{0} & 1 & \breve{0} & -1 & \\
 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & \\
 & & & & \vdots & & & &
\end{array}
$$

In an SL_2-tiling, $e \cdot \det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 0.$

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 & & & & \vdots & & & &
\end{array}
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In an SL_2-tiling, $e \cdot \det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 0.$

Wild entries are always 0s over $\mathbb{Z}$ (generally, zero divisors).


WILD ZERO

Tame SL_2-tilings correspond to pairs of paths in the Farey graph, what about wild ones?

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Idea: a 2×2 block in a wild SL_2-tiling is indistinguishable from a 2×2 block in a tame SL_2-tiling.

Definition. A *Farey edge* is a vertex-labelled graph consisting of a single edge joining vertices with labels $(i, \binom{a}{b})$ and $(i+1, \binom{a'}{b'})$, where $i \in \mathbb{Z}$ (to be referred as a *coordinate*) and a, b, a', b' are integers such that $\det \begin{pmatrix} a & a' \\ b & b' \end{pmatrix} = 1$.

Example:

$$(0, \binom{0}{1}) \text{ --- } (1, \binom{-1}{2})$$

Farey faces

Definition. A *Farey face* is a 2-dimensional cell complex consisting of a single 4-gonal cell, which is a direct product of two Farey edges.

Example:

$$\left(3, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) — \left(4, \begin{pmatrix} 2 \\ 5 \end{pmatrix}\right) \quad \times \quad \left(8, \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right) — \left(9, \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right) =$$

$$= \begin{array}{ccc} \left((3,8), \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}\right) & —— & \left((3,9), \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}\right) \\ | & & | \\ \left((4,8), \begin{pmatrix} 2 & -1 \\ 5 & 0 \end{pmatrix}\right) & —— & \left((4,9), \begin{pmatrix} 2 & 0 \\ 5 & -1 \end{pmatrix}\right) \end{array}$$

Definition. A *Farey surface* is a 2-dimensional cell complex consisting of Farey faces such that

- every edge belongs to exactly two faces, and
- for any two vertices labelled $((i,j),A)$ and $((i,j),B)$, $A \neq B$ and $\det A = \det B$.

This provides a well-defined way to turn a Farey surface into a bi-infinite matrix: $m_{i,j} = \det A$, where $((i,j),A)$ is a vertex in the Farey surface.
That bi-infinite matrix is always an SL_2-tiling.

Tame SL_2-tilings come from unbranched Farey surfaces

$$
\begin{array}{ccccc}
\left((-1,-1),\begin{pmatrix}3 & 2\\ -2 & 3\end{pmatrix}\right) & \text{———} & \left((-1,0),\begin{pmatrix}3 & 1\\ -2 & 2\end{pmatrix}\right) & \text{———} & \left((-1,1),\begin{pmatrix}3 & 0\\ -2 & 1\end{pmatrix}\right)\\
| & & | & & |\\
\left((0,-1),\begin{pmatrix}2 & 2\\ -1 & 3\end{pmatrix}\right) & \text{———} & \left((0,0),\begin{pmatrix}2 & 1\\ -1 & 2\end{pmatrix}\right) & \text{———} & \left((0,1),\begin{pmatrix}2 & 0\\ -1 & 1\end{pmatrix}\right)\\
| & & | & & |\\
\left((1,-1),\begin{pmatrix}1 & 2\\ 0 & 3\end{pmatrix}\right) & \text{———} & \left((1,0),\begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix}\right) & \text{———} & \left((1,1),\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)
\end{array}
$$

$$\downarrow \ \det$$

$$
\begin{array}{ccccc}
 & 13 & 8 & 3 & \\
\cdots & 8 & 5 & 2 & \cdots\\
 & 3 & 2 & 1 &
\end{array}
$$

The group $\mathrm{SL}_2(\mathbb{Z})$ acts on Farey surfaces by left multiplication by matrix labels.
Computing determinants of the matrix labels always gives an SL_2-tiling invariant under that action.

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Theorem. There is a bijection between

- the set of Farey surfaces, modulo $\mathrm{SL}_2(\mathbb{Z})$,
- and the set of SL_2-tilings.

What specifically happens around a 0 in an SL_2-tiling?
Whenever $m_{i,j} = 0$,

$$M_{i,j} = \begin{pmatrix} m_{i-1,j-1} & m_{i-1,j} & m_{i-1,j+1} \\ m_{i,j-1} & m_{i,j} & m_{i,j+1} \\ m_{i+1,j-1} & m_{i+1,j} & m_{i+1,j+1} \end{pmatrix} = \pm \begin{pmatrix} -u' + v & 1 & u' + v' + w_{i,j} \\ -1 & 0 & 1 \\ -u - v & -1 & u - v' \end{pmatrix}$$

for some $u, u', v, v', w_{i,j}$ and $\det M_{i,j} = \pm w_{i,j}$.

$$\begin{array}{ccccccccc} & & & & \vdots & & & & \\ & 0 & 1 & 0 & -1 & 0 & 1 & 0 & \\ & -1 & 0 & 1 & \breve{0} & -1 & \breve{0} & 1 & \\ & 0 & -1 & 0 & 1 & 9 & -1 & 0 & \\ \cdots & 1 & 0 & -1 & \breve{0} & 1 & \breve{0} & -1 & \cdots \\ & 0 & 1 & 0 & -1 & 0 & 1 & 0 & \\ & -1 & 0 & 1 & 0 & -1 & 0 & 1 & \\ & 0 & -1 & 0 & 1 & 0 & -1 & 0 & \end{array}$$

A portion of a Farey surface which is mapped to $M_{i,j}$ with $m_{i,j} = 0$ and $\det M_{i,j} = 0$ looks like this, up to the action of $\mathrm{SL}_2(\mathbb{Z})$:

$$
\begin{array}{ccccc}
\left((i-1,j-1), \begin{pmatrix} u' & \pm v \\ -1 & \mp 1 \end{pmatrix}\right) & \text{———} & \left((i-1,j), \begin{pmatrix} u' & \pm 1 \\ -1 & 0 \end{pmatrix}\right) & \text{———} & \left((i-1,j+1), \begin{pmatrix} u' & \pm v' \\ -1 & \pm 1 \end{pmatrix}\right) \\
\Big| & S_0 & \Big| & S_3 & \Big| \\
\left((i,j-1), \begin{pmatrix} 1 & \pm v \\ 0 & \mp 1 \end{pmatrix}\right) & \text{———} & \left((i,j), \begin{pmatrix} 1 & \pm 1 \\ 0 & 0 \end{pmatrix}\right) & \text{———} & \left((i,j+1), \begin{pmatrix} 1 & \pm v' \\ 0 & \pm 1 \end{pmatrix}\right) \\
\Big| & S_1 & \Big| & S_2 & \Big| \\
\left((i+1,j-1), \begin{pmatrix} u & \pm v \\ 1 & \mp 1 \end{pmatrix}\right) & \text{———} & \left((i+1,j), \begin{pmatrix} u & \pm 1 \\ 1 & 0 \end{pmatrix}\right) & \text{———} & \left((i+1,j+1), \begin{pmatrix} u & \pm v' \\ 1 & \pm 1 \end{pmatrix}\right)
\end{array}
$$

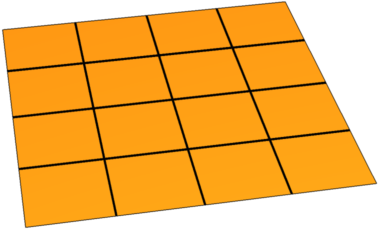
Now let $\det M_{i,j} = w_{i,j} \neq 0$. A few Farey faces adjacent to a vertex with coordinates (i,j) look like this (i.e. same as above), up to the action of $\mathrm{SL}_2(\mathbb{Z})$...

$$
\begin{array}{ccccc}
\left((i{-}1,j{-}1), \begin{pmatrix} u' & \pm v \\ -1 & \mp 1 \end{pmatrix}\right) & \text{———} & \left((i{-}1,j), \begin{pmatrix} u' & \pm 1 \\ -1 & 0 \end{pmatrix}\right) & & \\
\Big| & S_0 & \Big| & & \\
\left((i,j{-}1), \begin{pmatrix} 1 & \pm v \\ 0 & \mp 1 \end{pmatrix}\right) & \text{———} & \left((i,j), \begin{pmatrix} 1 & \pm 1 \\ 0 & 0 \end{pmatrix}\right) & \text{———} & \left((i,j{+}1), \begin{pmatrix} 1 & \pm v' \\ 0 & \pm 1 \end{pmatrix}\right) \\
\Big| & S_1 & \Big| & S_2 & \Big| \\
\left((i{+}1,j{-}), \begin{pmatrix} u & \pm v \\ 1 & \mp 1 \end{pmatrix}\right) & \text{———} & \left((i{+}1,j), \begin{pmatrix} u & \pm 1 \\ 1 & 0 \end{pmatrix}\right) & \text{———} & \left((i{+}1,j{+}1), \begin{pmatrix} u & \pm v' \\ 1 & \pm 1 \end{pmatrix}\right)
\end{array}
$$

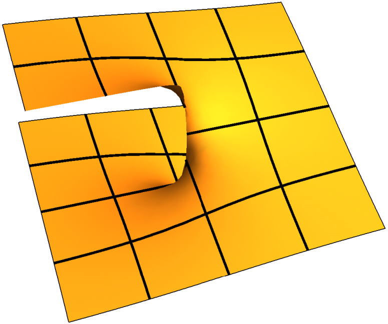
...and more faces look like this (S_2 is same as above, faces $\bar{S}_0$ and S_0 are different sheets of the same covering):

$$
\begin{array}{ccccc}
\left(\ \cdot\ , \begin{pmatrix} u'+w_{i,j} & \pm(v+w_{i,j}) \\ -1 & \mp 1 \end{pmatrix}\right) & \text{---} & \left(\ \cdot\ , \begin{pmatrix} u'+w_{i,j} & \pm 1 \\ -1 & 0 \end{pmatrix}\right) & \text{---} & \left(\ \cdot\ , \begin{pmatrix} u'+w_{i,j} & \pm v' \\ -1 & \pm 1 \end{pmatrix}\right) \\
| & \bar{S}_0 & | & \bar{S}_3 & | \\
\left(\ \cdot\ , \begin{pmatrix} 1 & \pm v \\ 0 & \mp 1 \end{pmatrix}\right) & \text{---} & \left((i,j), \begin{pmatrix} 1 & \pm 1 \\ 0 & 0 \end{pmatrix}\right) & \text{---} & \left(\ \cdot\ , \begin{pmatrix} 1 & \pm v' \\ 0 & \pm 1 \end{pmatrix}\right) \\
 & & | & S_2 & | \\
 & & \left(\ \cdot\ , \begin{pmatrix} u & \pm 1 \\ 1 & 0 \end{pmatrix}\right) & \text{---} & \left(\ \cdot\ , \begin{pmatrix} u & \pm v' \\ 1 & \pm 1 \end{pmatrix}\right)
\end{array}
$$

A Farey surface of a tame SL_2-tiling is the plane:

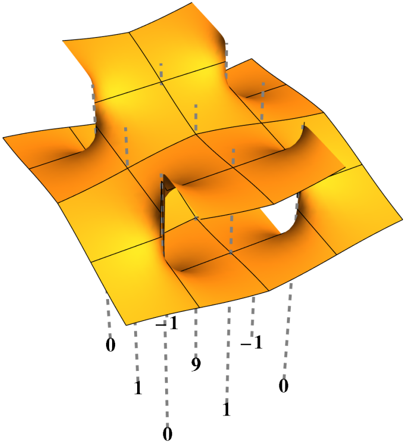


A Farey surface around a wild zero looks like the Riemann surface of $\mathrm{Log}(z)$:



What Farey surfaces look like

$$
\begin{array}{ccccccc}
 & & & \vdots & & & \\
 & 0 & -1 & 0 & 1 & 0 & \\
 & 1 & \breve{0} & -1 & \breve{0} & 1 & \\
\cdots & 0 & 1 & 9 & -1 & 0 & \cdots \\
 & -1 & \breve{0} & 1 & \breve{0} & -1 & \\
 & 0 & -1 & 0 & 1 & 0 & \\
 & & & \vdots & & &
\end{array}
$$



Very wild SL_2-tilings

How tightly can we pack wild zeros in an SL_2-tiling?

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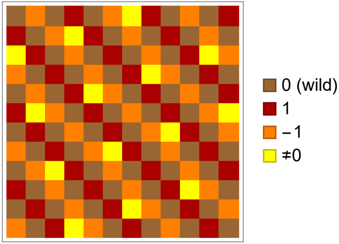
A maximally wild integer SL_2-tiling

How tightly can we pack wild zeros in an SL_2-tiling? ~~00~~



A maximally wild integer SL_2-tiling

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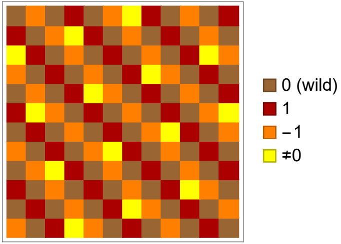
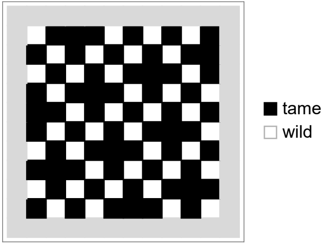
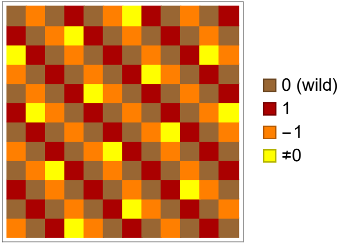
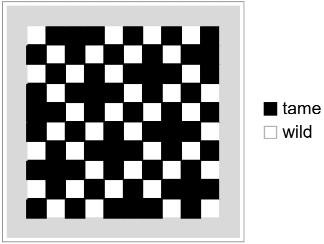




40% of entries are wild. Can we do better?

How tightly can we pack wild zeros in an SL_2-tiling? ~~00~~





40% of entries are wild. Can we do better?

Yes, but not over an integral domain!

Generally, in an SL_2-tiling,

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = (a + c + g + i) + (cg - ai)e.$$

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$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = (a + c + g + i) + (cg - ai)e.$$

Repeat

$$\begin{array}{ccccc} 3 & 2 & -3 & -2 & \\ 4 & 3 & -4 & -3 & \dots \\ -3 & -2 & 3 & 2 & \\ -4 & -3 & 4 & 3 & \\ & \vdots & & & \end{array}$$

Generally, in an SL_2-tiling,

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = (a+c+g+i) + (cg-ai)e.$$

Repeat and take entries modulo $N = 2^2 \cdot 3^2 = 36$

$$\begin{matrix} 3 & 2 & -3 & -2 & \\ 4 & 3 & -4 & -3 & \dots \\ N/3-3 & N/2-2 & 3 & 2 & \\ N/2-4 & N/3-3 & 4 & 3 & \\ & \vdots & & & \end{matrix}$$

The SL_2-tiling is over $\mathbb{Z}/36\mathbb{Z}$. Repeat:

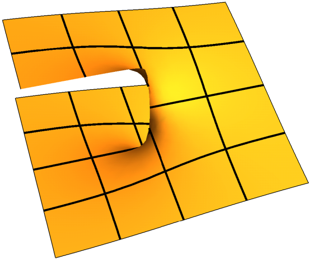
$$\begin{matrix} 3 & 2 & 33 & 34 & \\ 4 & 3 & 32 & 33 & \dots \\ 9 & 16 & 3 & 2 & \\ 14 & 9 & 4 & 3 & \\ & \vdots & & & \end{matrix}$$

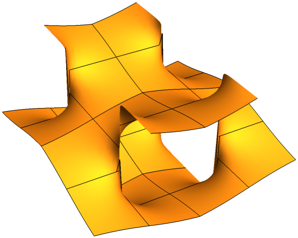
The SL_2-tiling is over $\mathbb{Z}/36\mathbb{Z}$. Repeat:

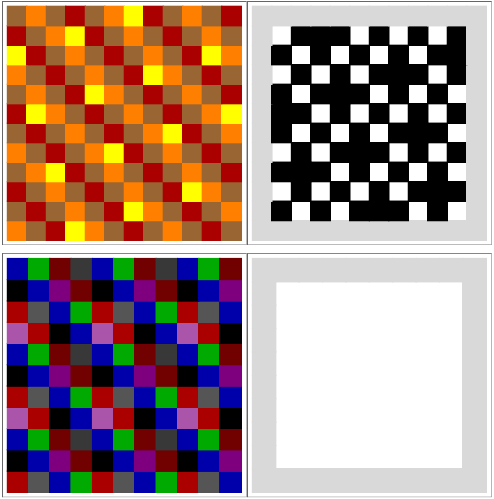
$$\begin{array}{ccccc} 3 & 2 & 33 & 34 & \\ 4 & 3 & 32 & 33 & \dots \\ 9 & 16 & 3 & 2 & \\ 14 & 9 & 4 & 3 & \\ & \vdots & & & \end{array}$$

All entries of this SL_2-tiling are wild, 3×3 determinants being

$$\begin{array}{ccccc} 12 & 18 & 12 & 18 & \\ 18 & 12 & 18 & 12 & \dots \\ 12 & 18 & 12 & 18 & \\ 18 & 12 & 18 & 12 & \\ & \vdots & & & \end{array}$$







- A closed, non-self-intersecting path in the Farey graph over $\mathbb{Z}/N\mathbb{Z}$ lifts to $\mathbb{Z}$ essentially non-uniquely when $N = 5$ and it is one of 9 distinguished Hamiltonian cycles.
- Farey graphs arise e.g. from Bianchi groups associated with $\mathbb{Q}(\sqrt{-d})$ for $d = 1, 2, 3, 7, 11$.
- Farey surface is a combinatorial model for any SL_2-tiling.
- There is an SL_2-tiling with all entries wild.

Thank you!
Questions, please.

`SL2tilings.github.io`
`mathstodon.xyz/PSL2Z`