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(1)

Infinite friezes of affine type D

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(1) Infinite friezes

Work over $\mathbb{Z}$ throughout

$\mathcal{F}$:

$$\begin{array}{ccccccccccccc} & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\ \cdots & & m_{-1,1} & & m_{0,2} & & m_{1,3} & & m_{2,4} & & m_{3,5} & & \cdots \\ \cdots & * & & * & & m_{0,3} & & m_{1,4} & & * & & \cdots & \\ & & \cdots & & & & m_{0,4} & & & \cdots & & & \end{array}$$

* diamond rule: for any
$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$$
in pattern: $ad-bc=1$

* all m_{ij} ($j\geq i+1$) are in $\mathbb{Z}_{>0}$

$\mathcal{F}$ is called periodic if $\exists\, n\in\mathbb{Z}_{>0}$: $m_{i,j} = m_{i+n,j+n}$ $\forall\,(i,j)$, $j\geq i+2$ } then $\mathcal{F}$ is of period n

If $\mathcal{F}$ is of period n: $Q=(m_{1,3},\dots,m_{n,n+2})$ a quiddity sequence, determines $\mathcal{F}$ (with diamond rule). If $\mathcal{F}$ is not periodic: $Q=(\dots,m_{i,i+2},m_{i+1,i+3},\dots)$ ∞ quidd. sequence

As finite (integer) friezes, ∞ friezes arise from triangulations of surfaces, as matching numbers:

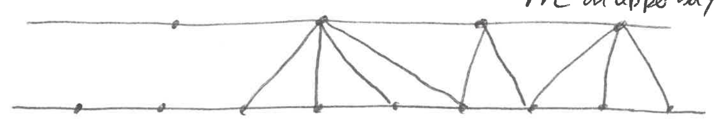
Thm (BParsons-Tschabold, Thms 5.2, 5.5, 3.7, 4.6)

1) If $\mathcal{F}$ is an ∞ frieze, $\exists$ triangulation of ∞-strip s.t. $m_{i-1,i+1}$ = # of triangles at i

(and every triangul. of an ∞-strip gives an ∞ frieze)

∞-strip:

$\mathcal{M}$ vertices on upper bdy



$i-2$ $i-1$ i $i+1$

lower bdy: indexed by $\mathbb{Z}$, for the $m_{i-1,i+1}$

e.g. $m_{i-1,i+1}=2$, $m_{i,i+2}=2$, $m_{i+1,i+3}=3$

2) If $\mathcal{F}$ is periodic, of period n: $\exists$ triangul. of an annulus with n pts on the outer boundary and $m\geq 0$ on inner bdy, s.t. $m_{i-1,i+1}$ = # of Δ's at vertex i (regions visible in a small neighbourhood) – some triangles get counted twice

[and every triangul. of $A_{n,m}$ (with $m>0$) gives two ∞ friezes – period n, – period m]

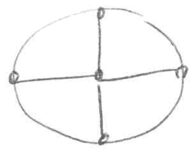
Examples

1)

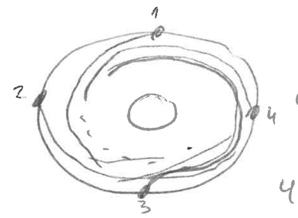
$$\begin{array}{ccccccccc} 1 & & 1 & & 1 & & 1 & & 1 \\ & 2 & & 2 & & 2 & & 2 & \\ 3 & & 3 & & 3 & & 3 & & \\ & & & \vdots & & & & & \end{array}$$

e.g. $Q = (2)$ or $Q = (2,2,2,2)$

triangulated surface: for $Q = (2,2,2,2)$

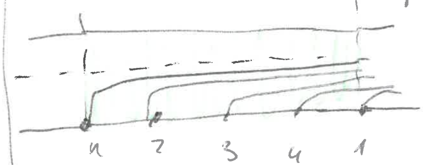


punctured disk



annulus

4 asymptotic (spiraling) arcs

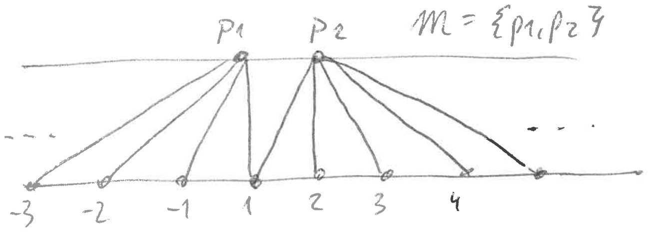


2) $Q = (\ldots, 2, 2, 3, 2, 2, \ldots)$ not periodic

$m_{i,i+2} = 2$ for all $i \neq 0$

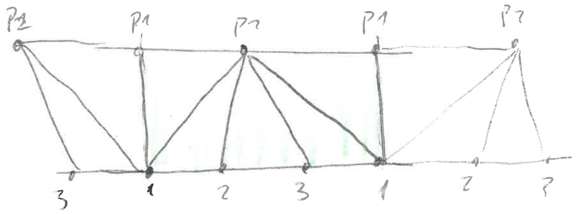
$m_{0,2} = 3$ $\mathcal{M} = \{p_1, p_2\}$

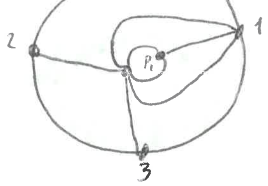
∞ strip



3) $Q = (2,2,4)$ or $Q' = (2,5)$

Q' = quidd. sequence from inner boundary of this triangulated annulus





(2) Growth behaviour

Fact: If $\mathcal{F}$ is n-periodic: difference of entries $m_{i,i+n+1} - m_{i+1,i+n} = m_{j,j+n+1} - m_{j+1,j+n}$ $\forall\, i,j$

(difference of an entry in row and the entry right above it is constant!)

[This is also true for finite friezes ... extend the frieze by negatives to see it]

If $\mathcal{F}$ is given by $Q = (a_1, \ldots, a_n)$ ($a_i = m_{i-1,i+1}$):

call $m_{i,i+n+1} - m_{i+1,i+n}$ the (first) growth coefficient of $\mathcal{F}$; $s(\mathcal{F}) = s_1(\mathcal{F})$

(and $\underbrace{m_{i,i+kn+1} - m_{i+1,i+kn}}_{s_k(\mathcal{F})}$ the k-th growth coeff. of $\mathcal{F}$)

Note: $s_0(Q) = 2$ for any periodic frieze

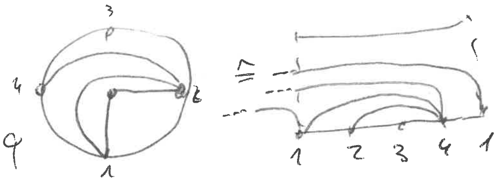
$s_0(Q) = 1 - (-1) = 2$

Linear growth (Tschabold '15)

$\mathcal{F}$ n-periodic, from punctured disk (or in annulus, with asymptotic arcs)

$\Rightarrow$ entries in each diagonal form an arithmetic progression of order n.

e.g. $Q = (2)$ or $Q = \cancel{(4,2,3,4)}$ $(3,2,1,4)$



0 0 0 0

1 1 1 1 1 1

3 2 1 4 3

11 5 1 3 11

18 2 2 8 18

13 7 3 5 13 7

$7-5=2$

$5-3=2$ ✓

Exponential growth (B-Felner-Parsons-Tschabold)

If $\mathcal{F}$ arises from a triangul. of an annulus but not from a triangul. of a punctured disk, then the entries in every diagonal grow asymptotically exponentially.

Growth coeff. (BFPT'19):

$$S_{k+2}(Q) = s_1(Q)\cdot S_{k+1}(Q) - S_k(Q)$$

$S_2(Q) = s_1(Q)^2 - 2$

(in other words: the sequence $S_k(Q)$ are given by a Chebyshev polyn. of the 2nd kind in ~~$s(Q) = s(Q) + k \sum \ldots$~~

$s(Q)=s_1(Q)$: $S_k(Q) = s(Q)^k + k\sum_{l=1}^{\lfloor k/2 \rfloor} (-1)^l \frac{1}{k-l}\binom{k-l}{l}\cdot s(Q)^{k-2l}$)

From above example (ex. 31):

$Q=(2,2,4)$

$s_1(Q)=8$

$s_2(Q)=62$

1 1 1

2 2 8 4 2 2

3 7 7 3

10 10 12 10 10

33 17 17 33

56 56 24 56 56

79 79

62

$Q'=(2,5)$

$s_1(Q')=8$

$s_2(Q')=62$

1 1 1 1

2 5 8

9 9 9 9

40 16 40

71 71 71 71

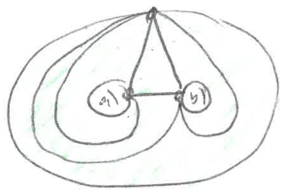
$\boxed{\text{Same growth!}}$

remarkable:

<u>Thm (BFPT '19, Thm 3.4)</u> The two growth coeff's of the friezes of a triangulated annulus are equal.

<u>Rem</u> (Gunawan-Musiker-Vogel): the growth coefficient of ~~an~~ (triangulated) annulus is equal to the # of submodules of the band (corresponding to the non-contractible closed loop)

<u>Rem</u>: * In general, the growth coeff. for a triangul. surface are not the same. Eg sphere w. 3 bdy components



$G_{outer} = (6)$

$G_{a)} = (4)$

$G_{b)} = (5)$

* For Grassmannian cluster categories $Gr(3,9)$ and $Gr(4,8)$: have 3 families of non-homog. tubes with different growth coeff's.

<u>Conj.</u> $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_m$, $m \geq 2$, from triangul. of surface with m bdy comp. (no puncture).

Then $s(\mathcal{F}_1) = s(\mathcal{F}_2) = \ldots = s(\mathcal{F}_m)$ iff $m = 2$.

<u>Qu</u>: How are the friezes linked for $m > 2$?

<u>Here</u>: show that for $\tilde{D}$ we also have uniform growth

(<u>Rem</u>: Annulus $\rightsquigarrow$ type $\tilde{A}$)

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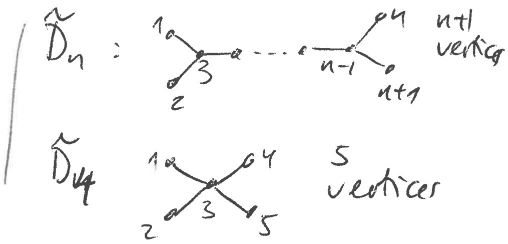
③ Friezes in type $\tilde{D}$

Cluster cat. of type $\tilde{D}_n$ $\leftrightarrow$ triangul. of disk with $n-2$ pts on bdy & 2 punctures

To a cluster category ~~cu~~ of type $\tilde{D}_n$ $(n\geq 4)$ we associate 3 ∞ friezes – from the three non-homogeneous tube in the Auslander-Reiten quiver.

They have period $n-2$, 2, 2



To get friezes from these tubes: take the Caldero-Chapoton map and specialise the cluster corresponding to the triangul. to 1.

Theorem (BBGTY) The 3 friezes arising from ~~the~~ a cluster cat. have the same growth coefficients.

Methods: Use the fact that we can work with (generalised) arcs in the twice punctured disk.

Translate the above into the surface set-up, count matching numbers to ~~com~~ get quidd-sequences.

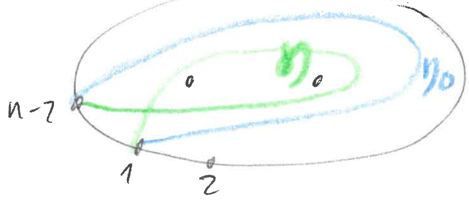
[cluster algebras from triangul. of surfaces: Fomin-Shapiro-Thurston '08
cl. categories from surfaces w.out punctures: Brüstle-Zhang
with punctures: Qiu-Zhou '17
geometric model for tubes: B-Marsh '09]

Notation: $\mathcal{T}_1$ the tube of rank $n-2$ $(n\geq 4)$

$\left.\begin{array}{l}\mathcal{T}_2\\ \mathcal{T}_3\end{array}\right\}$ the two tubes of rank 2 "the small tubes"

Fact: $\left\{\begin{array}{l}\text{Indecomposable}\\ \text{objects of } \mathcal{T}_1\end{array}\right\}/\text{iso} \longleftrightarrow$ arcs $\gamma_{i,j}^{(m)}$, $i,j\in\{1,\dots,n-2\}$, $m\geq 0$: starts at i, goes parallel to boundary, ~~clo~~ counterclockwise, m full turns, ends at j

If $m=0$: $j\notin\{i,i+1\}$

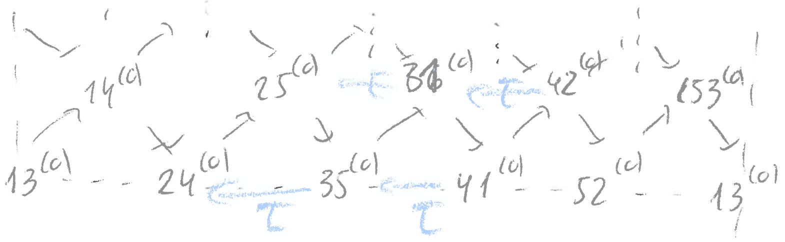


e.g. $\eta = \gamma_{n-2,1}^{(1)}$ and $\eta_0 = \gamma_{1,n-2}^{(0)}$

For $\mathcal{J}_1$:

eg $n=7$

write $ij^{(m)}$ for $\gamma_{ij}^{(m)}$



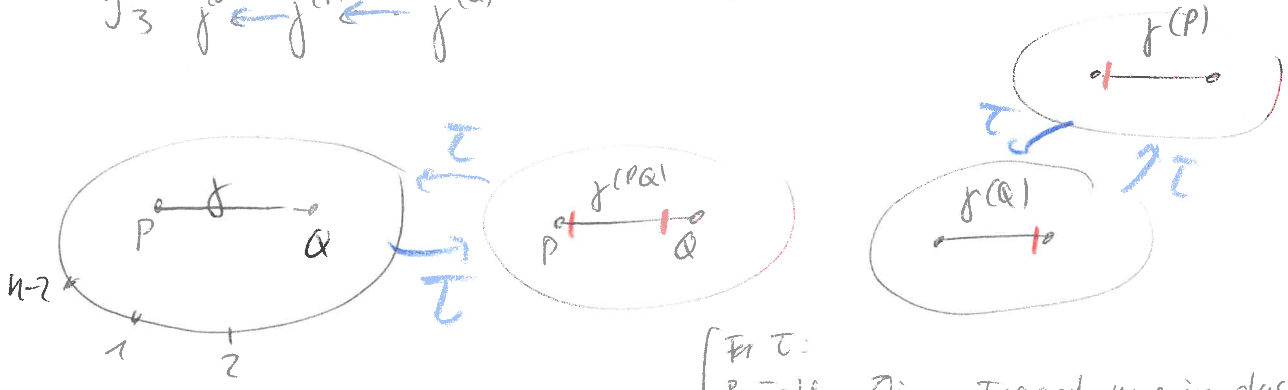
~~$\tau(\gamma_{ij}^{(m)}) = \tau$~~

$\tau(\gamma_{ij}^{(m)}) = \tau(\gamma_{i-1,j-1}^{(m)})$

Tubes $\mathcal{J}_2$ $\gamma^{(PQ)} \leftarrow \gamma \leftarrow \gamma^{(P,Q)}$

$\mathcal{J}_3$ $\gamma^{(Q)} \leftarrow \gamma^{(P)} \leftarrow \gamma^{(Q)}$

$\Big\}$ give the quasi-simple modules



[For τ: Brüstle-Qiu, Tagged mapping class group]

④ **To compute the growth coefficient of the frieze for $\mathcal{J}_1$.**

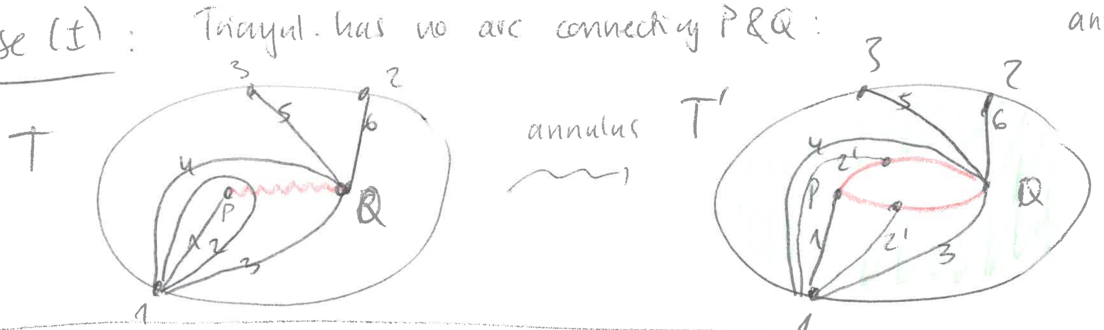
Translate set-up to $\tilde{A}$:

let $Q_1 = (a_1, \dots, a_{n-2})$ for $a_i = cc(\gamma^{(0)}_{i-1,i+1})$

= # of triangles in small neighbourhood of vertex i

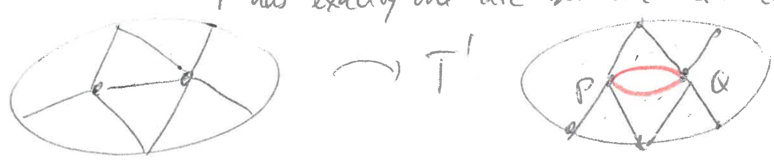
$\mapsto$ $\mathcal{F}(Q_1)$ as frieze $\rightsquigarrow$ lift triang. to a ~~tate~~ triangul. of annulus and compute growth coeff. there.

Case (I): Triangul. has no arc connecting P & Q:



any arc in T crossing ~~~ becomes 2 arcs in the annulus. All other arcs remain

case (II): T has exactly one arc between Q & P (no loop): make this into inner boundary (doubling the arc).



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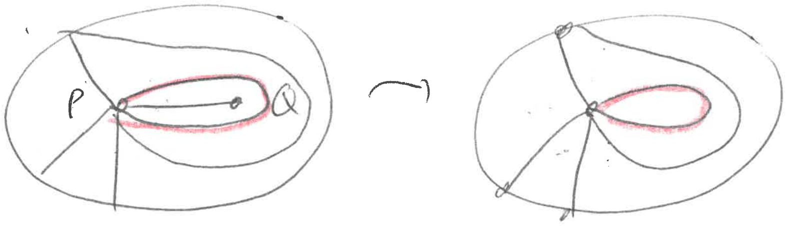
case III :



∃ loop & arc ~~at~~ P—Q
at P

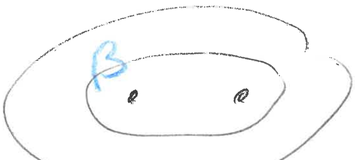
⤳ make loop into inner boundary

④

Computing growth coeff. in case I : either compute

of submodules for $M(\gamma)$ − # of submodules for $M(\gamma_0)$ using E Kantarci-Oguz's rank matrices

or : (by GMV) compute # of submodules of closed loop β } using $\overbrace{\text{—''—}}$ (and Kantarci-Oguz-Yildirim)



# of submodules is ~~entry in pos. (1,1)~~ the trace of ~~of~~ $\nabla(M(\tilde\beta)) = \nabla(U_{p-1})\nabla(\gamma)\nabla(U_{q-1})\nabla(\gamma^{-1})$ ($\tilde\beta$: open up β)

For the frieze of T_1:

Proposition : $s(Q_1) = a^2pq - 2$

→ a = # of submod's of $M(\gamma)$ $p \overset{\gamma}{—} c$

p = # of arcs of T incident with P, locally

q = # of arcs of T incident with Q, locally

Cases II, III : compute quidd. sequence

of inner boundary directly :

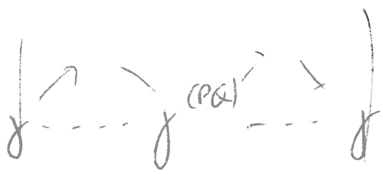
case II : $Q_1 = (p,q)$ ⤳ $s(Q_1) = pq - 2$ (and $a=1$) ✓

case III $Q_1 = (p-2)$ ⤳ $s(Q_1) = p-2$ (and $a=1, q=1$) ✓

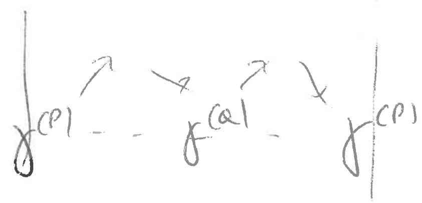
5 Components T_2, T_3

Work in the cluster algebra of the triangulated twice punctured disk, with tagged arcs (FST 08)

T_2 :



T_3



Lemma: $\mathcal{G}(T_2) = (a, a\cdot p\cdot q)$ $\mathcal{G}(T_3) = (ap, aq)$

(use Musiker-Schiffler-Williams 11, Prop 5.3)

Corollary: $s(\mathcal{G}_2) = a^2pq - 2 = s(\mathcal{G}_3)$

(Friezes:

$$\begin{array}{ccccccc} 1 & & 1 & & 1 & & 1 \\ & a & & apq & & a & \\ & & a^2pq-1 & & a^2pq-1 & & \\ & & & \vdots & & & \end{array} \qquad \begin{array}{ccccccc} 1 & & 1 & & 1 & & 1 \\ & ap & & aq & & ap & \\ & & a^2pq-1 & & a^2pq-1 & & \\ & & & \vdots & & & \end{array}$$