Cutting Sequences, Hecke Congruence Subgroups and the *p*-adic Littlewood Conjecture

John Blackman

Continued Fractions and SL_2-Tilings

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johnblackman.maths@gmail.com
https://johnblackmanmaths.wordpress.com/

Table of Contents

The p-adic Littlewood Conjecture (pLC)

Diophantine Approximation

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Alternatively, we can write $\nu(\alpha)$ as:

$$\nu(\alpha) := \liminf_{q \in \mathbb{N}} \{q \cdot ||q\alpha||\}$$

where $|| \cdot ||$ is the distance to the nearest integer function.

Diophantine Approximation

If $\nu(\alpha) = 0$, then we say that α is *well approximable*. Otherwise, we say that α is *badly approximable* and we denote the set of all badly approximable numbers as **Bad**, i.e.:

$$\mathbf{Bad} := \{\alpha \in \mathbb{R} : \nu(\alpha) > 0\}.$$

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Theorem (Hurwitz)

For all $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, *we have* $0 \leq c(\alpha) \leq \nu(\alpha) \leq \frac{1}{\sqrt{5}}$.

Continued Fractions

Definition

A (simple) *continued fraction* $\overline{\alpha}$ is an expression of the form:

$$\overline{\alpha} := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\dots}}},$$

where $a_0 \in \mathbb{Z}$ and $a_i \in \mathbb{N}$ for $i \geq 1$.

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To simplify notation, we write $\overline{\alpha} := [a_0; a_1, a_2, \dots]$. We refer to the a_i's as the *partial quotients*.

Convergents of Continued Fractions

Definition

Let $\overline{\alpha} = [a_0; a_1, a_2, \dots]$ be a continued fraction. We define the *k-th convergent* of $\overline{\alpha}$ to be $\frac{p_k}{q_k} := [a_0; a_1, \dots, a_k]$.

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We can define these terms iteratively:

$$p_{-1} = 1 \qquad p_0 = a_0 \qquad p_k = a_k p_{k-1} + p_{k-2}$$
$$q_{-1} = 0 \qquad q_0 = 1 \qquad q_k = a_k q_{k-1} + q_{k-2}$$

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Convergents provide us with very good rational approximations of real numbers. In fact, the convergents give the best possible rational approximations of α (in terms of the Markov constant and $c(\alpha)$).

Semi-Convergents of Continued Fractions

Definition

Let $\overline{\alpha} = [a_0; a_1, a_2, \ldots]$ be a continued fraction expansion of some real number α. We define the $\{k, m\}$-*th semi-convergent* of $\overline{\alpha}$ to be $\frac{p_{\{k,m\}}}{q_{\{k,m\}}} := [a_0; a_1, \ldots, a_k, m]$, where $0 \leq m \leq a_{k+1}$.

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Semi-convergents also provide us with good rational approximations of real numbers (but not as good as standard convergents).

Back to Rational Approximation

Since the convergents of a real number give the best possible rational approximations, for any $\alpha \in \mathbb{R}$ the value of $c(\alpha)$ is minimised by the sequence of convergent denominators $\{q_k\}_{k\in\mathbb{N}}$, i.e.:

$$c(\alpha) := \inf_{q\in\mathbb{N}}\{q \cdot \|q\alpha\|\} = \inf_{k\in\mathbb{N}}\{q_k \cdot \|q_k\alpha\|\}.$$

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If we let $B(\alpha) = \sup_{k\in\mathbb{N}}\{a_k : \overline{\alpha} = [a_0; a_1, \ldots]\}$, then (for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$) we can bound $c(\alpha)$ above and below as follows:

$$\frac{1}{B(\alpha)+2} < c(\alpha) < \frac{1}{B(\alpha)}.$$

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As a result, we can redefine the badly approximable numbers to be:

$$\textbf{Bad} := \{\alpha \in \mathbb{R} \setminus \mathbb{Q} : B(\alpha) < \infty\}.$$

The p-adic Littlewood Conjecture (pLC)

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The p-adic Littlewood Conjecture (de Mathan and Teulié 2004)

For every real number $\alpha \in \mathbb{R}$, we have:

$$m_p(\alpha) := \inf_{q \in \mathbb{N}} \left\{ q \cdot |q|_p \cdot ||q\alpha|| \right\} = 0,$$

where $|\cdot|_p$ is the p-adic norm and $||\cdot||$ is the distance to the nearest integer function.

Let $v(q) := sup\{j \in \mathbb{N} \cup \{0\} : p^j \mid q\}$, then $|q|_p = p^{-v(q)}$.

The p-adic Littlewood Conjecture (pLC)

If $\alpha \in \mathbf{Bad}$:

$$m_p(\alpha) = 0 \iff \inf_{k \in \mathbb{N} \cup \{0\}} \{c(p^k \alpha)\} = 0$$
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Sketch:

$$q \cdot |q|_p \cdot ||q\alpha|| = p^{v(q)} q' \cdot p^{-v(q)} \cdot ||q'(p^v(q)\alpha)||$$
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The set of counterexamples to pLC are (sometimes) referred to as the *multiplicatively badly approximable numbers*:

$$\mathbf{Mad}(p) := \{\alpha \in \mathbb{R} : m_p(\alpha) = 0\}.$$

Continued Fraction Arithmetic

- Hall 1947: Given a continued fraction x, described a process for computing:

$$z(x) = \frac{ax + b}{cx + d}$$

with $a, b, c, d \in \mathbb{Z}$ and $ad - bc \neq 0$.

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- Gosper 1972: HAKMEM Given continued fraction expansions for x and y, described a process for computing:

$$w(x, y) = \frac{axy + bx + cy + d}{exy + fx + gy + h}$$

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Theorem (Badziahin *et al.* 2015)

If α *has an eventually recurrent continued fraction expansion, then* α *satisfies pLC. Additionally, the complexity function of the continued fraction expansion a counterexample must grow sub-exponentially.*

Cutting Sequences

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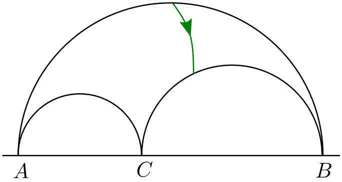
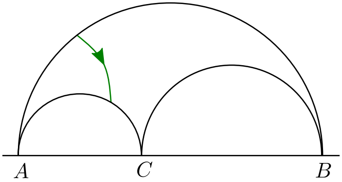
The *cutting sequence* (ζ, T) is then the potentially infinite word over the alphabet $\{L, R\}$ that tracks how ζ intersects each triangle in T.

Examples of left and right triangles

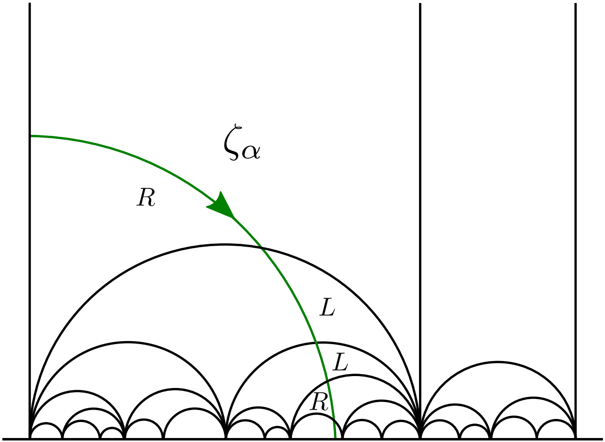


An example of a left triangle.



An example of a right triangle.

An Example of a Cutting Sequence



An example of a geodesic ray ζ_α intersecting a (truncated) triangulation T to form a cutting sequence. The cutting sequence starts $RLLR\cdots$.

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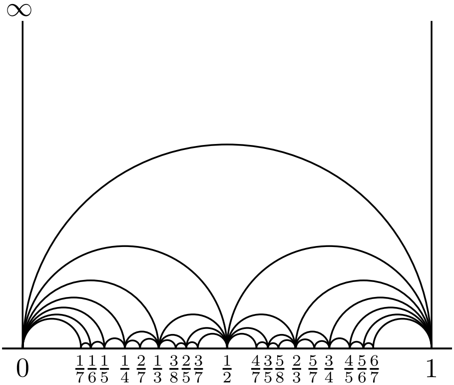

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- The Farey tessellation is preserved by the group $PSL_2(\mathbb{Z})$.
- The edges of $\mathcal{F}$ are covered by the space $PSL_2(\mathbb{Z}) \cdot I$, where I is the line between 0 and ∞.
- **Theorem**[Series 1985] If ζ_α is a geodesic ray starting at I and terminating at a point $\alpha \in \mathbb{R}$, then the cutting sequence $(\zeta_\alpha, \mathcal{F})$ is equal to the continued fraction expansion $\overline{\alpha}$.

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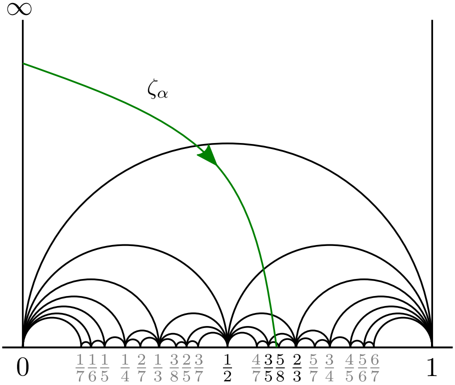
An example of a cutting sequence with the Farey tessellation



A truncated image of a geodesic ray ζ_α with endpoint $\alpha = \frac{\sqrt{5}-1}{2}$ intersecting the Farey tessellation $\mathcal{F}$ with convergents shown in bold. The cutting sequence is $RLRL\cdots$ and the corresponding continued fraction expansion is $[0; 1, 1, 1, \ldots]$

Integer Multiplication of Continued Fractions and Triangulation Replacement

Triangulation Replacement and Integer Multiplication

Idea: We can understand how multiplication affects continued fraction expansions by understanding how certain triangulation replacements affect cutting sequences.

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Therefore, replacing $\mathcal{F}$ with $\frac{1}{n}\mathcal{F}$ represents multiplication by n of continued fractions. $\overline{n} : (\cdot, \mathcal{F}) \to \left(\cdot, \frac{1}{n}\mathcal{F}\right)$.

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$$\begin{aligned} \Gamma_0(n) :&= \left\{ \left(\begin{smallmatrix} a & b \\ cn & d \end{smallmatrix} \right) \in PSL_2(\mathbb{Z}) \right\} \\ &= PSL_2(\mathbb{Z}) \cap \{(n^*)^{-1} \circ A \circ n^* : A \in PSL_2(\mathbb{Z})\} \end{aligned}$$

preserves both $\mathcal{F}$ and $\frac{1}{n}\mathcal{F}$.

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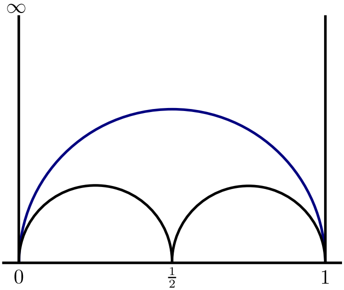
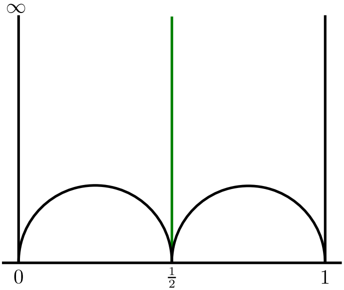
How these subpaths intersect $T_{\{1,n\}}$ and $T_{\{n,n\}}$ determines how ζ_α intersects $\mathcal{F}$ and $\frac{1}{n}\mathcal{F}$ and, therefore, how multiplication by n affects the underlying continued fraction $\overline{\alpha}$.

Example of Triangulation Replacement



An example of $T_{\{1,4\}}$.



An example of $T_{\{2,4\}}$.

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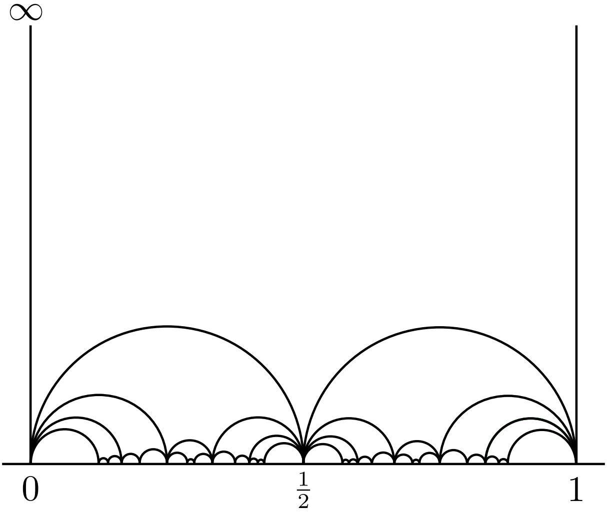

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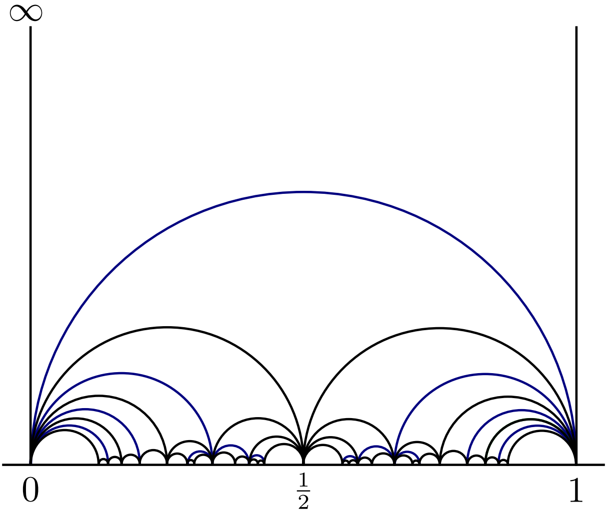

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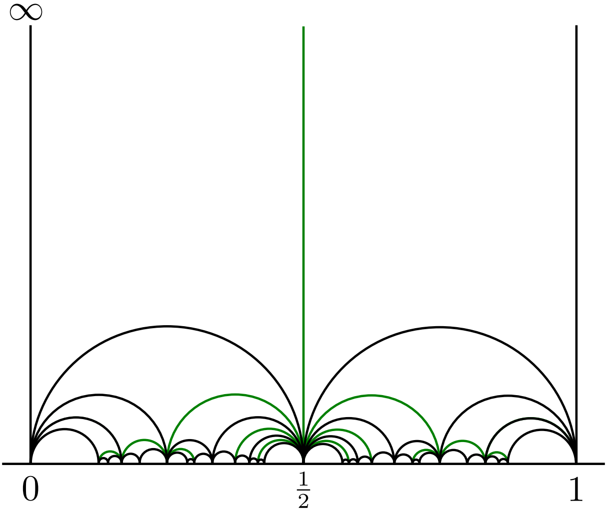

$\mathbb{H}$ tessellated by $T_{\{2,4\}}$. This is equivalent to $\frac{1}{2}\mathcal{F}$.

Cutting Sequences on Orbifolds

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Theorem

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Corollary

Let $\alpha \in \mathbb{R}$, let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be a non-trivial integer matrix (i.e. $a, b, c, d \in \mathbb{Z}$, $ad - bc \neq 0$), and let $\beta = M \cdot \alpha = \frac{a\alpha+b}{c\alpha+d}$. If the continued fraction expansion $\overline{\alpha}$ is eventually recurrent and $c\alpha + d \neq 0$, then the continued fraction $\overline{\beta}$ is eventually recurrent.

Cutting Sequences and the p-adic Littlewood Conjecture

Since replacing $\mathcal{F}$ with $\frac{1}{n}\mathcal{F}$ induces multiplication by n, it will be useful to look at the "common" structure of $\mathcal{F}$ and $\frac{1}{n}\mathcal{F}$.

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Lemma

Two points A and B are neighbours in both $\mathcal{F}$ and $\frac{1}{n}\mathcal{F}$ if and only if they have reduced form $\frac{a}{n_1 c_1}$ and $\frac{b}{n_2 d_1}$, with $n = n_1 n_2$ and $|a n_2 d_1 - b n_1 c_1| = 1$.

Splitting Cutting Sequences

If a geodesic ray ζ_α intersects an edge E in $\mathcal{F} \cap \frac{1}{n}\mathcal{F}$, then we can split ζ_α along E to form a geodesic segment $\zeta_{\alpha,1}$ and a geodesic ray $\zeta_{\alpha,2}$.

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If ζ_α doesn't intersect $\mathcal{F} \cap \frac{1}{n}\mathcal{F}$, no canonical map exists.

$\Gamma_0(n) \cdot I$

Since I is an edge of both $\mathcal{F}$ and $\frac{1}{n}\mathcal{F}$ and $\Gamma_0(n)$ preserves both $\mathcal{F}$ and $\frac{1}{n}\mathcal{F}$, it follows that $\Gamma_0(n) \cdot I \subseteq \mathcal{F} \cap \frac{1}{n}\mathcal{F}$.

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Proposition (B. 2023)

If ζ_α intersects $\Gamma_0(n) \cdot I$, then there is a tail $\overline{\beta}$ of $\overline{\alpha}$ such that $\overline{n\beta}$ is a tail of $\overline{n\alpha}$.

Infinite Loops mod n

Definition

Let ζ_α be a geodesic ray starting at the y-axis I and terminating at the point $\alpha \in \mathbb{R}_{>0}$. Then, ζ_α is an *infinite loop mod n*, if ζ_α is disjoint from $\Gamma_0(n) \cdot I$ except for the edges of the form $I + k$, for $k \in \mathbb{Z}_{\geq 0}$.

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Proposition (B. 2023)

If $n \in \mathbb{N}$ and $n \geq 4$, then there exist infinite loops mod n.

Infinite loops and pLC

If a real number is not an infinite loop mod n then the height function $B(\cdot)$ can not be small for both α and $n\alpha$.

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Assume that α is not an infinite loop mod n. Then we have:

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Corollary

*Let $\alpha \in$ **Bad** and assume there is some sequence of natural numbers $\{\ell_m\}_{m\in\mathbb{N}}$ such that $p^{\ell_m}\alpha$ is not an infinite loop mod p^m. Then α satisfies pLC.*

Cutting Sequences and pLC

On the other hand, if for some real number α and some natural number m, every value of $p^{\ell}\alpha$ is an infinite loop mod p^{m}, then $p^{\ell}\alpha$ is a counter-example to pLC.

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Lemma

*Let $\alpha \in$ **Bad** and assume there exists an $m \in \mathbb{N}$ such that $p^\ell\alpha$ is an infinite loop mod p^m for all $\ell \in \mathbb{N} \cup \{0\}$. Then α is a counterexample to pLC and $m_p(\alpha) \geq \frac{1}{p^m}$.*

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Combining these statements together, we get the main theorem of the talk: the following reformulation of pLC in terms of infinite loops mod n.

Theorem (B. 2023)

*Let $\alpha \in$ **Bad**. Then, α satisfies pLC if and only if there is a sequence of natural numbers $\{\ell_m\}_{m\in\mathbb{N}}$ such that $p^{\ell_m}\alpha$ is not an infinite loop mod p^m.*

Thank you for listening.
Any questions?