Arithmetic and Geometry of Markov Polynomials

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joint work with A.P. Veselov and B. Winn

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Markov Diophantine equation

$$X^2 + Y^2 + Z^2 = 3XYZ, \quad X, Y, Z \in \mathbb{Z}_+.$$

Markov 1880: Every solution can be found from $(1, 1, 1)$ by applying Vieta involution

$$(X, Y, Z) \to \left(X, Y, \frac{X^2 + Y^2}{Z}\right)$$

and permutations.

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Generalised Markov equation (**Propp et al. 2003**)

$$X^2 + Y^2 + Z^2 = k(x, y, z)XYZ, \qquad k(x, y, z) = \frac{x^2 + y^2 + z^2}{xyz}$$

Using the same procedure applied to $(X = x, Y = y, Z = z)$, we get the solutions, which are Laurent polynomials of the parameters x, y, z. Indeed, we can use the alternative Vieta involution

$$(X, Y, Z) \to (X, Y, k(x, y, z)XY - Z).$$

These Laurent polynomials are called **Markov polynomials**.

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$$\rho = \frac{a}{b} \to m_\rho \quad \text{(Markov number)}$$

Figure: Positive rationals and Markov numbers on the Conway topograph.

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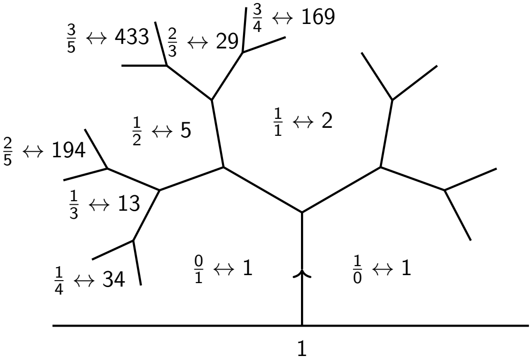

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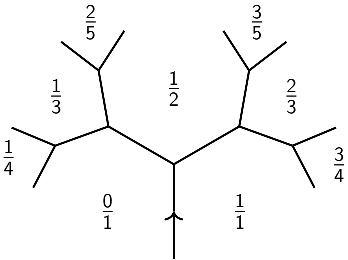
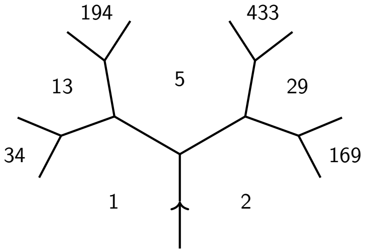
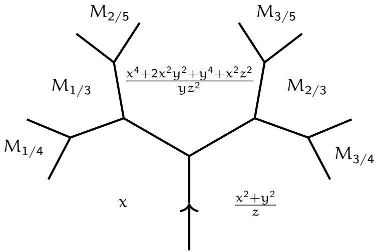
Conway Topograph and Frobenius Correspondence







$$M_\rho(x,y,z) = \frac{P_\rho(x,y,z)}{Q_\rho(x,y,z)}$$

Theorem 1 (EVW 2024)

The denominator of a Markov polynomial corresponding to the rational $\rho = \frac{a}{b}$ *is* $Q_\rho(x,y,z) = x^{(a-1)}y^{(b-1)}z^{(a+b-1)}$.

By homogeneity we have

$$P_\rho(x,y,z) = \sum A_{ij} x^{2i} y^{2j} z^{2(a+b-1-i-j)}.$$

Propp 2005: Markov polynomials have positive coefficients.

We define the Newton polygon Δ_ρ as follows

$$\Delta_\rho = \Delta(M_\rho) := Conv\{(i,j) : A_{ij} \neq 0\} \subset \mathbb{Z}^2.$$

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Newton Polygon Example

Figure: Newton polygon Δ_ρ.

Example:

$$\rho = \frac{2}{3}, \;\; m_\rho = 29.$$

$$\begin{aligned} P_\rho(x, y, 1) = \\ x^8 + 4x^6y^2 + 6x^4y^4 + 4x^2y^6 \\ + y^8 + 2x^6 + 5x^4y^2 \\ + 4x^2y^4 + y^6 + x^4 \end{aligned}$$

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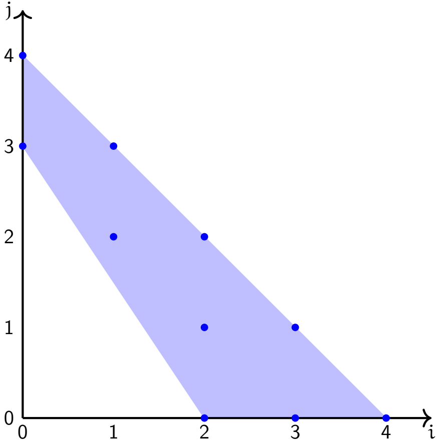

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Theorem 2 (EVW 2024)

Given a rational $\rho = \frac{a}{b}$, Δ_ρ *is the area (on the* ij*-plane with* $i, j \geqslant 0$*) satisfying the conditions*

$$\Delta_\rho = \begin{cases} \dfrac{i}{a} + \dfrac{j}{b} \geqslant 1 \\ i + j \leqslant a + b - 1 \end{cases}$$

Conjecture 3 (Saturation Conjecture, EVW 2024)

Terms that appear in the numerator of a Markov polynomial M_ρ *are precisely those corresponding to the set of integer lattice points on* Δ_ρ.

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Weighted Newton Polygon

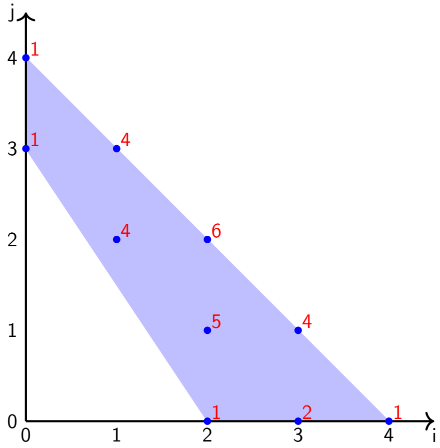

Figure: 'Weighted' Newton polygon Δ_ρ, $\rho = \frac{2}{3}$.

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We define the **Markov function** on the Newton polygon

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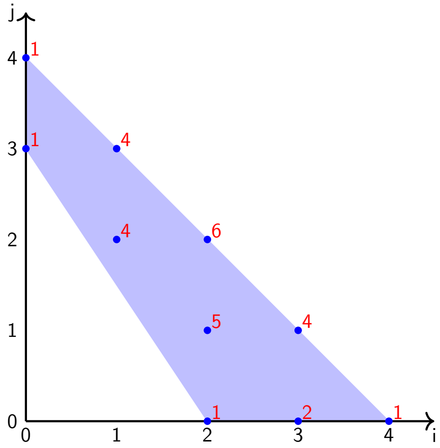

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Coefficients on Newton Polygon Boundary

Theorem 4 (EVW 2024)

Given a rational $\frac{a}{b}$ the coefficients on the boundary of the corresponding Markov polynomial's Newton polygon are binomial coefficients. In particular,

Line	*Coefficients*
$j = 0$	B_{a-1}
$i = 0$	B_{b-1}
$i + j = a + b - 1$	B_{a+b-1}

where B_k denote the kth row of Pascal's triangle.

Coefficients on Newton Polygon Diagonals

Coefficients of second upper-most diagonal of Δ_ρ

$$[2, 5, 4, 1] = [1, 3, 3, 1] + [1, 2, 1, 0]$$

Theorem 5 (EVW 2024)

Coefficients on the 2nd upper-most diagonal of the Newton polygon of Markov polynomials (the line $i + j = a + b - 2$) are

$$(a-1)B_{a+b-2} + (b-a)B_{a+b-3}.$$

Theorem 6 (EVW 2024)

Coefficients on the 3rd upper-most diagonal of the Newton polygon of Markov polynomials (the line $i + j = a + b - 3$) are:

$$\frac{(a-1)(a-2)}{2}B_{a+b-3} + [a(b-a) - a]B_{a+b-4} + \frac{1}{2}[(b-a)^2 + 5a - 3b]B_{a+b-5}.$$

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Coefficients on Newton Polygon Horizontals

Theorem 7 (EVW 2024)

Coefficients on the 2nd lower-most horizontal of the Newton polygon of Markov polynomials (the line $j = 1$) are

$$(3a - 1)B_{b-2} + (b - 2a)B_{b-3}.$$

Coefficients on Critical Triangle

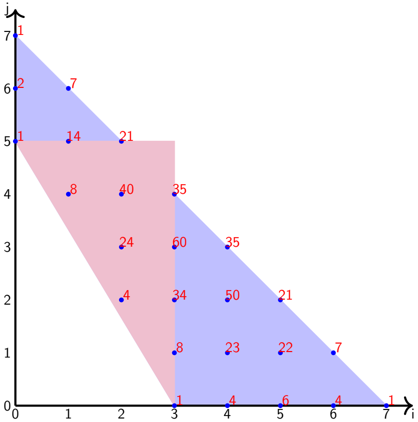

Figure: 'Weighted' Newton polygon Δ_ρ, $\rho = \frac{3}{5}(m_\rho = 433)$.

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The coefficients of the Markov polynomial M_ρ, $\rho = \frac{a}{b}$ lying inside the critical triangle of the Newton polygon are all multiples of 4.

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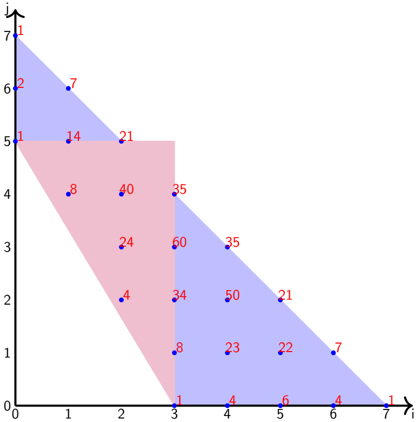

Figure: 'Weighted' Newton polygon $\Delta_\rho, \rho = \frac{3}{5} (m_\rho = 433)$.

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The coefficients of the Markov polynomial $M_\rho, \rho = \frac{a}{b}$ lying inside the critical triangle of the Newton polygon are all multiples of 4.

Markov polynomials M_ρ, with $\rho = \frac{1}{n+1}$, are a specialisation of the Fibonacci polynomials previously studied by **Caldero, Zelevinsky (2006)**.

Corollary 9

The Markov polynomials M_ρ, $\rho = \frac{1}{n+1}$ *have coefficients*

$$A_{ij} = \binom{n-j}{n+1-i-j}\binom{i+j}{j}.$$

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Log-Concavity of Coefficients

A sequence $x = (x_0, x_1, \ldots, x_n)$ is said to be **log-concave** if it satisfies the property

$$x_k^2 \geqslant x_{k-1}x_{k+1},$$

for $k \in \{1, 2, \ldots, n-1\}$.

Theorem 10 (EVW 2024)

The sequence of coefficients that appear on the 2nd upper diagonal of the Newton polygon associated to a Markov polynomial is (strictly) log-concave.

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We say that a weighted lattice is **weakly log-concave** if the sequence of weights in all principal directions are log-concave.

Conjecture 11 (EVW 2024)

Coefficients of Markov polynomials are weakly log-concave.

Theorem 12 (EVW 2024)

The above holds in the case $\rho = \frac{1}{n+1}$.

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Klein Diagram for Continued Fractions

Consider $\rho = \frac{5}{3} = [1, 1, 2]$. Table of convergents:

			1	1	2
p_k	0	1	1	2	5
q_k	1	0	1	1	3

We have sails $A_0A_1A_2\ldots$ and $B_0B_1B_2\ldots$

$$A_i = (q_{2i-1}, p_{2i-1}),$$
$$B_i = (q_{2i}, p_{2i}).$$

In our example,

$A_0 = (1, 0), A_1 = (1, 1), A_2 = (5, 3)$
$B_0 = (0, 1), B_1 = (2, 1), [B_2 = (5, 3)]$

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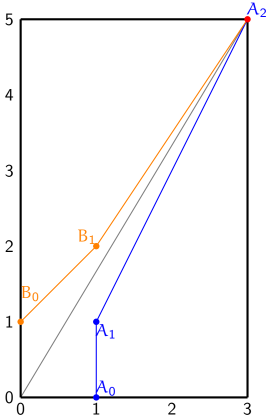

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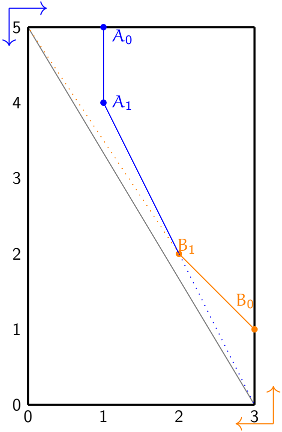
Karpenkov 2014: We have the following Edge-Angle Duality

$$\mathrm{l}\alpha(\angle A_i A_{i+1} A_{i+2}) = \mathrm{l}\ell(B_i B_{i+1}) \qquad (= a_{2i+2}),$$
$$\mathrm{l}\alpha(\angle B_i B_{i+1} B_{i+2}) = \mathrm{l}\ell(A_{i+1} A_{i+2}) \qquad (= a_{2i+3}),$$

Markov Sails



$$A_i := (q_{2i-1}, b - p_{2i-1}), \qquad B_i := (a - q_{2i}, p_{2i}),$$

Coefficients on the Markov Sail

Conjecture 13 (EVW 2024)

Coefficients on the edge $C_i C_{i+1}$ *of the Markov sail are arithmetic progressions with differences* $d(C_i C_{i+1})$ *satisfying*

$$d(B_i B_{i+1}) = -\mathcal{M}(A_{i+1}), \qquad d(A_{i+1} A_{i+2}) = -\mathcal{M}(B_{i+1}).$$

Conjecture 14 (EVW 2024)

Consider the continued fraction $\frac{b}{a} = [a_1, a_2, \ldots, a_n]$. *If* $n = 2m + 1$ *(odd) then* $\mathcal{M}(B_m) = 4$. *If* $n = 2m$ *(even) then* $\mathcal{M}(A_m) = 4$.

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$$\frac{b}{a} = \frac{18}{13} = [1, 2, 1, 1, 2]$$

Conjecture 14 $\implies \mathcal{M}(B_2) = 4$.

Now applying Conjecture 13 recursively,

$$\mathcal{M}(A_2) = \mathcal{M}(B_2) + (a_5 - 1)\mathcal{M}(B_2) = 8$$
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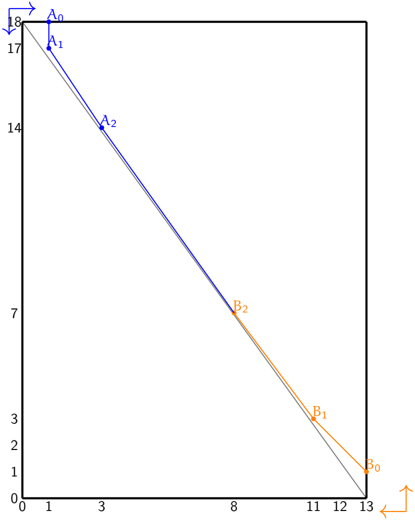

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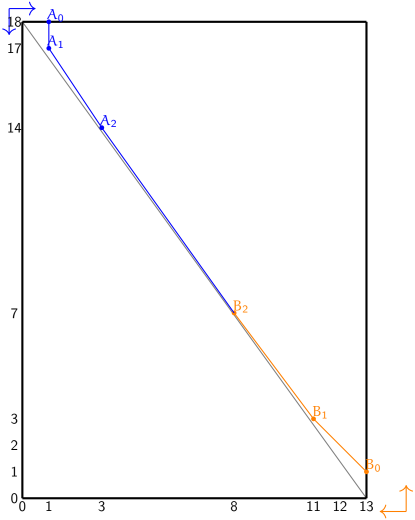

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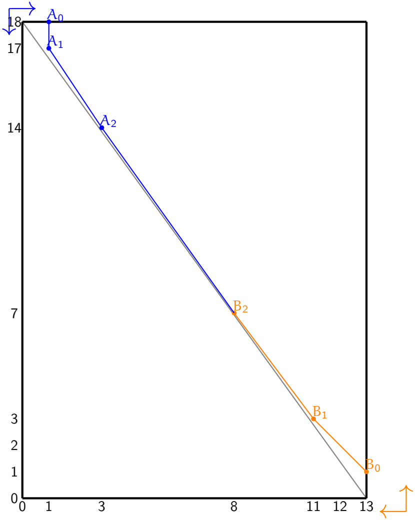

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