Continued Fraction approach to Gauss-Reduction theory

Oleg Karpenkov, University of Liverpool

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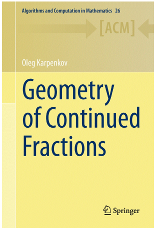
Contents

Contents

I. Gauss Reduction Theory.

II. Geometry of continued fractions.

III. Techniques to compute reduced matrices explicitly.



I. Gauss Reduction Theory

Formulation of a problem

Operators A and B are **conjugate** if there exists X such that

$$B = XAX^{-1}.$$

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Strategy: **find normal forms**.

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Problem

Describe conjugacy classes in $SL(2,\mathbb{Z})$.

Strategy: **find normal forms**.

Example

In the classical case of algebraically closed field any matrix is conjugate to Jordan normal form. The set of Jordan blocks is the complete invariant of a conjugacy class.

Formulation of a problem

Operators A and B are **conjugate** if there exists X such that

$$B = XAX^{-1}.$$

Problem

Describe conjugacy classes in $SL(2, \mathbb{Z})$.

To be more precise, we deal with $PSL(2, \mathbb{Z})$.

Problem

Describe explicitly conjugacy classes in $PSL(2, \mathbb{Z})$.

Here $A \sim -A$.

Current situation of the question

Gauss Reduction theory:
$SL(2,\mathbb{Z}) \to$ complete invariant $\to$ "almost" normal form.

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Gauss Reduction theory:
$SL(2,\mathbb{Z}) \to$ complete invariant $\to$ "almost" normal form.

In this presentation we show how to explicitly describe these "almost" normal forms.

The case of $SL(2,\mathbb{Z})$

- complex case: $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$.
- totally real case: Gauss Reduction Theory
- degenerate case of double roots: $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ for $n \geq 0$.

Continued fractions for 7/5

$$\frac{7}{5} =$$

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Ordinary continued fractions

The expression (finite or infinite)

$$a_0 + 1/(a_1 + 1/(a_2 + \ldots) \ldots))$$

is an *ordinary continued fraction* if $a_0 \in \mathbb{Z}$, $a_k \in \mathbb{Z}_+$ for $k > 0$.
Denote it $[a_0 : a_1; \ldots]$ (or $[a_0 : a_1; \ldots; a_n]$).

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Ordinary continued fraction is *odd* (*even*) if it has odd (even) number of elements.

$$\frac{7}{5} = 1 + \frac{1}{2 + \frac{1}{2}} = 1 + \frac{1}{2 + \frac{1}{1+1/1}}$$

$$\frac{7}{5} = [1 : 2; 2] = [1 : 2; 1; 1]$$

Ordinary continued fractions

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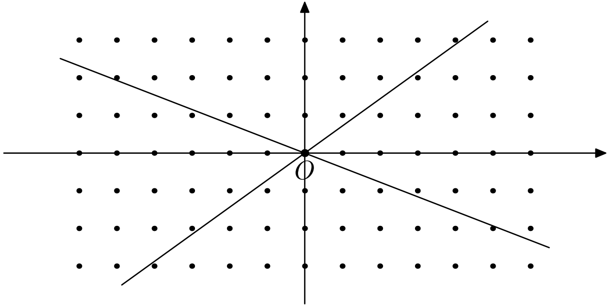
Proposition

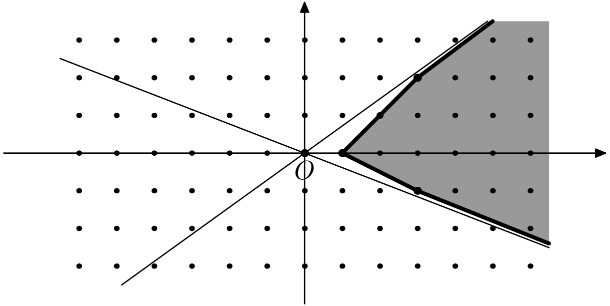
Any rational number has a unique odd and even ordinary continued fractions.
Any irrational number has a unique infinite ordinary continued fraction

The totally real case of $SL(2, \mathbb{Z})$



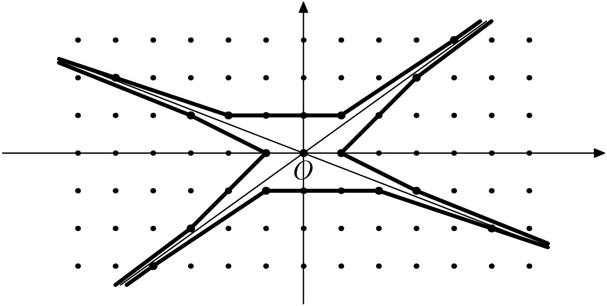
Eigenlines of an operator $\begin{pmatrix} 7 & 18 \\ 5 & 13 \end{pmatrix}$.

The totally real case of $SL(2,\mathbb{Z})$



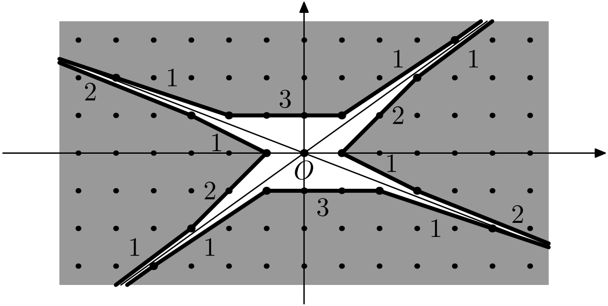
The *sail* for one of the octants, i.e. the boundary of the convex hull of all integer inner points.

The totally real case of $SL(2, \mathbb{Z})$



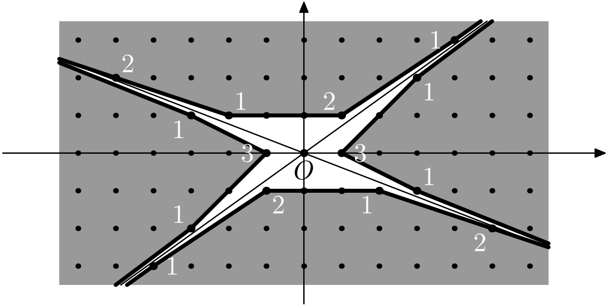
The set of all sails is called *geometric continued fraction* (in the sense of Klein).

The totally real case of $SL(2, \mathbb{Z})$

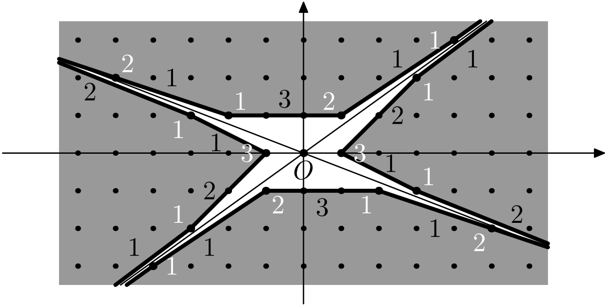


Integer length of a segment is the number of integer inner points in a segment plus one.

The totally real case of $SL(2, \mathbb{Z})$

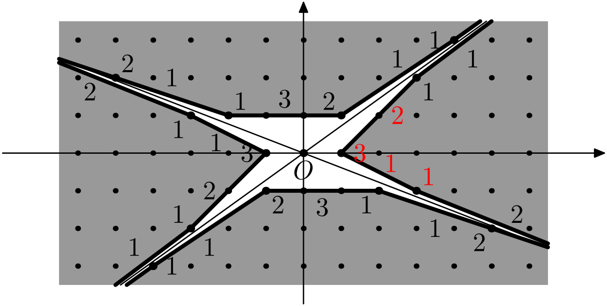


Integer angle is the index of the sublattice generated by points of the edges of the angle in the lattice of integer points.



Geometric continued fraction for the operator $\begin{pmatrix} 7 & 18 \\ 5 & 13 \end{pmatrix}$.

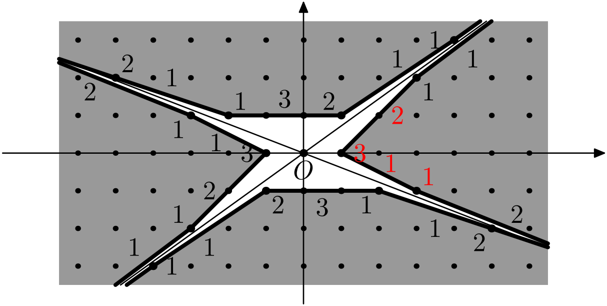
The totally real case of $SL(2,\mathbb{Z})$



In the case of $SL(2,\mathbb{Z})$ operators the sequences for the sails are periodic.

For instance, for $\begin{pmatrix} 7 & 18 \\ 5 & 13 \end{pmatrix}$ the period is: $(1,1,3,2)$.

The totally real case of $SL(2, \mathbb{Z})$



Theorem

A period (up to a shift) is a complete invariant of a conjugacy class of an operator in $SL(2, \mathbb{Z})$.

Definition

An operator $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is *reduced* if $d > b \geq a \geq 0$.

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The number of reduced matrices in a conjugacy class with minimal period $(a_1, \ldots, a_k)$ *is* k.

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If $A \in SL(2, \mathbb{Z})$*: take* **even** *continued fraction for* $\frac{d}{c}$*;*
If $A \in GL(2, \mathbb{Z}) \setminus SL(2, \mathbb{Z})$*: take* **odd** *continued fraction for* $\frac{d}{c}$*.*

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Let

$$d/c = [a_1; \ldots : a_n].$$

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Let

$$d/c = [a_1; \ldots : a_n].$$

Then *one of the periods of geometric continued fraction is*

$$(a_1, a_2, \ldots, a_n).$$

Example

For the operator $\begin{pmatrix} 1519 & 1164 \\ -1964 & -1505 \end{pmatrix}$ the period is $(1,2,1,2)$.

Hence minimal period is $(1,2)$.

The reduced operators conjugate to the given one are: $\begin{pmatrix} 3 & 8 \\ 4 & 11 \end{pmatrix}$ and $\begin{pmatrix} 3 & 4 \\ 8 & 11 \end{pmatrix}$.

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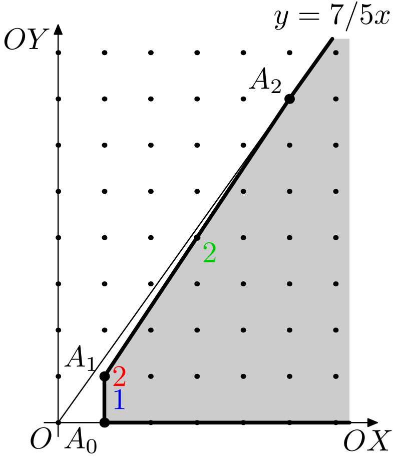
Question: How to compute a period for $GL(2,\mathbb{Z})$ matrics?

II. Geometry of continued fractions.

Geometry of continued fractions



$a_0 = \text{l}\ell(A_0A_1) = 1;$

$a_1 = \text{lsin}(A_0A_1A_2) = 2;$

$a_2 = \text{l}\ell(A_1A_2) = 2.$

$$7/5 = [1; 2 : 2].$$

$(a_0, \ldots, a_{2n})$ — lattice length-sine sequence (LLS-sequence).

Integer geometry

Integer geometry

Objects: Integer segments, integer angles, integer polygons.

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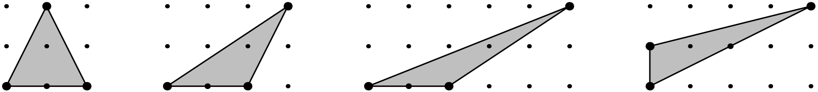
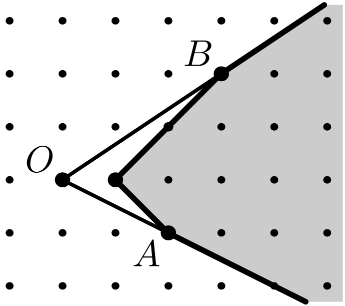
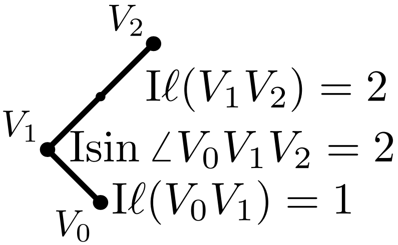
Transformations: Integer lattice preserving affine transformations in the plane.

$$(Aff(2,\mathbb{Z}) = GL(2,\mathbb{Z}) \rtimes \mathbb{Z}^2).$$

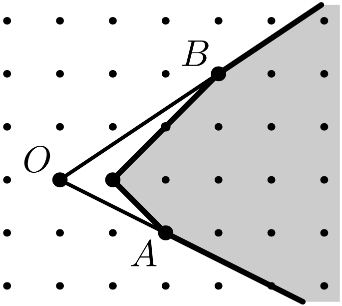


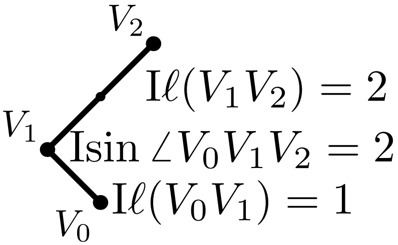




LLS-sequence for an arbitrary angle

Integer trigonometry (O.K. '08)

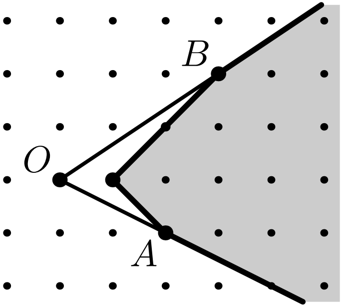


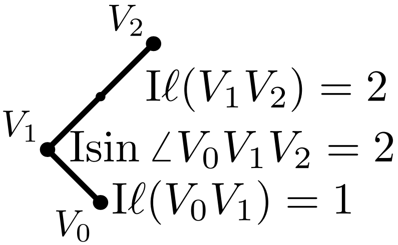


Theorem

LLS-sequence is a complete invariant of integer angles in integer geometry.

Integer trigonometry (O.K. '08)

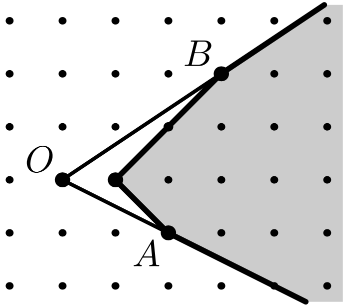


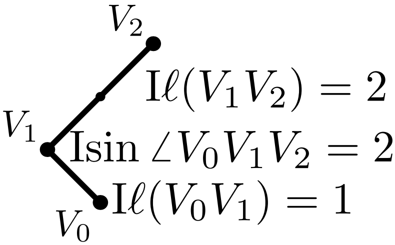


Definition

Let $(a_0, \ldots, a_{2n})$ be the LLS-sequence of α, then $\mathrm{Itan}\,\alpha = [a_0 : \ldots : a_{2n}]$.

Integer trigonometry (O.K. '08)

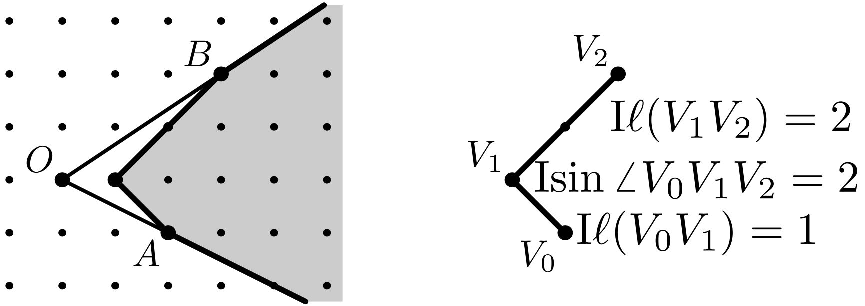




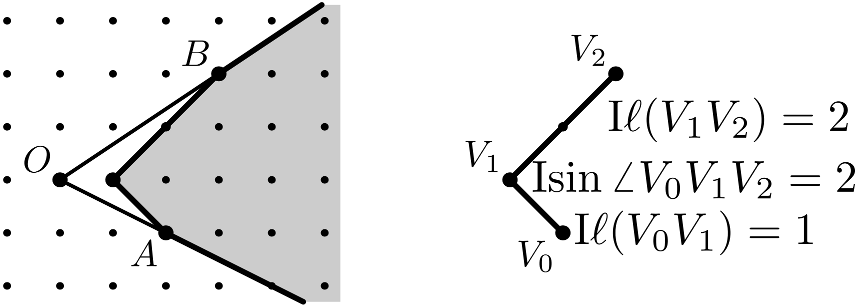
$$\mathrm{Itan}\,AOB = [1:2;2] = \frac{7}{5} \quad \Longrightarrow \quad \begin{cases} \mathrm{Isin}\,AOB = 7 \\ \mathrm{Icos}\,AOB = 5 \end{cases}$$

III. Techniques to compute reduced matrices explicitly.

How to compute the LLS of a rational angle (O.K. '21)



LLS-sequence for an arbitrary angle



LLS-sequence for an arbitrary angle

How to compute the LLS of a rational angle?

How to compute the LLS of a rational angle (O.K. '21)

Theorem

Given integer $A = (p, q)$ *and* $B = (r, s)$ *with non-zero entry.*

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Given integer $A = (p, q)$ *and* $B = (r, s)$ *with non-zero entry. W.l.o.g.* $\det(OA, OB) < 0$; $p, q, r, s > 0$ *(other cases similar).*

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W.l.o.g. $\det(OA, OB) < 0$*;* $p, q, r, s > 0$ *(other cases similar).*
Let

$$|q/p| = [a_0; a_1 : \ldots : a_{2m}],$$
$$|s/r| = [b_0; b_1 : \ldots : b_{2n}].$$

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Let

$$|q/p| = [a_0; a_1 : \ldots : a_{2m}],$$
$$|s/r| = [b_0; b_1 : \ldots : b_{2n}].$$

Denote also

$$\alpha = \Big|[-a_{2m} : -a_{2m-1} : \cdots : -a_1 : -a_0 : 0 : b_0 : b_1 : \cdots : b_{2n}]\Big|$$

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$$= [c_0; c_1 : \cdots : c_{2k}] \quad \textit{(odd regular c.f.)}$$

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$$= [c_0; c_1 : \cdots : c_{2k}] \quad (\textit{odd regular c.f.})$$

Set

- $S = (c_0, c_1, \ldots, c_{2k})$ *in the case* $c_0 \neq 0$*;*
- $S = (c_2, \ldots, c_{2k})$ *in the case* $c_0 = 0$*.*

How to compute the LLS of a rational angle (O.K. '21)

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Given integer $A=(p,q)$ *and* $B=(r,s)$ *with non-zero entry.*
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- $S=(c_0,c_1,\ldots,c_{2k})$ *in the case* $c_0 \neq 0$*;*
- $S=(c_2,\ldots,c_{2k})$ *in the case* $c_0 = 0$*.*

Then S is the LLS sequence for the angle $\angle AOB$.

How to compute the LLS of a rational angle (O.K. '21)

Example

Given $A = (12, 5)$ and $B = (7, 16)$.

How to compute the LLS of a rational angle (O.K. '21)

Example

Given $A = (12, 5)$ and $B = (7, 16)$.
Then

$$|5/12| = [0; 2 : 2 : 1 : 1],$$
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$$|5/12| = [0; 2 : 2 : 1 : 1],$$
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Denote also

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$$= [1; 1 : 2 : 1 : 1 : 1 : 3 : 2]$$

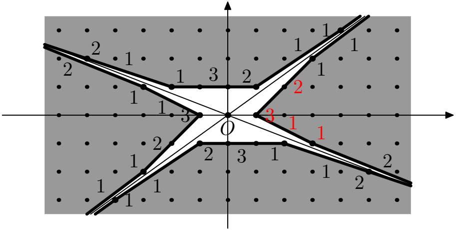
Then $LLS(\angle AOB) = (1; 1 : 2 : 1 : 1 : 1 : 3 : 2)$ **.**

How to compute the LLS of a algebraic angle (O.K. '21)

Definition

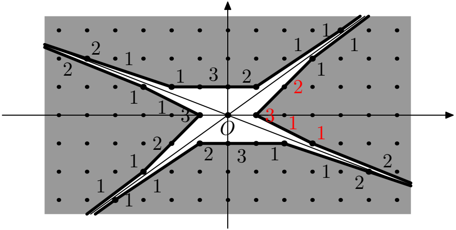
$$(s_1, s_2, s_3) - (s_1, s_3) = (s_2),$$

e.g., $(1, 2, 3, 4, 5, 6, 7, 8) - (1, 2, 3, 6, 7, 8) = (4, 5)$.



Proposition

Let $M \in GL(2, \mathbb{Z})$ matrix M with distinct irrational eigenvalues. Let also P_0 be any non-zero integer point.



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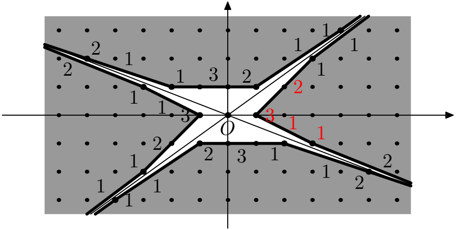
Denote $P_1 = M^4(P_0)$ and $P_2 = M^6(P_0)$.

How to compute the LLS of a algebraic angle (O.K. '21)



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Denote $P_1 = M^4(P_0)$ and $P_2 = M^6(P_0)$.

Then there exists a difference $LLS(\angle P_0OP_2) - LLS(\angle P_0OP_1)$, which is a period of the LLS sequence for M repeated twice.

Continuants

Definition

Let n be a positive integer. A *continuant* K_n is a polynomial defined recursively by

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$$
\begin{aligned}
&K_{-1}() = 0;\\
&K_0() = 1;\\
&K_1(a_1) = a_1;\\
&K_n(a_1, a_2, \ldots, a_n) = a_n K_{n-1}(a_1, a_2, \ldots, a_{n-1}) + K_{n-2}(a_1, a_2, \ldots, a_{n-2}).
\end{aligned}
$$

Continuants

Definition

Let n be a positive integer. A *continuant* K_n is a polynomial defined recursively by

$$K_{-1}() = 0;$$
$$K_0() = 1;$$
$$K_1(a_1) = a_1;$$
$$K_n(a_1, a_2, \ldots, a_n) = a_n K_{n-1}(a_1, a_2, \ldots, a_{n-1}) + K_{n-2}(a_1, a_2, \ldots, a_{n-2}).$$

Remark

$$[a_0; a_1 : \cdots : a_n] = \frac{K_{n+1}(a_0, a_1, \ldots, a_n)}{K_n(a_1, a_2, \ldots, a_n)}.$$

LLS to reduced operators (O.K. '21)

Claim. Let M be a $GL(2,\mathbb{Z})$ matrix with

$$LLS(M) = (a_1, \ldots, a_n).$$

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Let also m be the minimal length of the period of the LLS sequence.

Then the list of all reduced matrices $PGL(2,\mathbb{Z})$-conjugate to M consists of the following m matrices:

$$\begin{pmatrix} K_{n-2}(a_{2+k}, \ldots, a_{n-1+k}) & K_{n-1}(a_{2+k}, \ldots, a_{n-1+k}, a_{n+k}) \\ K_{n-1}(a_{1+k}, a_{2+k}, \ldots, a_{n-1+k}) & K_n(a_{1+k}, a_{2+k}, \ldots, a_{n-1+k}, a_{n+k}) \end{pmatrix},$$

for $k = 1, \ldots, m$.

Computing all reduce operators (O.K. '21)

Input: Find all reduced matrices for $M = \begin{pmatrix} 7 & -30 \\ -10 & 43 \end{pmatrix}$.

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$$P_1 = M^4(P_0) = (-2875199, 4119201) \quad \text{and}$$
$$P_2 = M^6(P_0) = (-7182245951, 10289762449).$$

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Using Theorem we have:

$$LLS(\angle P_0OP_1) = (\underline{1,1,2,3,4,1,2,3,4,1,2,3,4,1,2,3}, \overline{3}).$$
$$LLS(\angle P_0OP_2) = (\underline{1,1,2,3,4,1,2,3,4,1,2,3,4,1,2,3}, \boxed{4,\ 1,\ 2,\ 3,\ 4,\ 1,\ 2,\ 3}, \overline{3})$$

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Step 2. By Proposition

$$LLS(M) = \frac{1}{2}LLS(\angle P_0OP_2) - LLS(\angle P_0OP_1) = \frac{1}{2}(4,1,2,3,4,1,2,3)$$
$$= (4,1,2,3).$$

Computing all reduce operators (O.K. '21)

Input: Find all reduced matrices for $M = \begin{pmatrix} 7 & -30 \\ -10 & 43 \end{pmatrix}$.

Step 3. Write down the reduced matrices (using Claim) for

$$(4,1,2,3), \quad (1,2,3,4), \quad (2,3,4,1), \quad \text{and} \quad (3,4,1,2).$$

Output. The list of all reduced matrices $PGL(2,\mathbb{Z})$-conjugate to M:

$$\begin{pmatrix} K_2(1,2) & K_3(1,2,3) \\ K_3(4,1,2) & K_4(4,1,2,3) \end{pmatrix} = \begin{pmatrix} 3 & 10 \\ 14 & 47 \end{pmatrix}, \quad \begin{pmatrix} 7 & 30 \\ 10 & 43 \end{pmatrix},$$

$$\begin{pmatrix} 13 & 16 \\ 30 & 37 \end{pmatrix}, \quad \begin{pmatrix} 5 & 14 \\ 16 & 45 \end{pmatrix}.$$

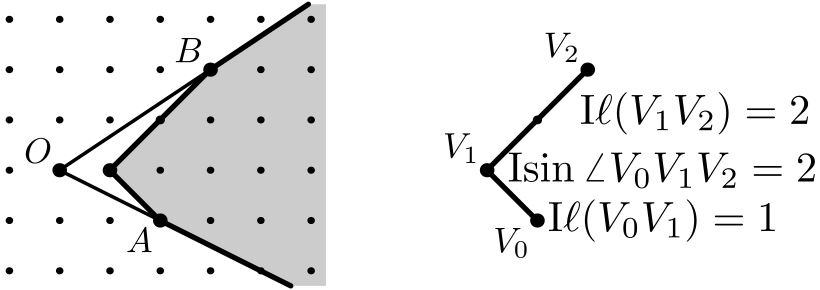
The end

Thank you.

Extra. Continued fractions for broken lines.

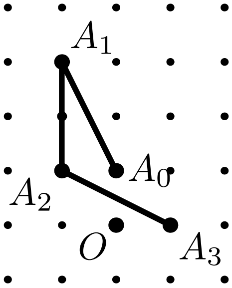
Generalization to broken lines

Generalization to broken lines



Is it possible to extend the LLS-sequence to arbitrary broken lines?

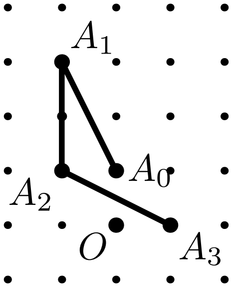
Generalization to broken lines



$$a_0 = 1;$$
$$a_1 = -1;$$
$$a_2 = 2;$$
$$a_3 = 2;$$
$$a_4 = -1.$$

Yes.

Generalization to broken lines



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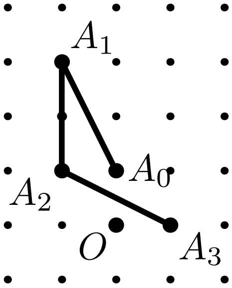
Definition

$$a_{2k} = |OA_k \times OA_{k+1}|, \quad k = 0, \ldots, n;$$

($|v \times w|$ — the oriented area of the parallelogram spanned by v and w)

Generalization to broken lines



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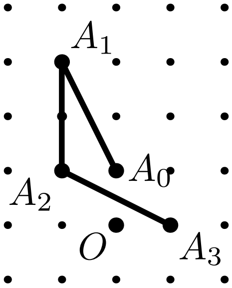

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The sequence $(a_0, \ldots, a_{2n})$ is called the *LLS-sequence*.

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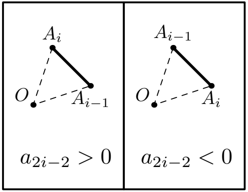

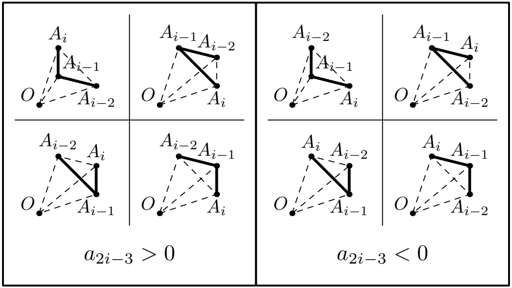

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($|v \times w|$ — the oriented area of the parallelogram spanned by v and w)

Generalized geometry of continued fractions

Theorem

Consider a broken line $A_0 \ldots A_n$ with LLS-sequence $(a_0, a_1, \ldots, a_{2n})$. Let $A_0 = (1,0)$, $A_1 = (1, a_0)$, and $A_n = (x, y)$. Then

$$\frac{y}{x} = [a_0 : a_1; \ldots ; a_{2n}].$$

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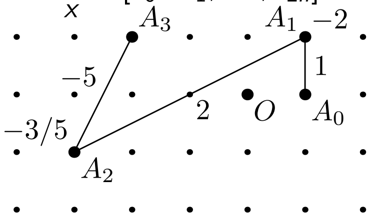
Example



$$[1; -2 : 2 : -3/5 : -5] = \frac{-1}{2}$$

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$$\frac{y}{x} = [a_0 : a_1; \ldots; a_{2n}].$$

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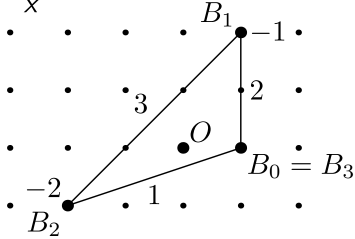
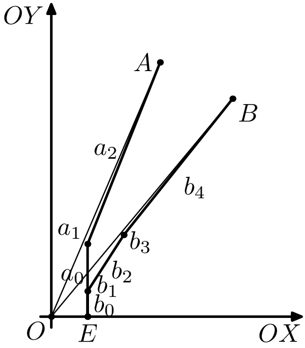


$$[2; -1 : 3 : -2 : 1] = \frac{0}{1}$$

Illustration to the proof of Theorem



The LLS of the broken line

$$(-a_{2m}, -a_{2m-1}, \ldots, -a_1, -a_0, 0, b_0, b_1, \ldots, b_{2n}).$$

Then the LLS of $\angle AOB$ is the sequence satisfies