Combinatorial aspects of continued fractions, $\mathrm{SL}(2,\mathbb{Z})$-tilings, and their q-analogs

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with Valentin Ovsienko:

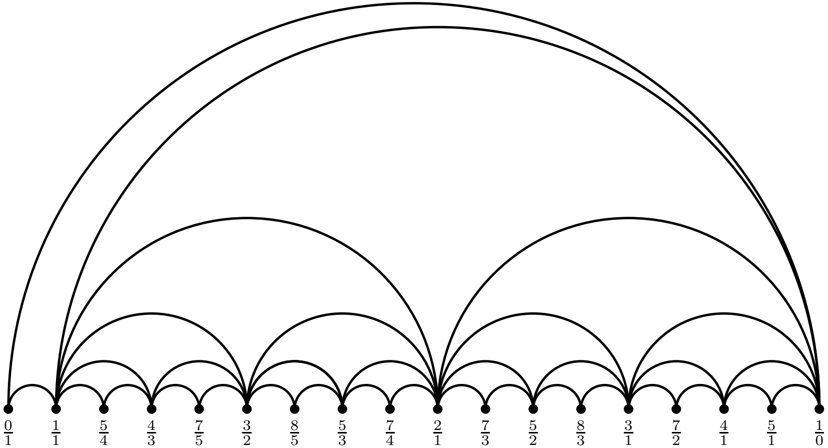
• Farey boat: continued fractions and triangulations, modular group and polygon dissections. Jahresber. Dtsch. Math.-Ver. (2019),
• q-deformed rationals and q-continued fractions. Forum Math. Sigma (2020)
• On q-deformed real numbers. Exp. Math. (2022)

with Ludivine Leclere:

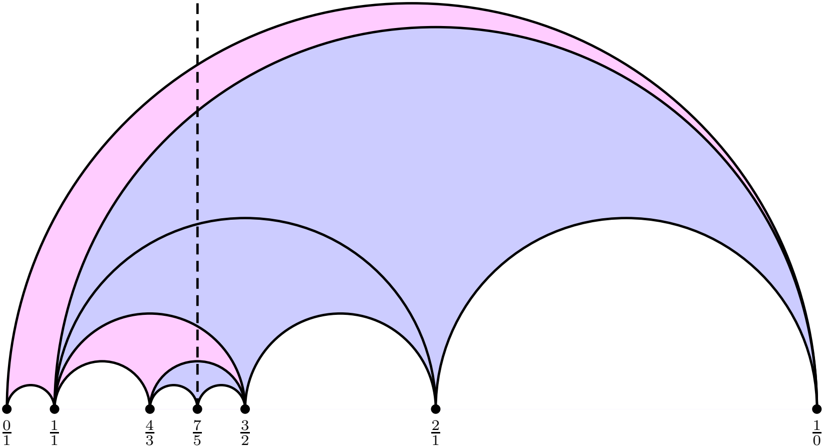
• q-deformations in the modular group and of the real quadratic irrational numbers. Adv. Appl. Math. (2021).
• Quantum continuants, quantum rotundus and triangulations of annuli. Electron. J. Combin. (2023)

1. Combinatorial models for rational numbers

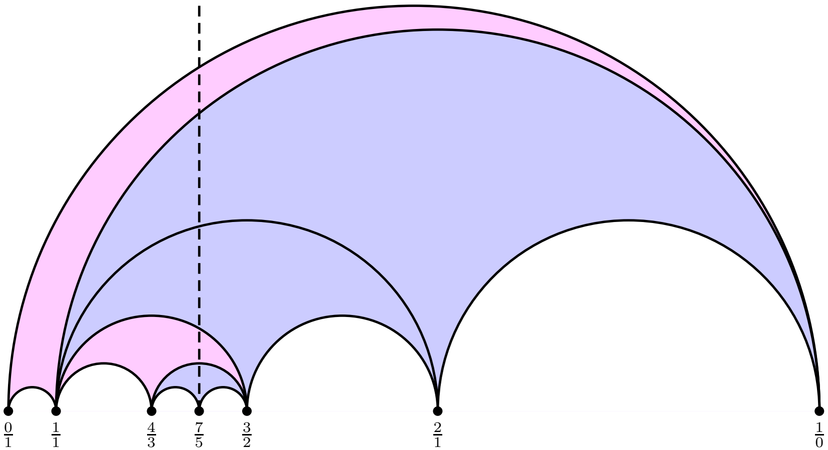
Combinatorics on the Farey graph



Combinatorics on the Farey graph

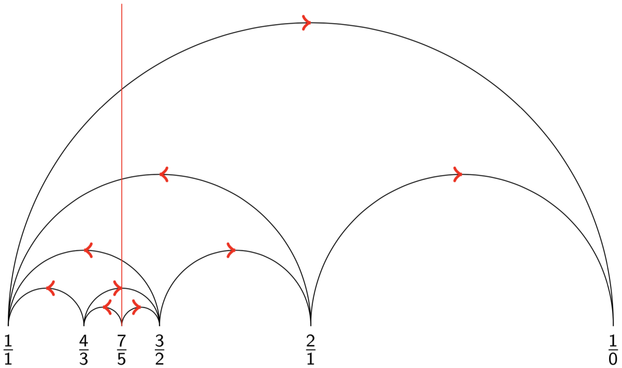


Combinatorics on the Farey graph

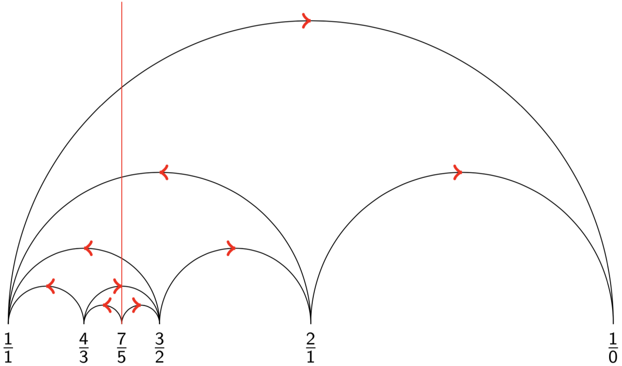


[Conway–Coxeter, 1970's] [C. Series, 1980's] [MG-O, 2019]

Enumerative interpretations of r and s

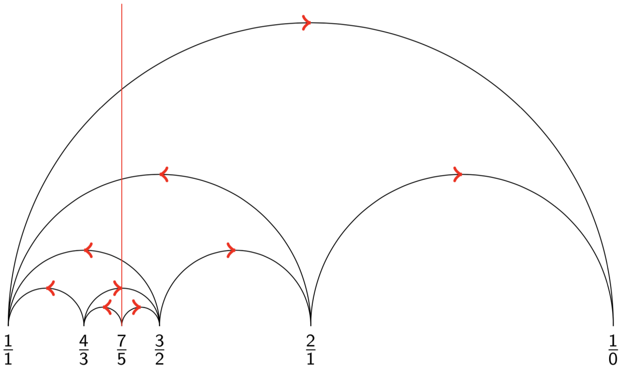


Enumerative interpretations of r and s



$$r = \#\{\text{paths } \tfrac{r}{s} \to \tfrac{1}{0}\}\,, \qquad s = \#\{\text{paths } \tfrac{r}{s} \to \tfrac{1}{1}\}.$$

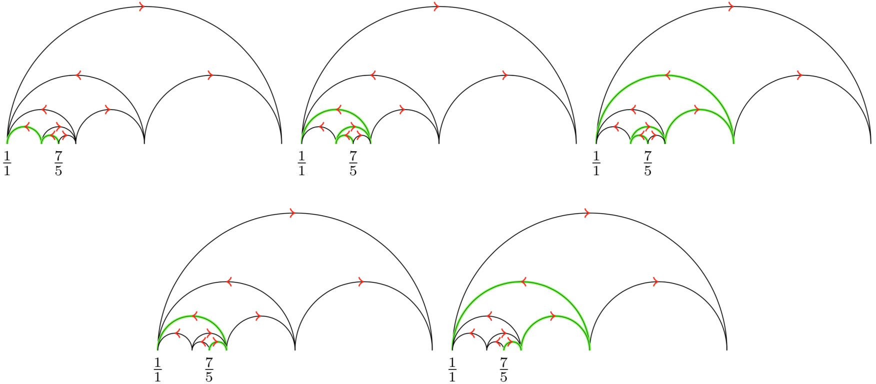
Enumerative interpretations of r and s



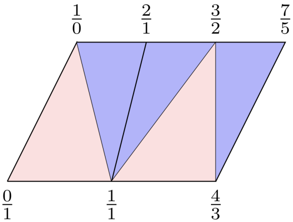
$$r = \#\{\text{paths } \tfrac{r}{s} \to \tfrac{1}{0}\}\ , \qquad s = \#\{\text{paths } \tfrac{r}{s} \to \tfrac{1}{1}\}.$$

[Garcia Barroso – Gonzalez Perez – Popescu-Pampu, 2020]

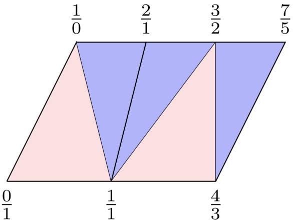
Enumerative interpretations

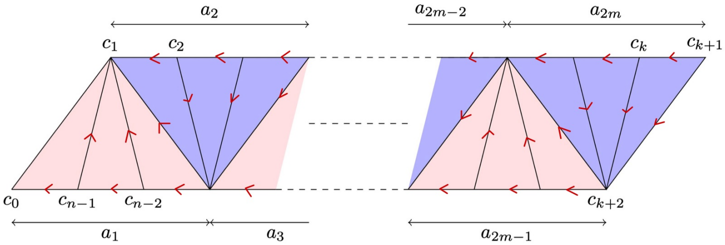


Triangulations of polygons



Triangulations of polygons





2. q-analogs

Interlude: q-analogs of integers

- q-integers [Euler, Gauss, ...]

$$[n]_q = 1 + q + \cdots + q^{n-1}, \qquad [-n]_q = -q^{-1} - q^{-2} - \cdots - q^{-n}.$$

- q-factorials

$$[n]_q! = [n]_q [n-1]_q \cdots [2]_q$$

- q-binomials

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q! [k]_q!}$$

Interlude: q-analogs of integers

- q-integers [Euler, Gauss, ...] $[6]_q = 1 + q + q^2 + q^3 + q^4 + q^5$

$$[n]_q = 1 + q + \cdots + q^{n-1}, \qquad [-n]_q = -q^{-1} - q^{-2} - \cdots - q^{-n}.$$

- q-factorials

$$[n]_q! = [n]_q[n-1]_q \cdots [2]_q \qquad [3]_q! = 1 + 2q + 2q^2 + q^3$$

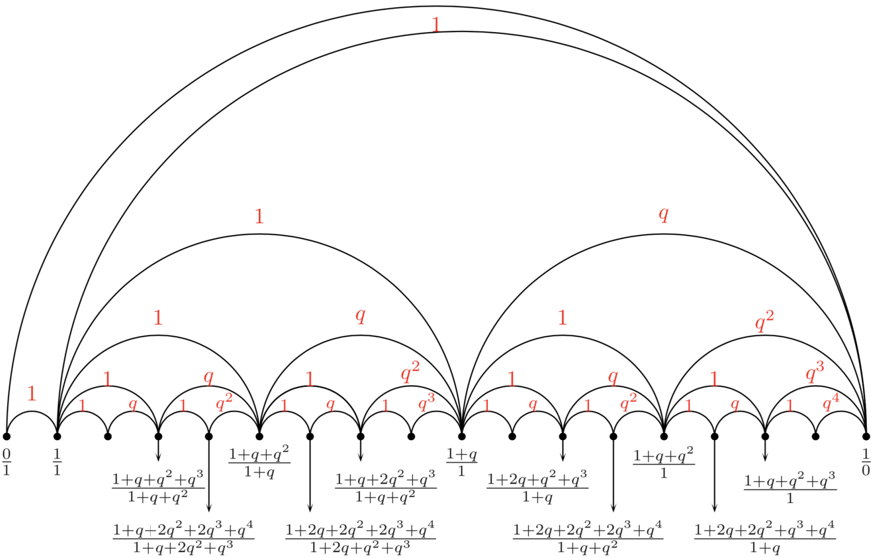
- q-binomials

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q![k]_q!} \qquad \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + q^3 + q^4$$

What can we expect from a q-analog?

- Natural definition
- q-analogs of positive integers are often polynomials
  - with positive integer coefficients
  - with symmetric sequence of coefficients (palindromes)
  - with unimodal sequence of coefficients
- Combinatorial interpretations
- Geometric interpretations

q-rationals


$\frac{0}{1}$
$\frac{1}{1}$
$\frac{1+q+q^2+q^3}{1+q+q^2}$
$\frac{1+q+2q^2+2q^3+q^4}{1+q+2q^2+q^3}$
$\frac{1+q+q^2}{1+q}$
$\frac{1+2q+2q^2+2q^3+q^4}{1+2q+q^2+q^3}$
$\frac{1+q+2q^2+q^3}{1+q+q^2}$
$\frac{1+q}{1}$
$\frac{1+2q+q^2+q^3}{1+q}$
$\frac{1+2q+2q^2+2q^3+q^4}{1+q+q^2}$
$\frac{1+q+q^2}{1}$
$\frac{1+2q+2q^2+q^3+q^4}{1+q}$
$\frac{1+q+q^2+q^3}{1}$
$\frac{1}{0}$

q-deformation of continued fractions

$$\frac{5}{3} = 2 - \frac{1}{3} \quad \to \quad 1 + q - \frac{q}{1 + q + q^2}$$

$$= 1 - \cfrac{1}{-1 - \cfrac{1}{2}} \quad \to \quad 1 - \cfrac{1}{-q^{-1} - \cfrac{q^{-2}}{1 + q}}$$

$$= -1 - \cfrac{1}{0 - \cfrac{1}{3 - \cfrac{1}{3}}} \quad \to \quad -q^{-1} - \cfrac{q^{-2}}{0 - \cfrac{q^{-1}}{1 + q + q^2 - \cfrac{q^2}{1 + q + q^2}}}$$

q-deformation of continued fractions

$$\frac{5}{3} = 2 - \frac{1}{3} \rightarrow 1 + q - \frac{q}{1+q+q^2}$$

$$=$$

$$= 1 - \cfrac{1}{-1 - \cfrac{1}{2}} \rightarrow 1 - \cfrac{1}{-q^{-1} - \cfrac{q^{-2}}{1+q}}$$

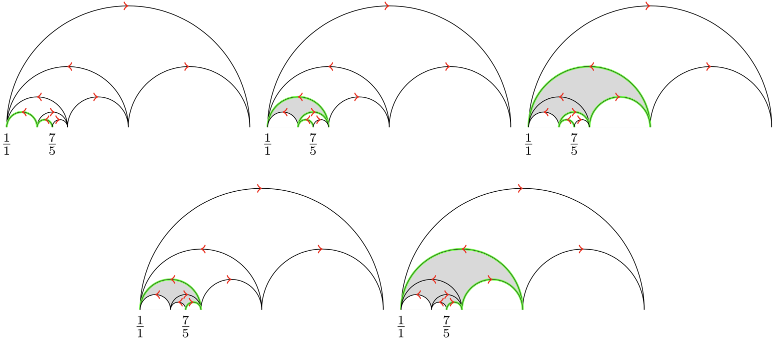
$$=$$

$$= -1 - \cfrac{1}{0 - \cfrac{1}{3 - \cfrac{1}{3}}} \rightarrow -q^{-1} - \cfrac{q^{-2}}{0 - \cfrac{q^{-1}}{1+q+q^2 - \cfrac{q^2}{1+q+q^2}}}$$

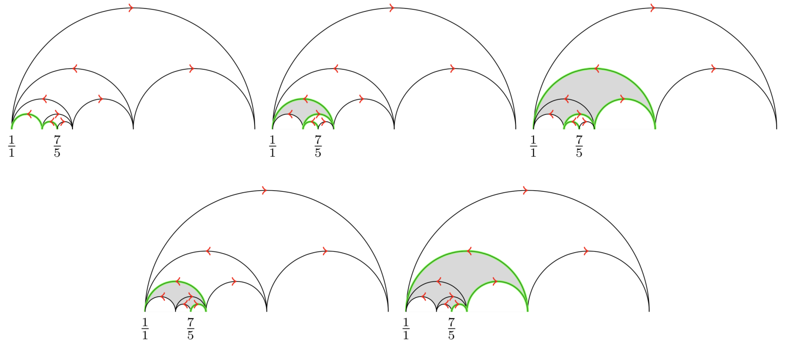
$$=$$

$$\frac{1+q+2q^2+q^3}{1+q+q^2} =: \left[\frac{5}{3}\right]_q$$

Enumerative interpretations



Enumerative interpretations



Theorem (Leclere-MG, Electron. J. Combin. 2023)

For the rational $\frac{r}{s} \geq 1$ *with q-deformation* $\frac{\mathcal{R}}{\mathcal{S}} = \left[\frac{r}{s}\right]_q$, *one has*

$$\mathcal{R} = \sum_{\pi : \frac{r}{s} \to \frac{1}{0}} q^{coar(\pi)}, \qquad \mathcal{S} = \sum_{\pi : \frac{r}{s} \to \frac{1}{1}} q^{coar(\pi)}$$

3. $\mathrm{SL}(2, \mathbb{Z})$

Deformation of matrices

- q-analogs of matrices

$$R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \to \quad R_q = \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \to \quad L_q = \begin{pmatrix} q & 0 \\ q & 1 \end{pmatrix}$$

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Well defined deformations:

$$M \in \mathrm{PSL}_2(\mathbb{Z}) \longrightarrow M_q \in \mathrm{GL}_2(\mathbb{Z}[q^{\pm 1}])/\{\pm q^k I_2\}$$

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Well defined deformations:

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Theorem (Leclere-MG, *Adv. Appl. Maths. 2022*)

The trace of a q-matrix M_q is a polynomial with positive integer coefficients forming a palindrome.

Examples of q-matrices

$$\begin{pmatrix}5 & 2\\ 2 & 1\end{pmatrix}_q = \begin{pmatrix} q+2q^2+q^3+q^4 & 1+q \\ q+q^2 & 1\end{pmatrix}$$

$$\begin{pmatrix}7 & 4\\ 5 & 3\end{pmatrix}_q = \begin{pmatrix} q+q^2+2q^3+2q^4+q^5 & 1+q+q^2+q^3 \\ q+q^2+2q^3+q^4 & 1+q+q^2\end{pmatrix}$$

$$\begin{pmatrix}12 & 5\\ 7 & 3\end{pmatrix}_q = \begin{pmatrix} q+2q^2+3q^3+3q^4+2q^5+q^6 & 1+q+2q^2+q^3 \\ q+2q^2+2q^3+q^4+q^5 & 1+q+q^2\end{pmatrix}$$

$$\begin{pmatrix}31 & 13\\ 19 & 8\end{pmatrix}_q = \begin{pmatrix} q+3q^2+5q^3+6q^4+7q^5+5q^6+3q^7+q^8 & 1+2q+3q^2+3q^3+3q^4+q^5 \\ q+3q^2+4q^3+4q^4+4q^5+2q^6+q^7 & 1+2q+2q^2+2q^3+q^4\end{pmatrix}$$

Examples of q-matrices

$$\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}_q = \begin{pmatrix} q + 2q^2 + q^3 + q^4 & 1 + q \\ q + q^2 & 1 \end{pmatrix}$$

$$\mathrm{Tr} = 1 + q + 2q^2 + q^3 + q^4$$

$$\begin{pmatrix} 7 & 4 \\ 5 & 3 \end{pmatrix}_q = \begin{pmatrix} q + q^2 + 2q^3 + 2q^4 + q^5 & 1 + q + q^2 + q^3 \\ q + q^2 + 2q^3 + q^4 & 1 + q + q^2 \end{pmatrix}$$

$$\mathrm{Tr} = 1 + 2q + 2q^2 + 2q^3 + 2q^4 + q^5$$

$$\begin{pmatrix} 12 & 5 \\ 7 & 3 \end{pmatrix}_q = \begin{pmatrix} q + 2q^2 + 3q^3 + 3q^4 + 2q^5 + q^6 & 1 + q + 2q^2 + q^3 \\ q + 2q^2 + 2q^3 + q^4 + q^5 & 1 + q + q^2 \end{pmatrix}$$

$$\mathrm{Tr} = 1 + 2q + 3q^2 + 3q^3 + 3q^4 + 2q^5 + q^6$$

$$\begin{pmatrix} 31 & 13 \\ 19 & 8 \end{pmatrix}_q =$$

$$\begin{pmatrix} q + 3q^2 + 5q^3 + 6q^4 + 7q^5 + 5q^6 + 3q^7 + q^8 & 1 + 2q + 3q^2 + 3q^3 + 3q^4 + q^5 \\ q + 3q^2 + 4q^3 + 4q^4 + 4q^5 + 2q^6 + q^7 & 1 + 2q + 2q^2 + 2q^3 + q^4 \end{pmatrix}$$

$$\mathrm{Tr} = 1 + 3q + 5q^2 + 7q^3 + 7q^4 + 7q^5 + 5q^6 + 3q^7 + q^8$$

Examples of q-matrices

$$\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}_q = \begin{pmatrix} q+2q^2+q^3+q^4 & 1+q \\ q+q^2 & 1 \end{pmatrix}$$

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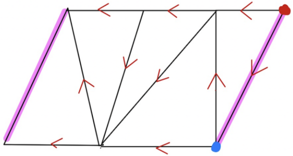
$$\mathrm{Tr} = 1+3q+5q^2+7q^3+7q^4+7q^5+5q^6+3q^7+q^8$$

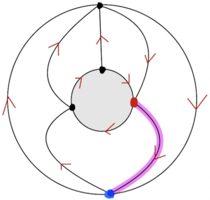
→ q-Markov equation [Kogiso, 2020]

→ q-Markov injectivity conjecture [Labbé–Lapointe, 2021], [L–L–Steiner, 2023]

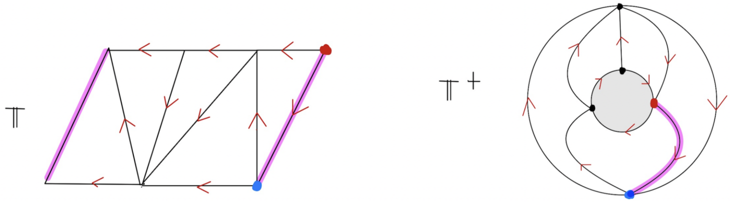
$\mathbb{T}$



$\mathbb{T}^+$



Triangulations of annuli



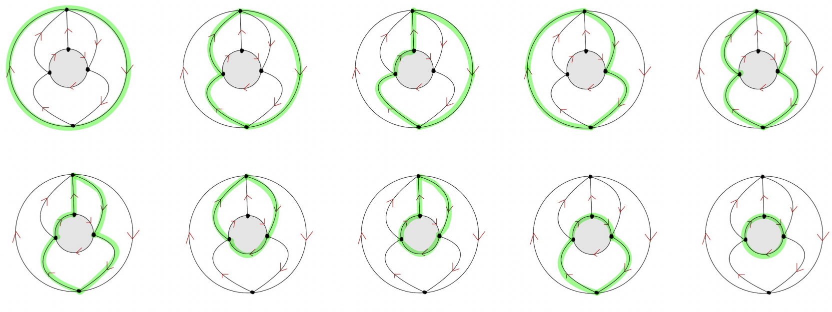
Theorem (Leclere–MG, Electron. J. Combin. 2023)

Let $(a_1, a_2, \ldots, a_{2m})$ be a sequence of positive integers

$$\mathrm{Tr}\ M^+(a_1, a_2, \ldots, a_{2m})_q = \sum_{\gamma \text{ in } \mathbb{T}^+} q^{ar(\gamma)}$$

where γ are closed curves with no self-intersection in $\mathbb{T}^+$.

Triangulations of annuli



$$M_q^+(1,2,1,1) = \begin{pmatrix} 7 & 4 \\ 5 & 3 \end{pmatrix}_q = \begin{pmatrix} q+q^2+2q^3+2q^4+q^5 & 1+q+q^2+q^3 \\ q+q^2+2q^3+q^4 & 1+q+q^2 \end{pmatrix}$$

$$\mathrm{Tr} \; = 1+2q+2q^2+2q^3+2q^4+q^5$$

4. q-reals

Stabilization Phenomenon

$$\frac{12}{5} \qquad \left[\frac{12}{5}\right]_q = \frac{1+2q+3q^2+3q^3+2q^4+q^5}{1+q+2q^2+q^3}$$

$$\frac{70}{29} \qquad \left[\frac{70}{29}\right]_q = \frac{1+3q+7q^2+11q^3+13q^4+13q^5+11q^6+7q^7+3q^8+q^9}{1+2q+5q^2+6q^3+6q^4+5q^5+3q^6+q^7}$$

$$\frac{408}{169} \qquad \left[\frac{408}{169}\right]_q = \frac{1+4q+12q^2+25q^3+41q^4+56q^5+65q^6+\ldots+4q^{12}+q^{13}}{1+3q+9q^2+16q^3+24q^4+29q^5+29q^6+\ldots+4q^{10}+q^{11}}$$

Stabilization Phenomenon

$\frac{12}{5} = 2.4 \qquad \left[\frac{12}{5}\right]_q = \frac{1+2q+3q^2+3q^3+2q^4+q^5}{1+q+2q^2+q^3}$

$\frac{70}{29} = 2.4137\ldots \qquad \left[\frac{70}{29}\right]_q = \frac{1+3q+7q^2+11q^3+13q^4+13q^5+11q^6+7q^7+3q^8+q^9}{1+2q+5q^2+6q^3+6q^4+5q^5+3q^6+q^7}$

$\frac{408}{169} = 2.41420\ldots \qquad \left[\frac{408}{169}\right]_q = \frac{1+4q+12q^2+25q^3+41q^4+56q^5+65q^6+\ldots+4q^{12}+q^{13}}{1+3q+9q^2+16q^3+24q^4+29q^5+29q^6+\ldots+4q^{10}+q^{11}}$

$\downarrow$

$1+\sqrt{2}$

$=2.41421\ldots$

Stabilization Phenomenon

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$$= 1 + q + q^4 - 2q^6 + q^7 + 3q^8 - 3q^9 - 4q^{10} + 7q^{11} + 4q^{12} \dots$$

$$\frac{70}{29} = 2.4137\dots \qquad \left[\frac{70}{29}\right]_q = \frac{1+3q+7q^2+11q^3+13q^4+13q^5+11q^6+7q^7+3q^8+q^9}{1+2q+5q^2+6q^3+6q^4+5q^5+3q^6+q^7}$$

$$= 1 + q + q^4 - 2q^6 + q^7 + 4q^8 - 5q^9 - 7q^{10} + 18q^{11} + 6q^{12} \dots$$

$$\frac{408}{169} = 2.41420\dots \qquad \left[\frac{408}{169}\right]_q = \frac{1+4q+12q^2+25q^3+41q^4+56q^5+65q^6+\dots+4q^{12}+q^{13}}{1+3q+9q^2+16q^3+24q^4+29q^5+29q^6+\dots+4q^{10}+q^{11}}$$

$$= 1 + q + q^4 - 2q^6 + q^7 + 4q^8 - 5q^9 - 7q^{10} + 18q^{11} + 7q^{12} \dots$$

$\downarrow$

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Stabilization Phenomenon

$\frac{12}{5} = 2.4$

$$\left[\frac{12}{5}\right]_q = \frac{1+2q+3q^2+3q^3+2q^4+q^5}{1+q+2q^2+q^3}$$

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$$= 1 + q + q^4 - 2q^6 + q^7 + 4q^8 - 5q^9 - 7q^{10} + 18q^{11} + 6q^{12} \dots$$

$\frac{408}{169} = 2.41420\dots$

$$\left[\frac{408}{169}\right]_q = \frac{1+4q+12q^2+25q^3+41q^4+56q^5+65q^6+\dots+4q^{12}+q^{13}}{1+3q+9q^2+16q^3+24q^4+29q^5+29q^6+\dots+4q^{10}+q^{11}}$$

$$= 1 + q + q^4 - 2q^6 + q^7 + 4q^8 - 5q^9 - 7q^{10} + 18q^{11} + 7q^{12} \dots$$

$\downarrow$ $\qquad$ $\downarrow$

$1 + \sqrt{2}$
$= 2.41421\dots$

$$[1+\sqrt{2}]_q = 1 + q + q^4 - 2q^6 + q^7 + 4q^8 - 5q^9 - 7q^{10} + 18q^{11} + 7q^{12} - 55q^{13} + 18q^{14} + 146q^{15} \dots$$

$$= \frac{q^3 + 2q - 1 + \sqrt{q^6 + 4q^4 - 2q^3 + 4q^2 + 1}}{2q}$$

q-analogs of irrationals

- q-real numbers

Definition/Theorem [MG-Ovsienko, *Exp. Math. 2022*]

The q-irrationals are formal Laurent series obtained from the convergence of q-rationals.

Examples of q-irrationals

$$
\begin{aligned}
[\varphi]_q \;=\;& 1+q^2-q^3+2q^4-4q^5+8q^6-17q^7+37q^8\\
&-82q^9+185q^{10}-423q^{11}+978q^{12}-2283q^{13}\\
&+5373q^{14}-12735q^{15}+30372q^{16}-72832q^{17}\\
&+175502q^{18}-424748q^{19}+1032004q^{20}\cdots
\end{aligned}
$$

$$
\begin{aligned}
[\pi]_q \;=\;& 1+q+q^2+q^{10}-q^{12}-q^{13}+q^{15}+q^{16}\\
&-q^{20}-2q^{21}-q^{22}+2q^{23}+4q^{24}+q^{25}\\
&-4q^{27}-4q^{28}-2q^{29}+q^{30}+5q^{31}+8q^{32}+3q^{33}\\
&-3q^{34}-10q^{35}-12q^{36}-5q^{37}+8q^{38}+19q^{39}+20q^{40}\\
&+2q^{41}-18q^{42}-32q^{43}-25q^{44}+31q^{46}+51q^{47}\\
&+45q^{48}-7q^{49}-65q^{50}-94q^{51}-57q^{52}+35q^{53}\cdots
\end{aligned}
$$

$$
\begin{aligned}
[e]_q \;=\;& 1+q+q^3-q^5+2q^6-3q^7+3q^8-q^9\\
&-3q^{10}+9q^{11}-17q^{12}+25q^{13}-29q^{14}+23q^{15}+2q^{16}\\
&-54q^{17}+134q^{18}-232q^{19}+320q^{20}-347q^{21}+243q^{22}+71q^{23}\\
&-660q^{24}+1531q^{25}-2575q^{26}+3504q^{27}-3804q^{28}+2747q^{29}+488q^{30}\cdots
\end{aligned}
$$

Related work...

- Knots and Braids
  [Lee–Schiffler, 2019] [Kogiso–Wakui, 2019] [MG-O-Veselov, 2023]
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- Enumerative interpretations
  [Canakci–Schiffler, 2018] [Kantarci Oguz—Ravichandran, 2021, 2021]
  [Garcia B–Gonzalez P–Popescu P, 2020] [Kantarci Oguz–Yildirim, 2022]
  [McConville–Sagan–Smyth, 2021] [Ovenhouse 2021]
- Categorical and homological interpretations
  [Bapat–Becker–Licata, 2022] [L.Fan–Y.Qiu, 2023]
- q-Markov
  [Kogiso, 2020] [Labbé–Lapointe, 2021] [Labbé–Lapointe–Steiner, 2023]
- Power series of q-reals, q-calculus, q-algebra
  [L-MG-O-Veselov, 2023] [Ren, 2022] [Ovsienko–Pedon, 2023]
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