Continued fractions and Hankel determinants for q-metallic numbers

Emmanuel Pedon (Univ. Reims)
in collaboration with Valentin Ovsienko (CNRS & Univ. Reims)

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Continued Fractions and SL_2-tilings
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Teaser

Fix k a positive integer. Take the k-th *metallic number* (or *metallic ratio*)

$$y_k := k + \cfrac{1}{k + \cfrac{1}{k + \cfrac{1}{k + \cfrac{1}{\ddots}}}}$$

and consider its q-deformation in the sense of S. Morier-Genoud & V. Ovsienko

$$[y_k]_q := [k]_q + \cfrac{q^k}{[k]_{q^{-1}} + \cfrac{q^{-k}}{[k]_q + \cfrac{q^k}{[k]_{q^{-1}} + \cfrac{q^{-k}}{\ddots}}}}$$

where $[k]_q = 1 + q + \cdots + q^{k-1}$.

Teaser

Expand it into a Taylor series $[y_k]_q = \sum_{i=0}^{\infty} f_i q^i$ around $q = 0$ and compute its *(shifted) Hankel determinants*

$$\Delta_n^{(\ell)} := \det(f_{i+j+\ell})_{i,j=0}^{n-1}$$

where $n, \ell = 0, 1, 2 \ldots$

Our main results: we prove, for $k = 1$ and $k = 2$, and conjecture, in general, that:

- The first $k + 2$ sequences $\Delta_n^{(0)}, \Delta_n^{(1)}, \ldots, \Delta_n^{(k+1)}$ consist of $-1, 0, 1$ only.
- They are $2k(k+1)$-periodic when k is even and $2k(k+1)$-antiperiodic (hence $4k(k+1)$-periodic) when k is odd.
- They satisfy the following three-term *Somos-Gale-Robinson recurrence*

$$\Delta_{n+2k+2}^{(\ell)} \Delta_n^{(\ell)} = \Delta_{n+2k+1}^{(\ell)} \Delta_{n+1}^{(\ell)} - \left(\Delta_{n+k+1}^{(\ell)}\right)^2 \quad \text{for all } n \geqslant 0.$$

Why do we care? Because the situation resembles others which are much better known: when the power series $\sum f_i q^i$ is the generating function for *Catalan numbers* or *Motzkin numbers*.

How do we do that? We find "nice" continued fractions for $[y_k]_q$.

Outline

1. **Some classical material**
   Hankel determinants
   The example of Catalan and Motzkin sequences
   C-fractions and J-fractions

2. **A very short introduction to q-real numbers**

3. **Hankel determinants for q-metallic numbers**
   Introducing the q-metallic numbers
   Our results
   Super δ-fractions
   Conjectures

Hankel determinants

To a power series $f(q) = \sum_{i=0}^{\infty} f_i q^i$ or a sequence of numbers $f = (f_i)_{i \geqslant 0}$, one can associate a sequence $(H_n)_{n \geqslant 1}$ of **Hankel matrices** defined as follows:

$$H_n(f) = (f_{i+j})_{i,j=0}^{n-1} = \begin{pmatrix} f_0 & f_1 & \cdots & f_{n-1} \\ f_1 & f_2 & \cdots & f_n \\ \vdots & \vdots & & \vdots \\ f_{n-1} & f_n & \cdots & f_{2n-2} \end{pmatrix}.$$

Their determinants $\Delta_n(f) = \det H_n(f)$ are called **Hankel determinants** of f. More generally, we can introduce a "shift" $\ell = 0, 1, 2\ldots$ and consider the determinants

$$\Delta_n^{(\ell)}(f) := \det(f_{i+j+\ell})_{i,j=0}^{n-1}.$$

Hankel matrices and determinants have important applications in combinatorics, Padé approximation, coding theory, probability... e.g.

- *Kronecker's theorem:* The power series f is a rational function if and only if $\Delta_n(f) = 0$ for n large enough.
- *Hamburger's (resp. Stieltjes') Moment problem:* is a given sequence (μ_n) of numbers the moment sequence $\int_I x^n \mathrm{d}\mu(x)$ for some measure μ? When $I = \mathbb{R}$ (resp. $I = (0, +\infty)$) a necessary and sufficient condition involves Hankel matrices (resp. Hankel determinants).

Catalan numbers

The **Catalan numbers** C_n are the integers defined by $C_n := \frac{1}{n+1}\binom{n}{2n}$ for $n \geqslant 0$:

$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440 \ldots$

Some interpretations:

- (Euler, 1751) number of triangulations of a convex $(n+2)$-gon.
- (Catalan, 1838) number of ways $n+1$ factors can be parenthesized in a set equipped with a binary operation, e.g. for $n = 3$:

$$((ab)c)d \quad (a(bc))d \quad (ab)(cd) \quad a((bc)d) \quad a(b(cd))$$

- number of *Dyck paths*, i.e. paths from $(0,0)$ to $(2n,0)$ in $\mathbb{Z} \times \mathbb{Z}$ which never dip below the x-axis and are made up only of the two steps $\nearrow$ and $\searrow$.

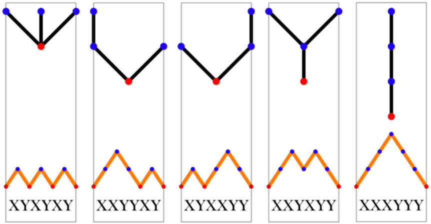

Figure: ©wikipedia

Motzkin numbers

The **Motzkin numbers** M_n are the integers defined by $M_n =: \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k$ for $n \geqslant 0$:

$$1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798, 15511, 41835, 113634\ldots$$

Some interpretations:

- (Motzkin, 1948) number of different ways of drawing non-intersecting chords between n points on a circle.
- number of *Motzkin paths*, i.e. paths from $(0, 0)$ to $(n, 0)$ in $\mathbb{Z} \times \mathbb{Z}$ which never dip below the x-axis and are made up only of the three steps $\rightarrow$, $\nearrow$, $\searrow$.

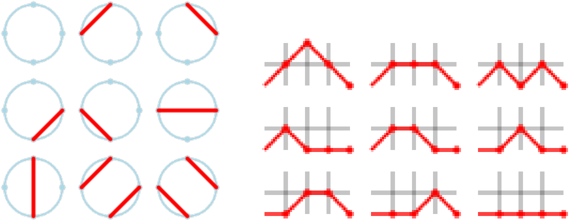

Figure: ©wikipedia

Hankel determinants for Catalan and Motzkin sequences

Facts

- *For the Catalan sequence:*

$$\Delta_n(C) = 1,\, 1,\, 1,\, 1 \ldots \qquad \Delta_n^{(1)}(C) = 1,\, 1,\, 1,\, 1 \ldots \tag{1}$$

$$\Delta_n^{(2)}(C) = 1,\, 2,\, 3,\, 4 \ldots \qquad \Delta_n^{(\ell)}(C) = \prod_{1 \leqslant i \leqslant j \leqslant \ell-1} \frac{2n+i+j}{i+j} \quad (\ell \geqslant 2).$$

  (Last formula: Desainte-Catherine & Viennot, 1986.)

  Moreover (C_n) *is the unique sequence of real numbers s.t.* (1) *holds.*

- *For the Motzkin sequence (Aigner, 1998):*

$$\Delta_n(M) = 1,\, 1,\, 1,\, 1 \ldots \qquad \Delta_n^{(1)}(M) = 1,\, 1,\, 0, -1, -1,\, 0, \ldots \tag{2}$$

$$\Delta_n^{(2)}(M) = 1,\, 2,\, 2,\, 3,\, 4,\, 4,\, 5,\, 6,\, 6,\, 7,\, 8,\, 8, \ldots$$

  Moreover, (M_n) *is the unique sequence of real numbers s.t.* (2) *holds.*

Remark

The shifted Hankel sequence $\Delta_n^{(1)}(M)$ satisfies the recurrence $\Delta_{n+2}\Delta_n = \Delta_{n+1}^2 - 1$.

C-fractions and J-fractions

Two classical families of algebraic continued fractions:

- the **C-fractions**:

$$\cfrac{b_0}{1 - \cfrac{b_1 q^{p_1}}{1 - \cfrac{b_2 q^{p_2}}{\ddots}}}$$

  Here (p_i) is a sequence of integers $\geqslant 1$, and (b_i) is a sequence of real or complex numbers.

  ▶ A fraction having $p_i \equiv 1$ is called a **regular C-fraction** (aka **Stieltjes continued fraction** or **S-fraction**).

- The *generalized Jacobi continued fractions*, or **J-fractions**:

$$\cfrac{b_0}{1 + qA_1(q) - \cfrac{b_1 q^{p_1}}{1 + qA_2(q) - \cfrac{b_2 q^{p_2}}{\ddots}}},$$

  where $A_i(q)$ are polynomials with $\deg(A_i) < p_i - 1$.

  ▶ A fraction having $p_i \equiv 2$ is called a **regular J-fraction**.

C-fractions and J-fractions

C/J-fractions are naturally related with orthogonal polynomials and Hankel determinants through the question of the existence of such expansions for a given power series.

Facts

- *Any power series can be written as a C-fraction (not always regular), in a unique way.*
- *Any power series with non-zero Hankel determinants can be written as a regular J-fraction, in a unique way.*

Example

The formal Catalan series $C(q) := \sum C_n q^n$ satisfies $C(q) = 1 + qC(q)^2$. This equation gives the expansions as regular C-fraction and J-fraction, respectively:

$$C(q) = \cfrac{1}{1 - \cfrac{q}{1 - \cfrac{q}{\ddots}}} \quad = \quad 1 + \cfrac{q}{1 - 2q - \cfrac{q^2}{1 - 2q - \cfrac{q^2}{\ddots}}}$$

Notice the 1-periodicity.

C-fractions and J-fractions

Example

The formal Motzkin series $M(q) := \sum M_n q^n$ satisfies
$M(q) = 1 + qM(q) + q^2 M(q)^2$.
This equation gives the expansions as non-regular C-fraction and regular J-fraction, respectively:

$$M(q) = \cfrac{1}{1 - \cfrac{q}{1 - \cfrac{q}{1 - \cfrac{q^2}{1 - \cfrac{q}{1 - \cfrac{q}{1 - \cfrac{q^2}{\ddots}}}}}}} \quad = \quad \cfrac{1}{1 - q - \cfrac{q^2}{1 - q - \cfrac{q^2}{\ddots}}}$$

Notice the 3- and 1-periodicities.

Hankel determinants for regular C-fractions and J-fractions

Heilermann's formula for regular J-fractions: if

$$f(q) = \cfrac{b_0}{1 + a_0 q - \cfrac{b_1 q^2}{1 + a_1 q - \cfrac{b_2 q^2}{1 + a_2 q - \cfrac{b_3 q^2}{\ddots}}}}$$

then $\Delta_n(f) = b_0^n b_1^{n-1} b_2^{n-2} \cdots b_{n-1}^2 b_n$.

$+$ Similar formulas for $\Delta_n^{(1)}(f)$ and $\Delta_n^{(2)}(f)$.

Example

Since

$$M(q) = \cfrac{1}{1 - q - \cfrac{q^2}{1 - q - \cfrac{q^2}{\ddots}}}$$

we find that $\Delta_n(M) \equiv 1$.

Hankel determinants for regular C-fractions and J-fractions

For regular C-fractions: if

$$f(q) = \cfrac{b_0}{1 + \cfrac{b_1 q}{1 + \cfrac{b_2 q}{\ddots}}}$$

then $\Delta_n(f) = b_0^n (b_1 b_2)^{n-1} (b_3 b_4)^{n-2} \cdots (b_{2n-5} b_{2n-4})^2 (b_{2n-3} b_{2n-2})$.

$+$ Similar formulas for $\Delta_n^{(1)}(f)$ and $\Delta_n^{(2)}(f)$.

Example

Recall that

$$C(q) = \cfrac{1}{1 - \cfrac{q}{1 - \cfrac{q}{\ddots}}}$$

hence $\Delta_n(C) \equiv 1$.

q-integers

- Very classical (Euler, Gauss): any integer $n \geqslant 0$ can be quantized as a polynomial
$$[n]_q := \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}.$$
This definition is equivalent to the recurrence formula
$$[n+1]_q = q[n]_q + 1 \qquad (3)$$
with initial term $[0]_q = 0$.
- Also classical are: q-factorials, q-binomials, q-hypergeometric functions... used in combinatorics, number theory, fractals, mathematical physics...
- Extension to rationals ? The naïve idea $\frac{m}{n} \to \frac{[m]_q}{[n]_q}$ lack crucial properties, such as (3).

  From 2018, S. Morier-Genoud & V. Ovsienko proposed a construction of q-analogues for rational, and then for real and complex numbers. Their work gave rise to a beautiful theory connected to many topics: cluster algebras, Markov-Hurwitz approximation theory, braid groups, combinators of posets, Calabi-Yau triangulated categories, Lie algebras of differential operators, supergeometry...

q-numbers: the continued fraction model

Several equivalent models are available for MGO's deformation. One of them involves continued fractions.

Definition

Let $x \in \mathbb{R}$, consider its regular continued fraction expansion (finite iff $x \in \mathbb{Q}$)

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}},$$

with $a_i \geqslant 1$. The **q-deformation** or **q-analogue** of x is the following algebraic continued fraction:

$$[x]_q := [a_0]_q + \cfrac{q^{a_0}}{[a_1]_{q^{-1}} + \cfrac{q^{-a_1}}{[a_2]_q + \cfrac{q^{a_2}}{[a_3]_{q^{-1}} + \cfrac{q^{-a_3}}{\ddots}}}}$$

(When infinite, it always converges in the field of formal Laurent series.)

q-numbers: examples

$$\left[\frac{3}{2}\right]_q = \frac{1+q+q^2}{1+q}, \qquad \left[\frac{5}{2}\right]_q = \frac{1+2q+q^2+q^3}{1+q}, \qquad \left[\frac{5}{3}\right]_q = \frac{1+q+2q^2+q^3}{1+q+q^2}$$

$$\left[\frac{8}{5}\right]_q = \frac{1+2q+2q^2+2q^3+q^4}{1+2q+q^2+q^3}$$
$$= 1+q^2-q^3+2q^4-4q^5+7q^6-12q^7+21q^8-37q^9+65q^{10}-114q^{11}+\cdots$$

$$\left[\frac{13}{8}\right]_q = \frac{1+2q+3q^2+3q^3+3q^4+q^5}{1+2q+2q^2+2q^3+q^4},$$
$$= 1+q^2-q^3+2q^4-3q^5+3q^6-3q^7+4q^8-5q^9+5q^{10}-5q^{11}+\cdots$$

$$\left[\frac{21}{13}\right]_q = \frac{1+3q+4q^2+5q^3+4q^4+3q^5+q^6}{1+3q+3q^2+3q^3+2q^4+q^5}$$
$$= 1+q^2-q^3+2q^4-4q^5+8q^6-17q^7+36q^8-75q^9+156q^{10}-325q^{11}+\cdots$$

$$\left[\frac{55}{34}\right]_q = \ldots = 1+q^2-q^3+2q^4-4q^5+8q^6-17q^7+37q^8-82q^9+184q^{10}-414q^{11}+\cdot$$

$$\ldots \left[\frac{F_{n+1}}{F_n}\right]_q \longrightarrow 1+q^2-q^3+2q^4-4q^5+8q^6-17q^7+37q^8-82q^9+185q^{10}-423q^{11}+$$
$$= 1+\cfrac{q}{1+\cfrac{q^{-1}}{1+\cfrac{q}{1+\cfrac{q^{-1}}{\ddots}}}} = \left[\frac{1+\sqrt{5}}{2}\right]_q$$

q-numbers: some important properties

References. Morier-Genoud & Ovsienko (2020, 2022), Leclere & Morier-Genoud (2021).

- For any $x \in \mathbb{R}$, the Laurent series $[x]_q$ has integer coefficients. More precisely,

$$[\cdot]_q : \quad \mathbb{Q}^+ \to \mathbb{Z}^+(q), \quad \mathbb{Q} \to \mathbb{Z}(q), \quad \mathbb{R}^+ \to \mathbb{Z}[[q]], \quad \mathbb{R} \to \mathbb{Z}((q))$$

  For $x \in \mathbb{Z}$, $[x]_q$ coincides with the classical definition of Euler and Gauss.

- We have, for any $x \in \mathbb{R}$,

$$[x+1]_q = q[x]_q + 1, \qquad \left[-\frac{1}{x}\right]_q = -\frac{1}{q[x]_q},$$

  i.e. the q-deformation $[\cdot]_q$ commutes with the action of the (deformed) modular group $PSL_q(2, \mathbb{Z})$ on $\mathbb{Z}((q)) \cup \{\infty\}$ by Möbius transformations.

- Special case : when x is a quadratic irrational number,
  - $[x]_q$ is solution of a quadratic equation with coefficients in $\mathbb{Z}[q]$ $\rightsquigarrow$ explicit generating function.
  - $[x]_q$ has a *periodic* continued fraction expansion.

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   - The example of Catalan and Motzkin sequences
   - C-fractions and J-fractions
2. **A very short introduction to q-real numbers**
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   - Introducing the q-metallic numbers
   - Our results
   - Super δ-fractions
   - Conjectures

q-metallic numbers

Let k be a positive integer. Consider the **k-th metallic number**

$$y_k := k + \cfrac{1}{k + \cfrac{1}{k + \cfrac{1}{k + \cfrac{1}{\ddots}}}} = \frac{k + \sqrt{k^2+4}}{2}$$

By definition, its q-deformation is the continued fraction

$$[y_k]_q := [k]_q + \cfrac{q^k}{[k]_{q^{-1}} + \cfrac{q^{-k}}{[k]_q + \cfrac{q^k}{[k]_{q^{-1}} + \cfrac{q^{-k}}{\ddots}}}}$$

Remark

$$[y_k]_q = \frac{1}{2q}\left(q[k]_q + (q^k+1)(q-1) + \sqrt{(q[k]_q + (q^k+1)(q-1))^2 + 4q}\right).$$

q-metallic numbers

Vocabulary and notation:

- $k = 1$: $y_1 = \frac{1+\sqrt{5}}{2}$ is the *golden ratio*; $[y_1]_q$ will be denoted by $G(q)$.
- $k = 2$: $y_2 = \sqrt{2} + 1$ is the *silver ratio*; $[y_2]_q$ will be denoted by $S(q)$.

Power series expansions:

$$G(q) = 1 + q^2 - q^3 + 2q^4 - 4q^5 + 8q^6 - 17q^7 + 37q^8 - 82q^9 \\ + 185q^{10} - 423q^{11} + 978q^{12} - 2283q^{13} + 5373q^{14} - 12735q^{15} \\ + 30372q^{16} - 72832q^{17} + 175502q^{18} - 424748q^{19} + 1032004q^{20} + \cdots$$

$$S(q) = 1 + q + q^4 - 2q^6 + q^7 + 4q^8 - 5q^9 - 7q^{10} + 18q^{11} + 7q^{12} - 55q^{13} \\ + 18q^{14} + 146q^{15} - 155q^{16} - 322q^{17} + 692q^{18} + 476q^{19} - 2446q^{20} + \cdots$$

The coefficients of $G(q)$ are 'almost' those of sequence A004148 in OEIS, which are called "generalized Catalan numbers" and admit many combinatorial interpretations; e.g. enumeration of peakless Motzkin paths of length n and secondary structures of RNA molecules.

The big surprise: computer experimentations showed unexpected beautiful properties of Hankel determinants of q-metallic numbers G, S and other $[y_k]_q$'s.

The q-golden number

Theorem

1. *The first four sequences of Hankel determinants $\Delta_n^{(\ell)}(G)$, for $\ell = 0, 1, 2, 3$, corresponding to the series $G(q)$ of the q-golden number are 4-antiperiodic (thus 8-periodic):*

$$\Delta_{n+4}^{(\ell)}(G) = -\Delta_n^{(\ell)}(G) \qquad \ell = 0, 1, 2, 3,$$

and they consist of $0, 1$, and -1 only. Periods are:

$$\begin{array}{rcrrrrrrrr} \Delta_n(G) & = & 1, & 1, & 1, & 0, & -1, & -1, & -1, & 0 \\ \Delta_n^{(1)}(G) & = & 1, & 0, & -1, & 1, & -1, & 0, & 1, & -1 \\ \Delta_n^{(2)}(G) & = & 1, & 1, & 1, & 0, & -1, & -1, & -1, & 0 \\ \Delta_n^{(3)}(G) & = & 1, & -1, & 0, & 0, & -1, & 1, & 0, & 0 \end{array} \qquad (4)$$

for $n = 0, 1, 2, \ldots, 7$.

2. *The first three rows are interconnected:* $\Delta_n(G) = (-1)^n \Delta_{n-2}^{(1)}(G) = \Delta_n^{(2)}(G)$.
3. *Invertibility: the series $G(q)$ is the only series whose first four Hankel determinants are 8-periodic and given by (4).*

The q-golden number

Remark (computer experimentation)

Higher shifted sequences $\Delta_n^{(\ell)}(G)$ with $\ell \geqslant 4$ are not periodic, but have interesting patterns, e.g.

$$\Delta_n^{(4)}(G) = 1,\ 2,\ 0, -2, -3, -4,\ 0,\ 4,\ 5,\ 6,\ 0, -6, -7, -8,\ 0,\ 8, \ldots$$

Compare with

$$\Delta_n^{(2)}(M) = 1,\ 2,\ 2,\ 3,\ 4,\ 4,\ 5,\ 6,\ 6,\ 7,\ 8,\ 8, \ldots$$

Corollary

The first three Hankel determinants sequences $\Delta_n(G), \Delta_n^{(1)}(G), \Delta_n^{(2)}(G)$ all satisfy a Somos-4 recurrence

$$\Delta_{n+4}\Delta_n = \Delta_{n+3}\Delta_{n+1} - \Delta_{n+2}^2.$$

Somos sequences

Definition

For $k \geqslant 2$ an integer, a **Somos-k sequence** is a solution of a quadratic recurrence of the form

$$S_{n+k}S_n = \sum_{j=1}^{\lfloor k/2 \rfloor} \alpha_j S_{n+k-j} S_{n+j}$$

for arbitrary parameters α_i.

Remark

- Somos sequences arose from elliptic function theory. Their properties are best understood with Fomin-Zelevinsky's cluster algebras (and the *Laurent phenomenon*). For instance, Somos-k sequences for $k = 4, 5, 6, 7$ with coefficients $\alpha_j = 1$ and initial data $S_0 = S_1 = \cdots = S_{k-1} = 1$ have the property of *integrality*.
- Somos sequences exhibit solutions to discrete dynamical systems integrable in the sense of Liouville-Arnold (Fordy & Hone, 2014).
- Many examples of Somos sequences are produced by Hankel determinants.

The q-silver number

Theorem

1. *The first four sequences of Hankel determinants corresponding to the series* $S(q)$ *of the q-silver number are* 12*-periodic:*

$$\Delta_{n+12}^{(\ell)}(S) = \Delta_n^{(\ell)}(S), \qquad \ell = 0, 1, 2, 3,$$

and consist in $-1, 0, 1$ *only. Periods are:*

$$\begin{array}{rcrrrrrrrrrrrr}
\Delta_n(S) & = & 1, & 1, & -1, & -1, & 1, & 0, & -1, & 0, & 0, & 1, & 0, & -1 \\
\Delta_n^{(1)}(S) & = & 1, & 1, & 0, & -1, & 0, & 0, & -1, & 0, & 1, & 1, & -1, & -1 \\
\Delta_n^{(2)}(S) & = & 1, & 0, & 0, & -1, & 0, & 1, & -1, & -1, & 1, & 1, & -1, & 0 \\
\Delta_n^{(3)}(S) & = & 1, & 0, & -1, & -1, & 1, & 1, & -1, & -1, & 0, & 1, & 0, & 0
\end{array}$$

for $n = 0, 1, 2, \ldots, 11$*.*

2. *These four rows are interconnected:* $\Delta_n^{(\ell+1)}(S) = (-1)^{n-1}\Delta_{n+3}^{(\ell)}(S)$ *for* $\ell = 0, 1, 2$*.*

The q-silver number

Corollary

The first four Hankel determinants sequences $\Delta_n^{(\ell)}(S)$, $\ell = 0, 1, 2, 3$, all satisfy a Somos-6 recurrence

$$\Delta_{n+6}\Delta_n = \Delta_{n+5}\Delta_{n+1} - \Delta_{n+3}^2.$$

Remark (computer experimentation)

Next sequence is also 12-periodic with period

$$\Delta_n^{(4)}(S) = 1,\ 1, -2, -1,\ 2, -1, -2,\ 1,\ 1,\ 0,\ 0,\ 0.$$

Strategy to prove the results: find continued fractions for which a Hankel determinant formula exists!

Problem (at first sight): no hope to write q-metallic numbers as regular C/J fractions, because of the zeros in the Hankel sequences.

Still... we have remarkable (non-regular) C-fractions for $G(q)$.

C-fractions for $G(q)$

$G(q)$ is represented by two C-fractions:

$$G(q) = \cfrac{1}{1 - \cfrac{q^2}{1 + \cfrac{q}{1 - \cfrac{q^2}{1 + \cfrac{q}{\ddots}}}}} \qquad = \quad 1 + \cfrac{q^2}{1 + \cfrac{q}{1 + \cfrac{q}{1 + \cfrac{q^3}{1 + \cfrac{q}{1 + \cfrac{q}{1 + \cfrac{q^3}{\ddots}}}}}}}$$

Remind Catalan and Motzkin C-fractions:

$$C(q) = \cfrac{1}{1 - \cfrac{q}{1 - \cfrac{q}{\ddots}}} \qquad M(q) = \cfrac{1}{1 - \cfrac{q}{1 - \cfrac{q}{1 - \cfrac{q^2}{1 - \cfrac{q}{1 - \cfrac{q}{1 - \cfrac{q^2}{\ddots}}}}}}}$$

J-fractions for $G(q)$

$$\frac{G(q)-1}{q^2} = \cfrac{1}{1+q-\cfrac{q^2}{1+q+\cfrac{q^3}{1+q-q^2+\cfrac{q^3}{1+q-\cfrac{q^2}{\ddots}}}}} \tag{5}$$

$$\frac{G(q)-1-q^2}{q^3} = \cfrac{1}{1+2q-\cfrac{q^4}{1+q-q^2+2q^3-\cfrac{q^4}{1+2q-\cfrac{q^4}{1+q-q^2+2q^3-\cfrac{q^4}{\ddots}}}}} \tag{6}$$

Remind regular J-fractions for the Catalan and Motzkin series:

$$\frac{C(q)-1}{q} = \cfrac{1}{1-2q-\cfrac{q^2}{1-2q-\cfrac{q^2}{\ddots}}} \qquad M(q) = \cfrac{1}{1-q-\cfrac{q^2}{1-q-\cfrac{q^2}{\ddots}}}$$

Fact. (5) and (6) are not regular but are “super 2-fractions”.

J-fractions for $G(q)$

The last continued fraction we give is also the simplest:

$$\frac{G(q)-1}{q^2} = \cfrac{1}{1+q-q^2+\cfrac{q^3}{1+q-q^2+\cfrac{q^3}{\ddots}}} \tag{7}$$

More generally:

Theorem

If y_k is a metallic number, then we have the following 1-periodic expansion

$$[y_k]_q = [k]_q + \cfrac{q^{2k}}{\langle k\rangle_q + \cfrac{q^{2k+1}}{\langle k\rangle_q + \cfrac{q^{2k+1}}{\ddots}}}$$

where $\langle k\rangle_q := q[k]_q + (1+q^k)(1-q)$.

Fact. The function $\dfrac{[y_k]_q - [k]_q}{q^{k+1}}$ is represented by a "super 3-fraction" (e.g. (7) for $k=1$).

Super δ-fractions

Definition (G.N. Han, 2016)

Let δ be a positive integer. A **super δ-fraction** is a continued fraction of the form

$$H(q) = \cfrac{v_0 q^{k_0}}{1 + q\,U_1(q) - \cfrac{v_1 q^{k_0+k_1+\delta}}{1 + q\,U_2(q) - \cfrac{v_2 q^{k_1+k_2+\delta}}{1 + q\,U_3(q) - \cfrac{v_3 q^{k_2+k_3+\delta}}{\ddots}}}}$$

where $v_i \neq 0$ are constants, $k_i \in \mathbb{Z}_{\geqslant 0}$, and $U_i(q)$ are polynomials such that $\deg(U_i) \leqslant k_{i-1} + \delta - 2$.

Facts

- *Super δ-fractions include regular C-fractions (for $\delta = 1$ and $k_i \equiv 0$) and regular J-fractions (for $\delta = 2$ and $k_i \equiv 0$).*
- *For every $\delta \geqslant 1$, any power series can be expanded as a unique super δ-fraction.*
- *When $\delta = 2$, Hankel determinants of $H(q)$ can be computed explicitly! Super 2-fractions are called **Hankel continued fractions**.*

Determinant formula for Hankel continued fractions

Let

$$H(q) = \cfrac{v_0 q^{k_0}}{1 + q\,U_1(q) - \cfrac{v_1 q^{k_0+k_1+2}}{1 + q\,U_2(q) - \cfrac{v_2 q^{k_1+k_2+2}}{1 + q\,U_3(q) - \cfrac{v_3 q^{k_2+k_3+2}}{\ddots}}}}$$

be a Hankel continued fraction; introduce the notation

$$s_n := \sum_{i=0}^{n-1} k_i + n, \qquad \varepsilon_n := \sum_{i=0}^{n-1} \frac{k_i\,(k_i+1)}{2}, \qquad \text{for } n \geqslant 1.$$

Then

$$\begin{cases} \Delta_{s_n}(H(q)) = (-1)^{\varepsilon_n} v_0^{s_n} v_1^{s_n - s_1} v_2^{s_n - s_2} \cdots v_{n-1}^{s_n - s_{n-1}}, \\ \Delta_m(H(q)) = 0 \quad \text{if } m \notin \{s_n,\ n \geqslant 1\}. \end{cases}$$

This formula + our H-fractions for $G(q)$ + some calculation $\Rightarrow$ **Theorem for $G(q)$**.

H-fractions for $S(q)$

We have the following 8-periodic H-fraction presentations:

$$S^{(1)}(q) = \cfrac{1}{1 - \cfrac{q^3}{1 + 2q^2 + \cfrac{q^5}{1 + 2q^2 - q^3 + \cfrac{q^5}{1 + 2q^2 - \cfrac{q^3}{1 + \cfrac{q^2}{1 + \cfrac{q^2}{1 + q + \cfrac{q^2}{1 + q^2\, S^{(1)}(q)}}}}}}}$$

where $S^{(1)}(q) := \dfrac{S(q) - 1}{q}$, as well as

$$S(q) = \cfrac{1}{1 - q + \cfrac{q^2}{1 + q + \cfrac{q^2}{1 + q^2\, S^{(1)}(q)}}}$$

These formulas + the determinant formula + some calculation $\Rightarrow$ **Theorem for $S(q)$**.

Other q-metallic numbers

Facts (computer experimentation)

1. *Consider the bronze ratio* $y_3 = \frac{3+\sqrt{13}}{2}$.
   - *The first five sequences of Hankel determinants* $\Delta_n^{(\ell)}$ *associated with the series* $[y_3]_q$*, consist of* $-1, 0$ *and* 1 *only.*
   - *These sequences are* 24*-antiperiodic, thus* 48*-periodic.*
   - *They are interconnected and satisfy a three-term Gale-Robinson recurrence*
   $$\Delta_{n+8}\Delta_n = \Delta_{n+7}\Delta_{n+1} - \Delta_{n+4}^2.$$
   - *Next sequence* $\Delta_n^{(5)}$ *is also* 24*-antiperiodic, and takes values in* $\{0, \pm1, \pm2\}$.
2. *Consider* $y_4 = 2 + \sqrt{5}$.
   - *The first six sequences of Hankel determinants of* $[y_4]_q$ *consist of* $-1, 0, 1$ *only.*
   - *They are* 40*-periodic.*
   - *They are interconnected and satisfy a three-term Gale-Robinson recurrence*
   $$\Delta_{n+10}\Delta_n = \Delta_{n+9}\Delta_{n+1} - \Delta_{n+5}^2.$$
   - *Next sequence* $\Delta_n^{(6)}$ *is also* 40*-antiperiodic, and takes values in* $\{0, \pm1, \pm2\}$.

Other q-metallic numbers

Conjecture

Let $k \geqslant 1$ and $\ell \geqslant 0$ be two integers, and let $\Delta_n^{(\ell)} := \Delta_n^{(\ell)}\big([y_k]_q\big)$ denote as before the ℓ-shifted sequence of Hankel determinants associated with the q-deformation of the metallic number y_k.

(a) The $k+2$ sequences $\Delta_n^{(0)}, \Delta_n^{(1)}, \ldots, \Delta_n^{(k+1)}$

- **(i)** consist of $-1, 0, 1$ only,
- **(ii)** are $2k(k+1)$-periodic when k is even and $2k(k+1)$-antiperiodic (hence $4k(k+1)$-periodic) when k is odd,
- **(iii)** satisfy the Somos-Gale-Robinson recurrence

$$\Delta_{n+2k+2}^{(\ell)} \, \Delta_n^{(\ell)} = \Delta_{n+2k+1}^{(\ell)} \, \Delta_{n+1}^{(\ell)} - \left(\Delta_{n+k+1}^{(\ell)}\right)^2 \quad \text{for all } n \geqslant 0.$$

(b) Next sequence $\Delta_n^{(k+2)}$ also satisfies (ii) and takes values in $\{0, \pm 1, \pm 2\}$.

(c) The $k+1$ pairs of consecutive shifted sequences $(\Delta_n^{(\ell)}, \Delta_n^{(\ell-1)})$ with $\ell = 1, 2, \ldots, k+1$ are interconnected by the formula:

$$\Delta_n^{(\ell)} = (-1)^{n + \frac{k(k+2\ell+1)}{2}} \, \Delta_{n+k+1}^{(\ell-1)} \quad \text{for all } n \geqslant 0.$$