Friezes, reduction and categorification

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Université de Versailles Saint-Quentin

March 26, 2024

Durham

Continued fractions and SL_2-tilings

0 0 0 0 0 0 0 0 0 0 0 0 0

1 1 1 1 1 1 1 1 1 1 1 1

1 3 2 2 1 4 2 1 3 2 2 1

1 2 5 3 1 3 7 1 2 5 3 1

1 3 7 1 2 5 3 1 3 7 1 2

1 4 2 1 3 2 2 1 4 2 1

1 1 1 1 1 1 1 1 1 1 1 1

0 0 0 0 0 0 0 0 0 0 0 0

0 0 0 0 0 0 0 0 0 0 0 0 0

1 1 1 1 1 1 1 1 1 1 1 1

1 3 2 2 1 4 2 1 3 2 2 1

1 2 5 3 1 3 7 1 2 5 3 1

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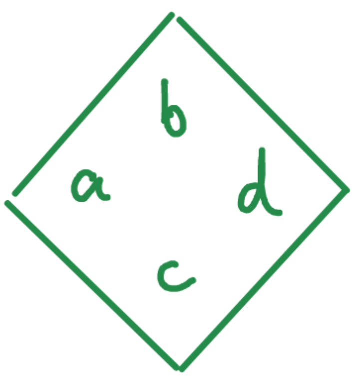
1 4 2 1 3 2 2 1 4 2 1

1 1 1 1 1 1 1 1 1 1 1 1

0 0 0 0 0 0 0 0 0 0 0 0

Rule:



$ad - bc = 1$

0 0 0 0 0 0 0 0 0 0 0 0 0

1 1 1 1 1 1 1 1 1 1 1 1

1 3 2 2 1 4 2 1 3 2 2 1

1 2 5 3 1 3 7 1 2 5 3 1

1 3 7 1 2 5 3 1 3 7 1 2

1 4 2 1 3 2 2 1 4 2 1

1 1 1 1 1 1 1 1 1 1 1 1

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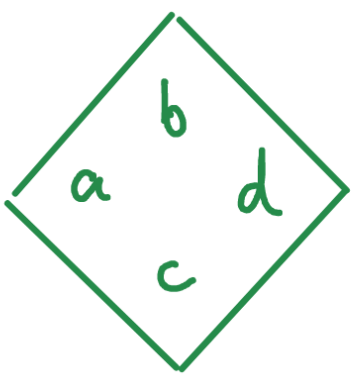
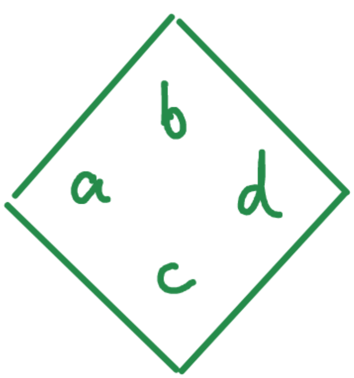
Rule:



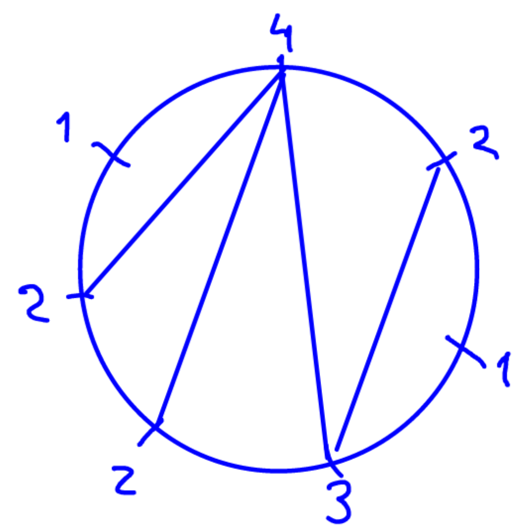
$ad - bc = 1$

$$
\begin{array}{ccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \\
1 & 3 & 2 & 2 & 1 & 4 & 2 & 1 & 3 & 2 & 2 & 1 & \\
1 & 2 & 5 & 3 & 1 & 3 & 7 & 1 & 2 & 5 & 3 & 1 & \\
1 & 3 & 7 & 1 & 2 & 5 & 3 & 1 & 3 & 7 & 1 & 2 & \\
1 & 4 & 2 & 1 & 3 & 2 & 2 & 1 & 4 & 2 & 1 & & \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 &
\end{array}
$$

Rule:



$ad - bc = 1$



0 0 0 0 0 0 0 0 0 0 0 0 0

1 1 1 1 1 1 1 1 1 1 1 1

1 3 2 2 1 4 2 1 3 2 2 1

1 2 5 3 1 3 7 1 2 5 3 1

1 3 7 1 2 5 3 1 3 7 1 2

1 4 2 1 3 2 2 1 4 2 1

1 1 1 1 1 1 1 1 1 1 1 1

0 0 0 0 0 0 0 0 0 0 0 0

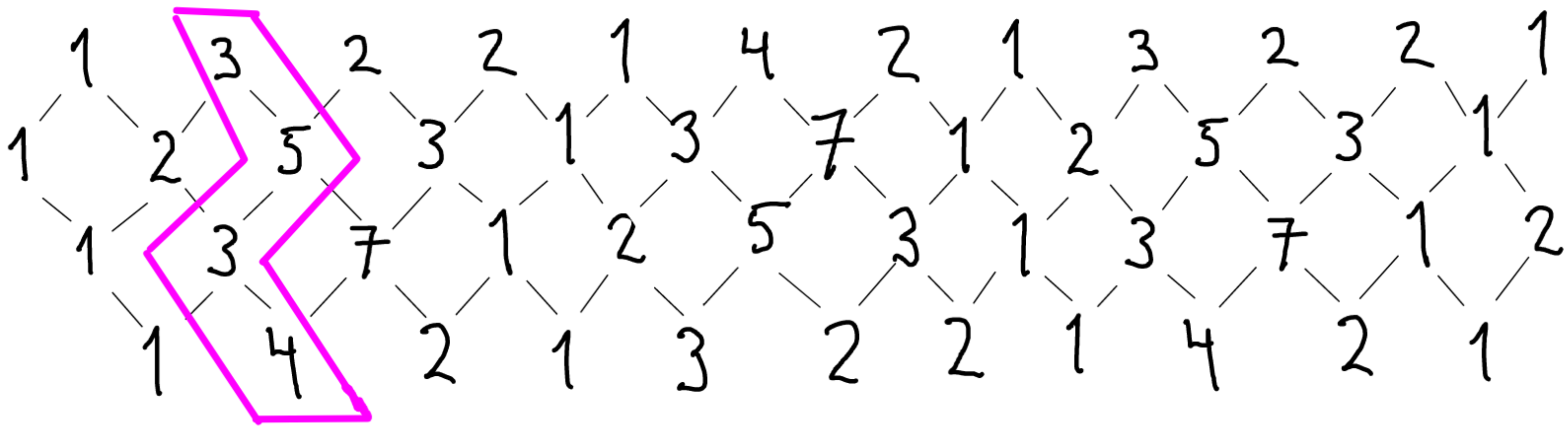
1 3 2 2 1 4 2 1 3 2 2 1
1 2 5 3 1 3 7 1 2 5 3 1
1 3 7 1 2 5 3 1 3 7 1 2
1 4 2 1 3 2 2 1 4 2 1

1 3 2 2 1 4 2 1 3 2 2 1

1 2 5 3 1 3 7 1 2 5 3 1

1 3 7 1 2 5 3 1 3 7 1 2

1 4 2 1 3 2 2 1 4 2 1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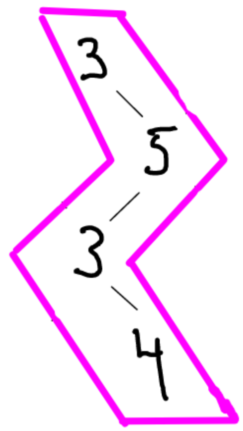
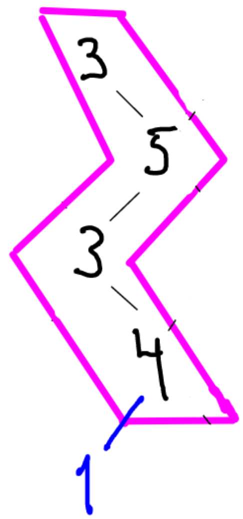
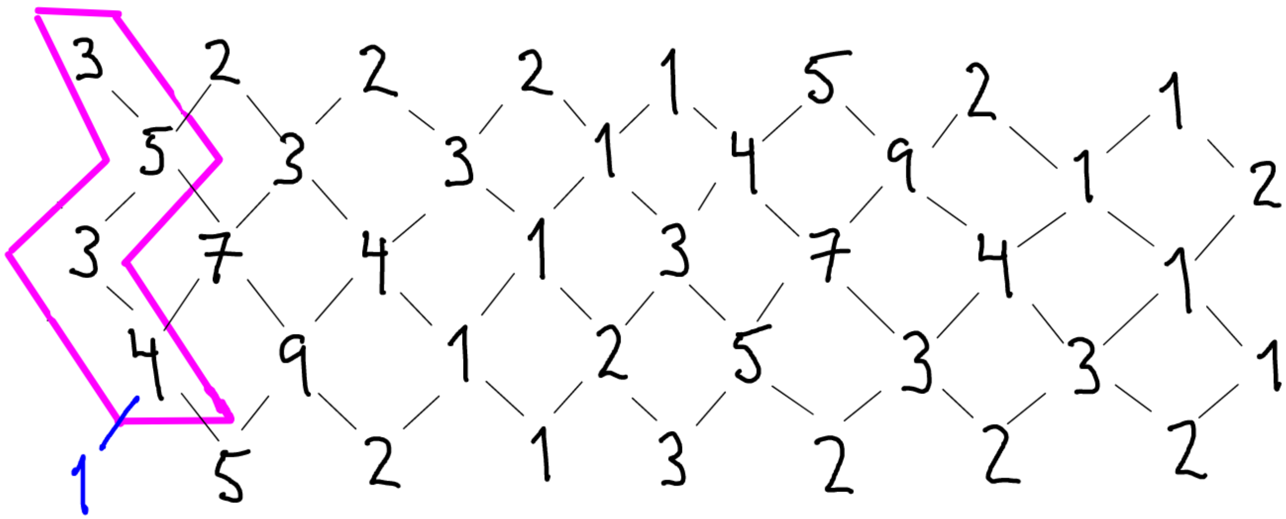
1 3 7 1 2 5 3 1 3 7 1 2
1 4 2 1 3 2 2 1 4 2 1


3
5
3
4


3
5
3
4
1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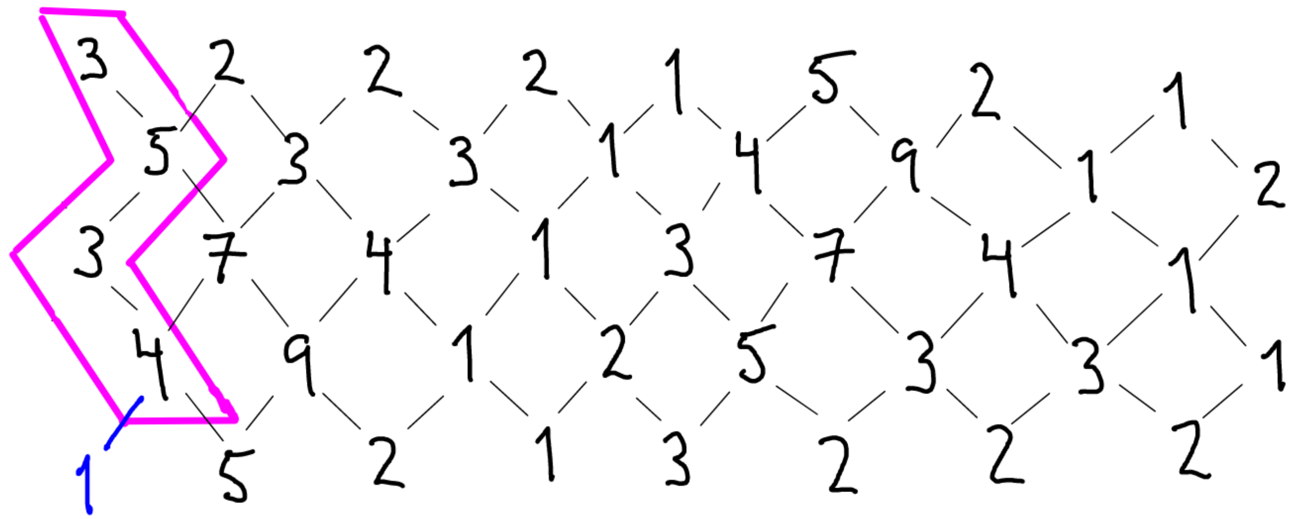
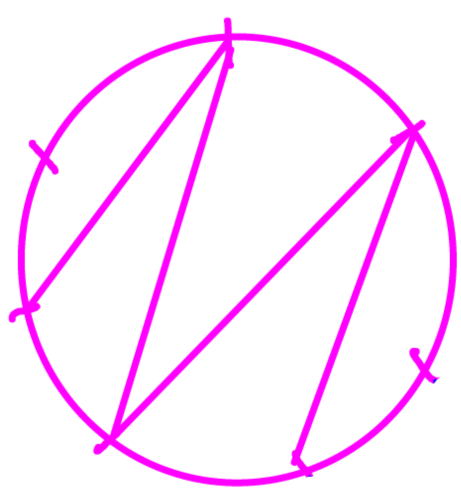
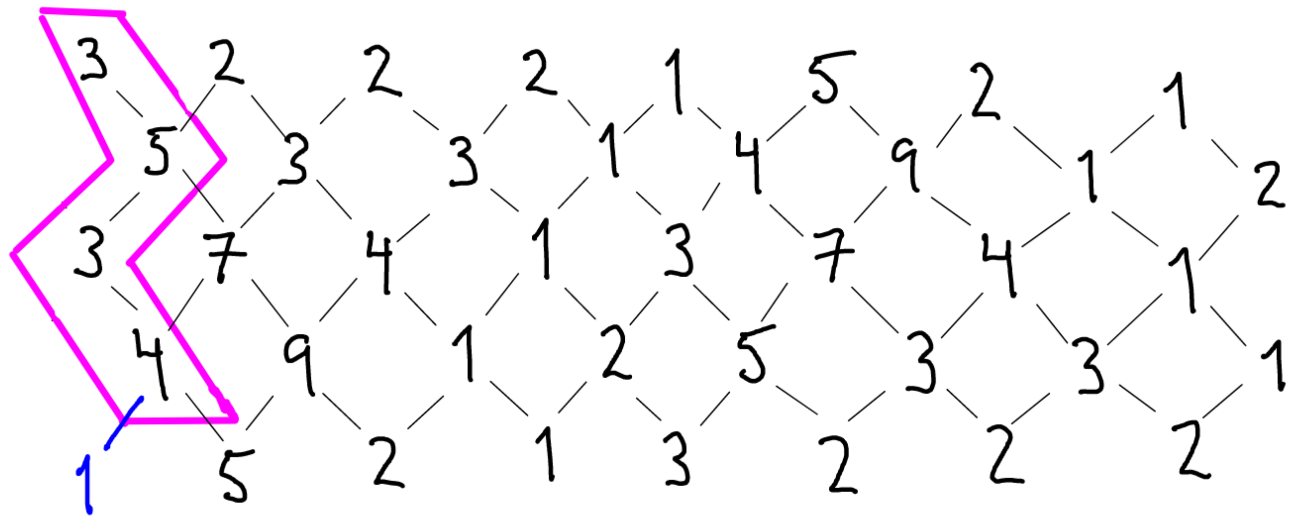
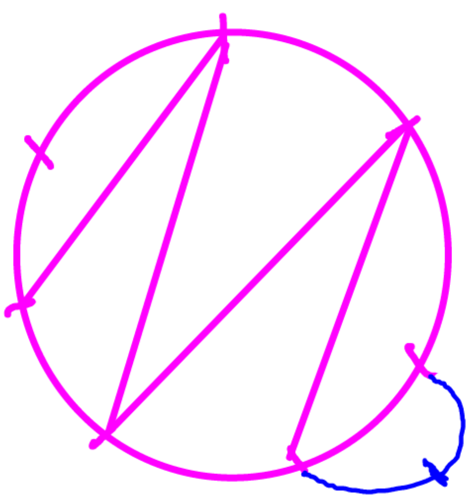
3 7 4 1 3 7 4 1
4 9 1 2 5 3 3 1
1 5 2 1 3 2 2 2



7-gon







~~7-gon~~

8-gon

Friezes from quivers (without oriented cycles)

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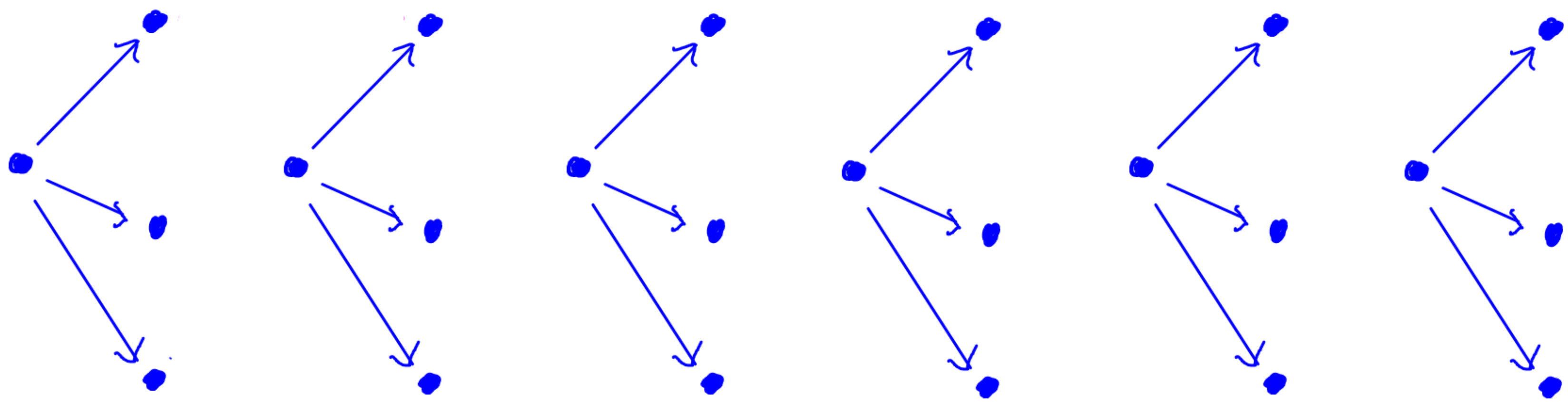
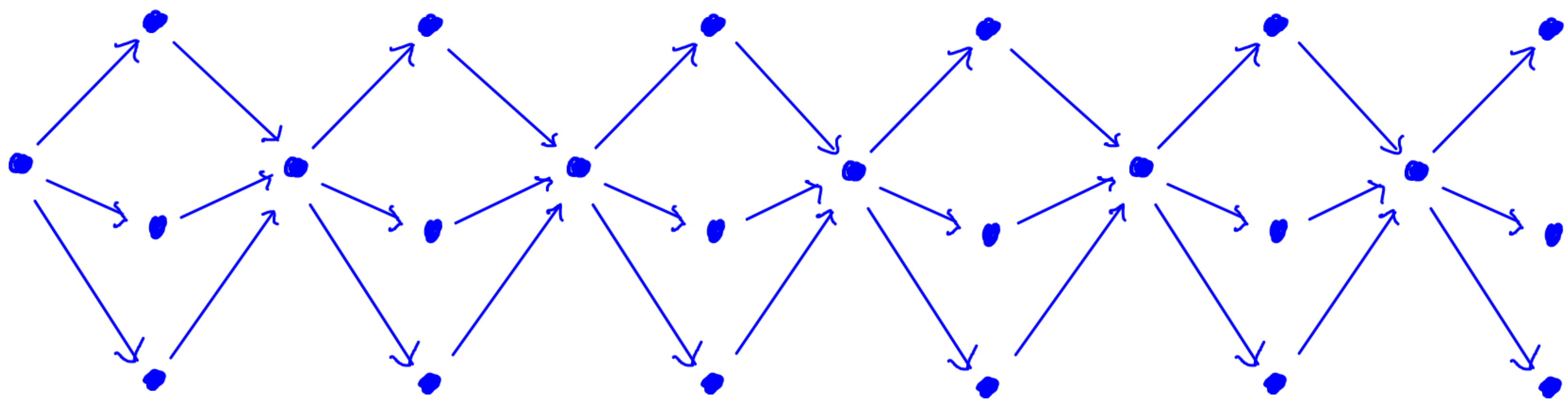


Friezes from quivers (without oriented cycles)



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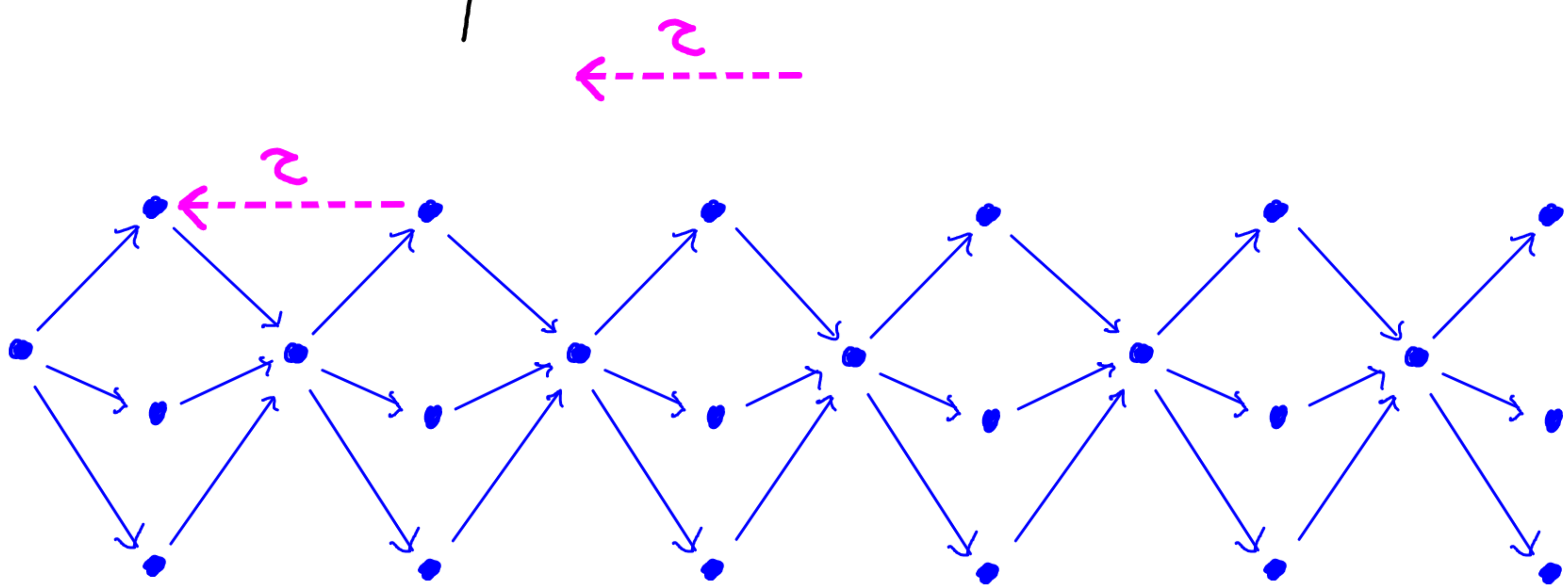
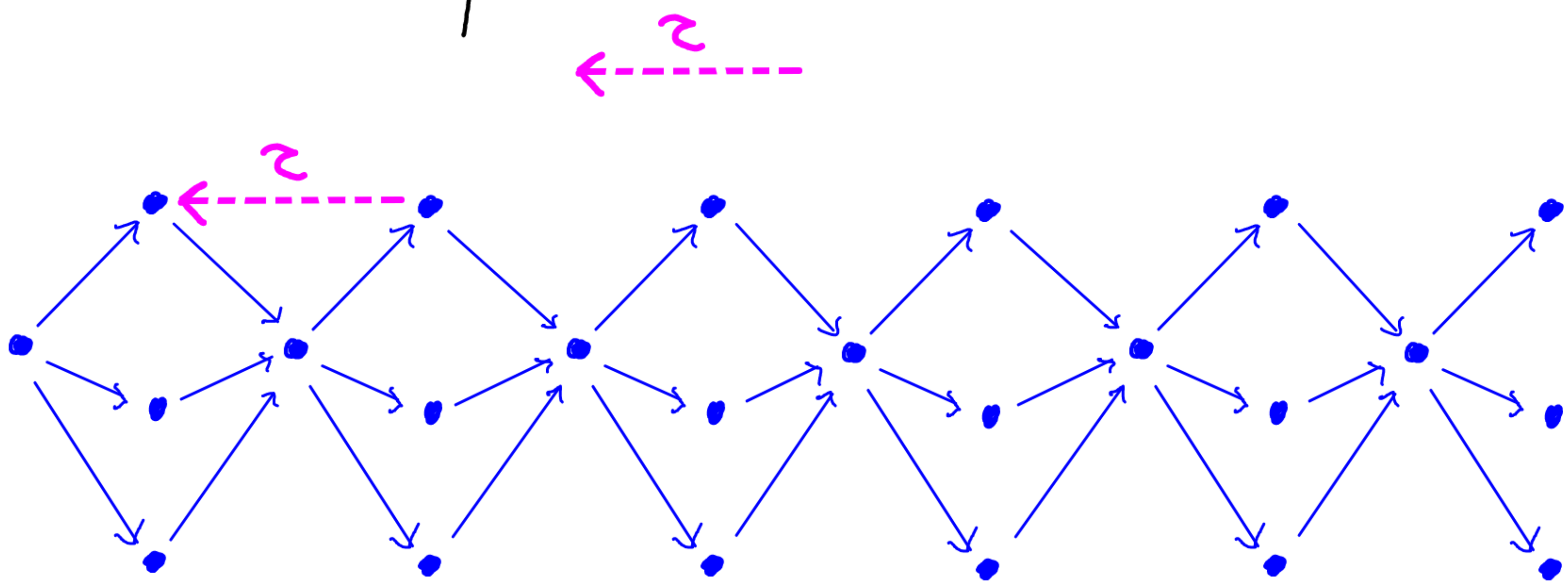
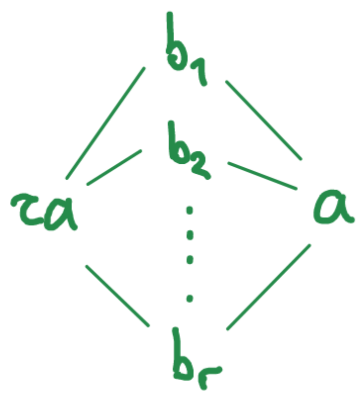


Friezes from quivers (without oriented cycles)

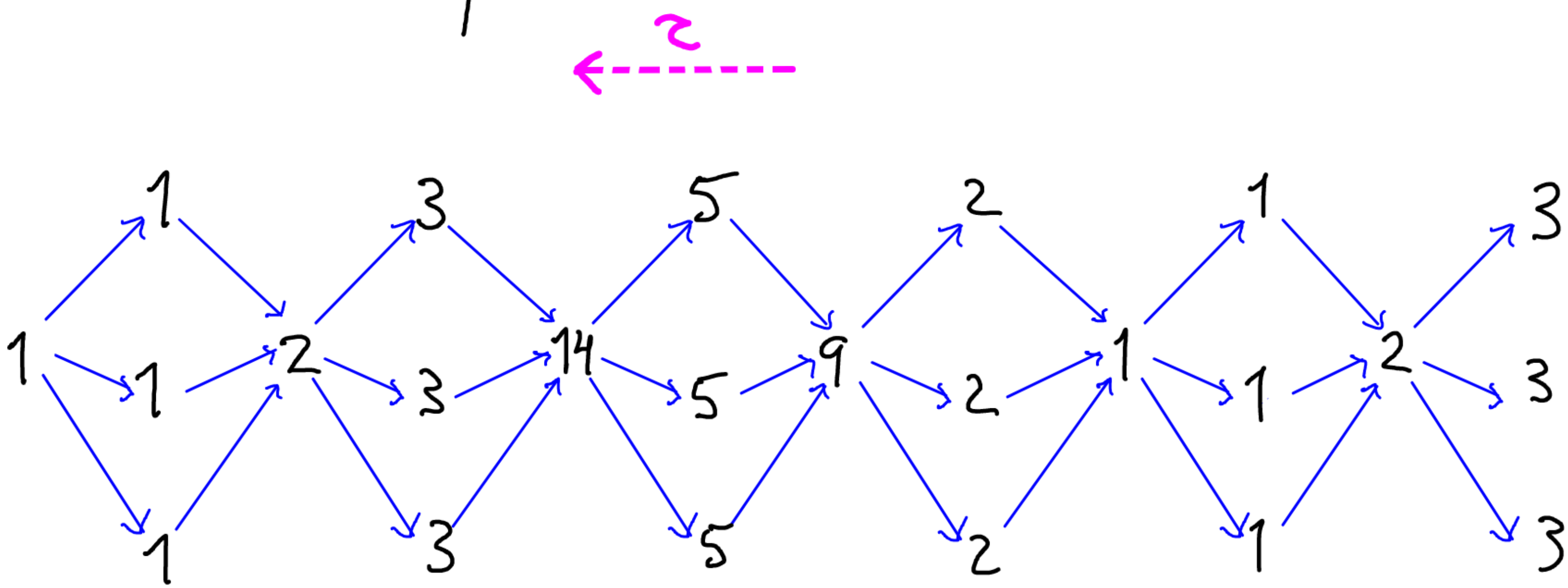


Rule:

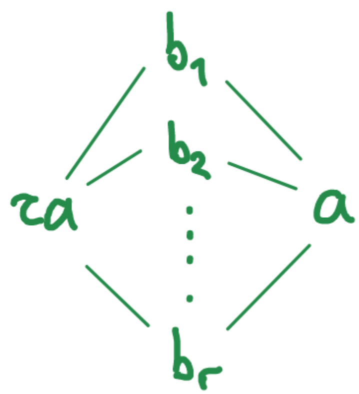


$$\tau a = \frac{\prod_{i=1}^{r} b_i + 1}{a}$$

Friezes from quivers (without oriented cycles)



Rule:



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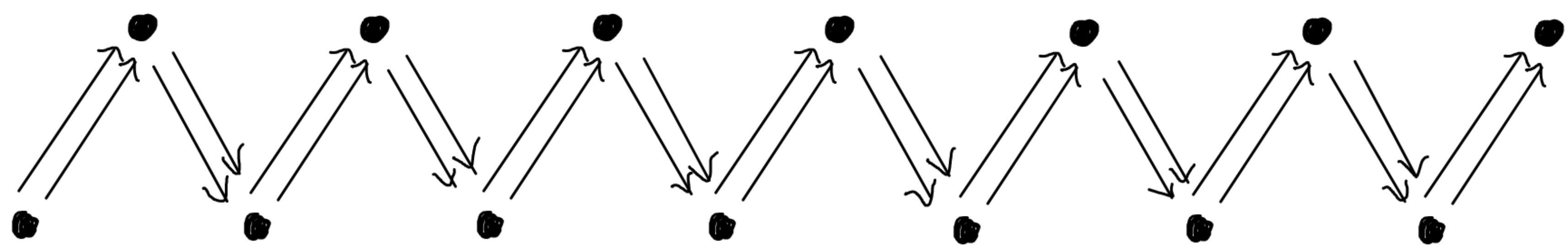
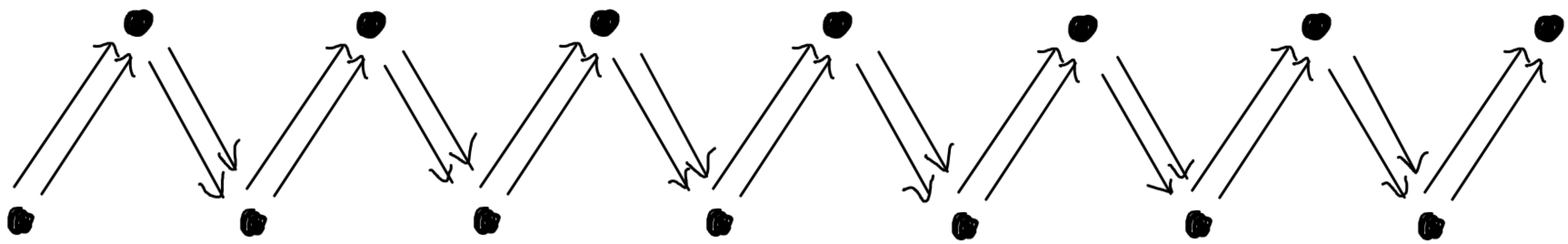


Friezes from quivers (without oriented cycles)



Rule:

$$\begin{tikzcd} & b \arrow[dr, shift left] \arrow[dr, shift right] & \\ \tau a \arrow[ur, shift left] \arrow[ur, shift right] & & a \end{tikzcd} \qquad \tau a = \frac{b^2+1}{a}$$

Friezes from quivers (without oriented cycles)

$$\begin{tikzcd}[column sep=small]
& 1 \arrow[dr, shift left] \arrow[dr, shift right] & & 5 \arrow[dr, shift left] \arrow[dr, shift right] & & 34 \arrow[dr, shift left] \arrow[dr, shift right] & & 233 \arrow[dr, shift left] \arrow[dr, shift right] & & 1597 \arrow[dr, shift left] \arrow[dr, shift right] & & 10946 \arrow[dr, shift left] \arrow[dr, shift right] & & \cdots \\
1 \arrow[ur, shift left] \arrow[ur, shift right] & & 2 \arrow[ur, shift left] \arrow[ur, shift right] & & 13 \arrow[ur, shift left] \arrow[ur, shift right] & & 89 \arrow[ur, shift left] \arrow[ur, shift right] & & 610 \arrow[ur, shift left] \arrow[ur, shift right] & & 4181 \arrow[ur, shift left] \arrow[ur, shift right] & & \cdots \arrow[ur, shift left] \arrow[ur, shift right] &
\end{tikzcd}$$

Rule:

$$\begin{tikzcd}
& b \arrow[dr, shift left] \arrow[dr, shift right] & \\
\tau a \arrow[ur, shift left] \arrow[ur, shift right] & & a
\end{tikzcd}
\qquad \tau a = \frac{b^2+1}{a}$$

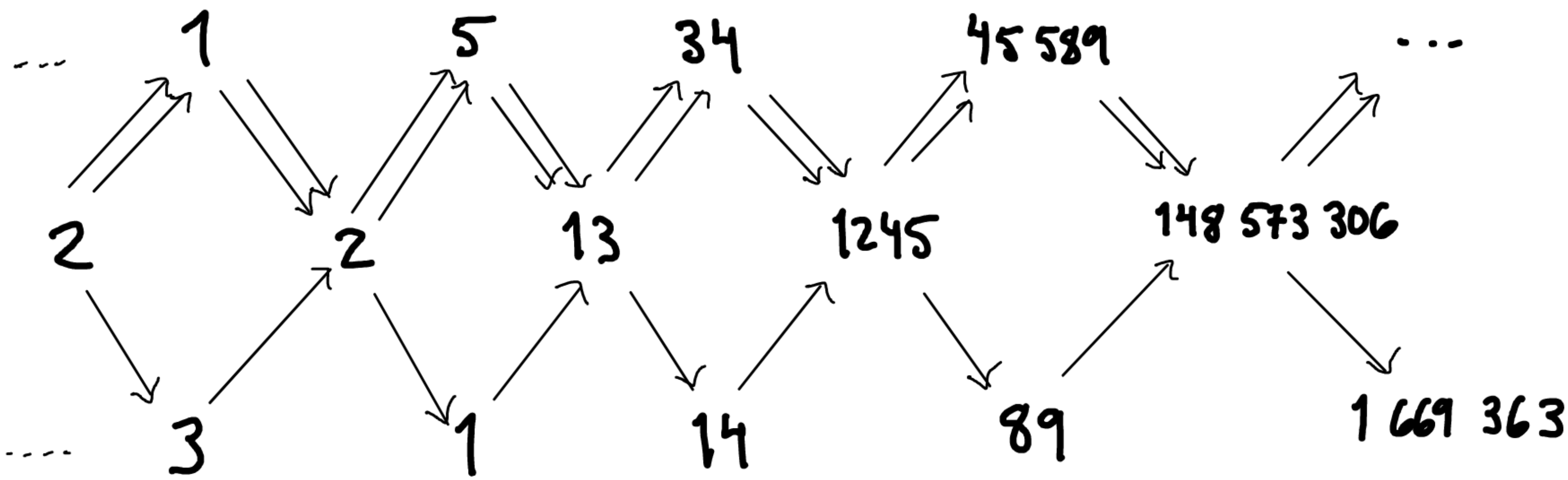
Friezes from quivers (without oriented cycles)

$$\begin{tikzcd} & 5 \\ 2 \arrow[ur, shift left] \arrow[ur, shift right] & \end{tikzcd}$$

Friezes from quivers (without oriented cycles)

$$
\begin{tikzcd}
 & 5 \\
2 \arrow[ur, shift left] \arrow[ur, shift right] \arrow[dr] & \\
 & 1
\end{tikzcd}
$$

Friezes from quivers (without oriented cycles)



Theorem [Keller – P. – Qin]

Let
- Q be an acyclic quiver
- $*$ be a vertex of Q
- $Q' = Q \setminus \{*\}$.

If $f : Q'_0 \longrightarrow \mathbb{N}$ defines a frieze, then putting $f(*) = 1$ defines a frieze on Q.

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This "reverses" a reduction procedure studied by Baur–Faber–Gratz–Serhiyenko–Todorov