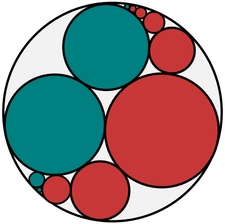
Continued fractions, SL_2-tilings, and the Farey graph

Ian Short



Monday 25 March 2024

With Margaret Stanier, Matty van Son, and Andrei Zabolotskii
EPSRC EP/W002817/1 & EP/W524098/1

The Open University

Integer continued fractions

Euclid's algorithm

$$\frac{31}{13}$$

Euclid's algorithm

$$\frac{31}{13} = 2 + \frac{5}{13}$$

Euclid's algorithm

$$
\begin{aligned}
\frac{31}{13} &= 2 + \frac{5}{13} \\
&= 2 + \frac{1}{\frac{13}{5}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{31}{13} &= 2 + \frac{5}{13} \\
&= 2 + \frac{1}{\frac{13}{5}} \\
&= 2 + \frac{1}{2 + \frac{3}{5}}
\end{aligned}
$$

Euclid's algorithm

$$\frac{31}{13} = 2 + \frac{5}{13} = 2 + \frac{1}{\frac{13}{5}} = 2 + \frac{1}{2 + \frac{1}{\frac{5}{3}}}$$

Euclid's algorithm

$$
\begin{aligned}
\frac{31}{13} &= 2 + \frac{5}{13} \\
&= 2 + \frac{1}{\frac{13}{5}} \\
&= 2 + \cfrac{1}{2 + \frac{1}{\frac{5}{3}}} \\
&= 2 + \cfrac{1}{2 + \cfrac{1}{1 + \frac{2}{3}}}
\end{aligned}
$$

Euclid's algorithm

$$
\begin{aligned}
\frac{31}{13} &= 2 + \frac{5}{13} \\
&= 2 + \frac{1}{\frac{13}{5}} \\
&= 2 + \frac{1}{2 + \frac{1}{\frac{5}{3}}} \\
&= 2 + \frac{1}{2 + \frac{1}{1 + \frac{1}{\frac{3}{2}}}}
\end{aligned}
$$

$$\frac{31}{13} = 2 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2}}}}$$

$$\frac{31}{13}$$

The nearest-integer algorithm

$$\frac{31}{13} = 2 + \frac{1}{\frac{13}{5}}$$

$$\begin{aligned}\frac{31}{13} &= 2+\frac{1}{\frac{13}{5}}\\ &= 2+\cfrac{1}{3-\cfrac{2}{5}}\end{aligned}$$

The nearest-integer algorithm

$$\frac{31}{13} = 2 + \frac{1}{\frac{13}{5}}$$

$$= 2 + \cfrac{1}{3 + \cfrac{1}{-\frac{5}{2}}}$$

The nearest-integer algorithm

$$\begin{aligned}
\frac{31}{13} &= 2+\frac{1}{\frac{13}{5}} \\
&= 2+\cfrac{1}{3+\cfrac{1}{-\frac{5}{2}}} \\
&= 2+\cfrac{1}{3+\cfrac{1}{-3+\cfrac{1}{2}}}
\end{aligned}$$

Another expansion

$$\frac{31}{13} = 3 + \cfrac{1}{-2 + \cfrac{1}{3 + \cfrac{1}{-3}}}$$

Integer continued fractions

Positive integer continued fractions

Finite continued fractions for rationals.

Infinite continued fractions for irrationals.

Unique expansions in both cases.

$$b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cfrac{1}{b_4 + \cdots}}}$$

Integer continued fractions

Finite continued fractions for rationals.

Infinite continued fractions may represent rational or irrationals, or may diverge.

No uniqueness.

Minus continued fractions

$$b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cfrac{1}{b_4 + \cdots}}} \quad \longrightarrow \quad b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 - \cfrac{1}{b_4 - \cdots}}}$$

$$\frac{3}{4} = 2 + \cfrac{1}{-3 + \cfrac{1}{1 + \cfrac{1}{-2 + \cfrac{1}{6}}}} \quad \longrightarrow \quad \frac{3}{4} = 2 - \cfrac{1}{3 - \cfrac{1}{1 - \cfrac{1}{2 - \cfrac{1}{6}}}}$$

Continued fraction approximants

Convergents

$$\frac{A_n}{B_n} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 - \cfrac{1}{b_4 - \cdots - \cfrac{1}{b_n}}}}$$

Calculating convergents

$$\begin{pmatrix} A_n & A_{n-1} \\ B_n & B_{n-1} \end{pmatrix} = \begin{pmatrix} b_1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b_2 & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} b_n & -1 \\ 1 & 0 \end{pmatrix}$$

Modular group

All these matrices belong to the *modular group*

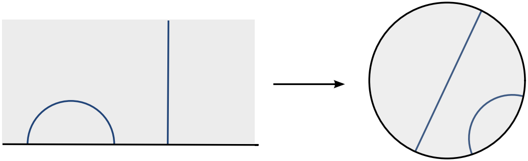
$$\mathsf{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

The hyperbolic plane

Definition The *hyperbolic plane* is the upper half-plane

$$\mathbb{H} = \{z : \operatorname{Im} z > 0\}$$

endowed with the Riemannian metric $\dfrac{|dz|}{\operatorname{Im} z}$.



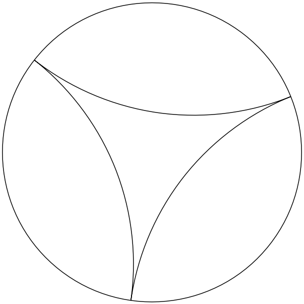
$$\frac{|dz|}{\operatorname{Im} z} \qquad\qquad \frac{2|dz|}{1-|z|^2}$$

Remark The group $\mathsf{SL}_2(\mathbb{Z})$ acts on $\mathbb{H}$ as a group of isometries.

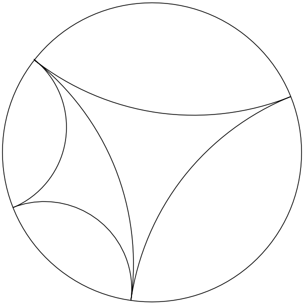
Definition The *ideal boundary* of $\mathbb{H}$ is $\mathbb{R} \cup \{\infty\}$. It is not part of the hyperbolic plane.

Ideal triangles

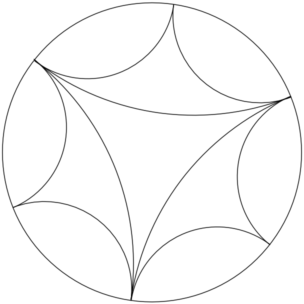
Ideal triangles



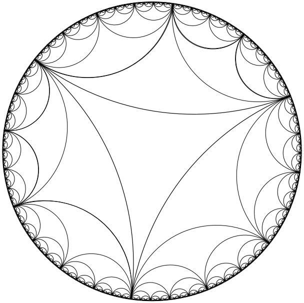
Ideal triangles



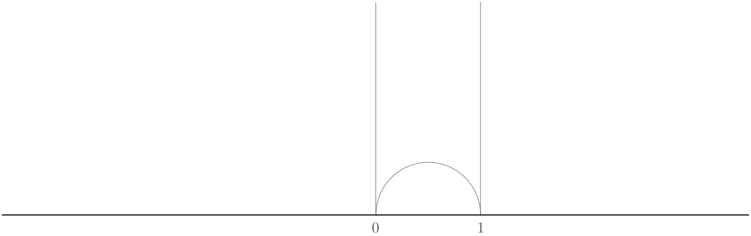
Ideal triangles



Ideal triangles



Farey graph



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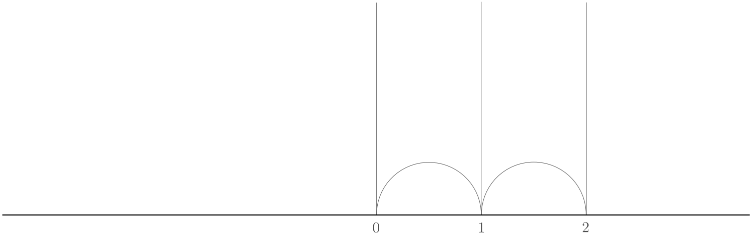

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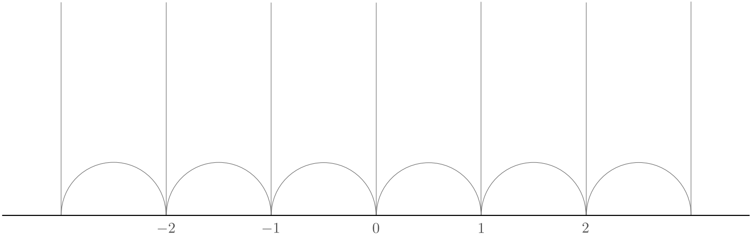

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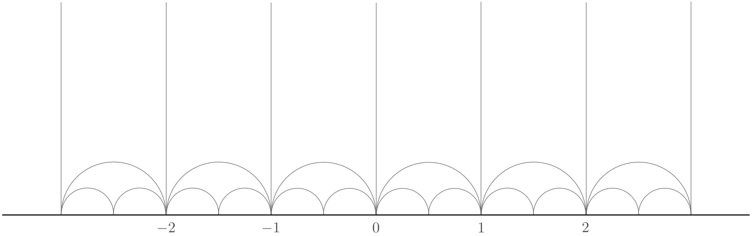
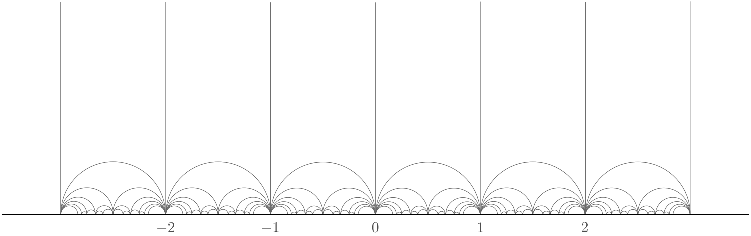
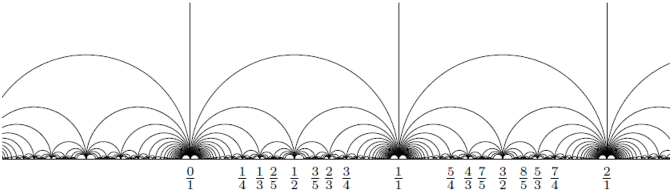


Farey graph



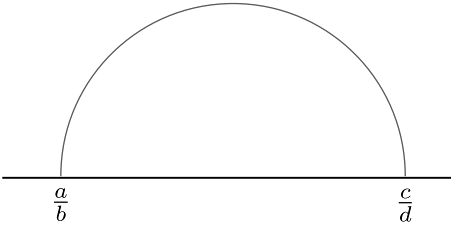
Farey graph



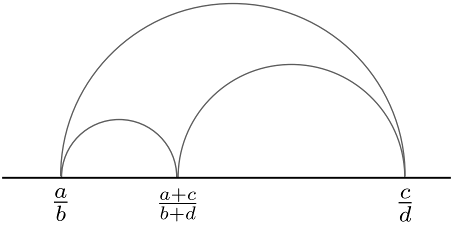
Definition The *Farey graph* is the graph with vertices $\mathbb{Q} \cup \{\infty\}$ and with edges comprising pairs of vertices a/b and c/d that satisfy $ad - bc = \pm 1$.

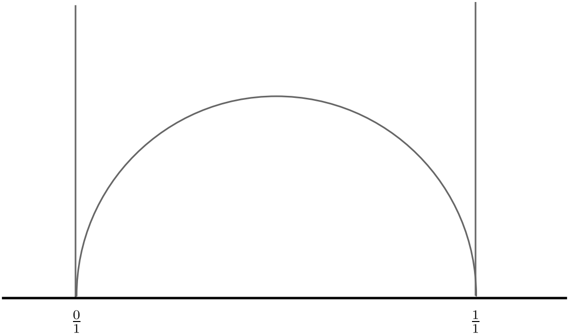
The edges are represented by hyperbolic lines.

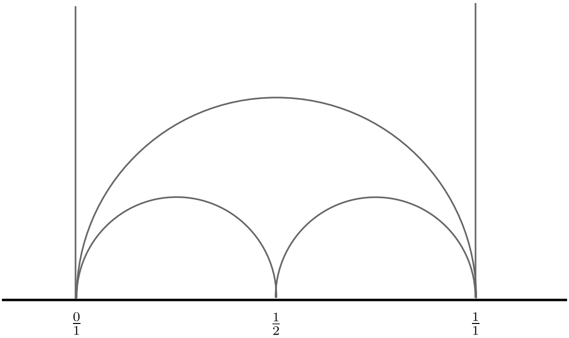
Farey addition

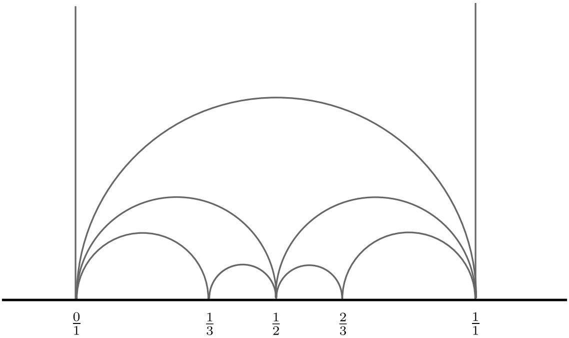


Farey addition

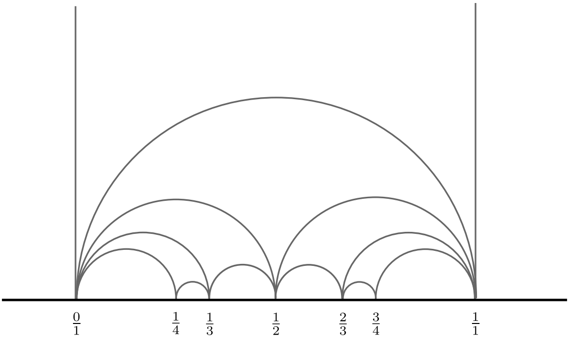



$\frac{0}{1}$
$\frac{1}{1}$

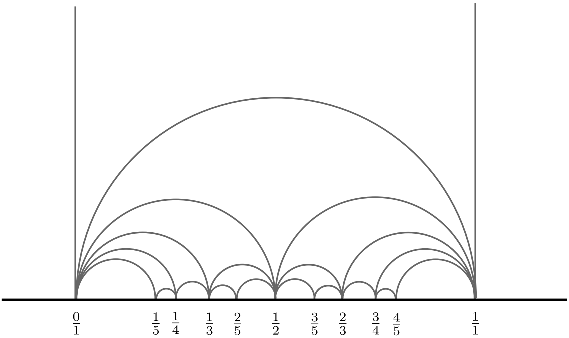

$\frac{0}{1}$
$\frac{1}{2}$
$\frac{1}{1}$



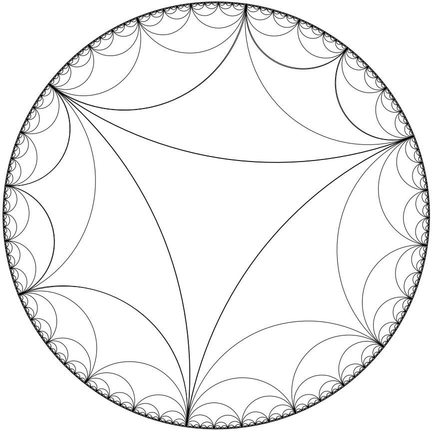
$\frac{0}{1}$
$\frac{1}{3}$
$\frac{1}{2}$
$\frac{2}{3}$
$\frac{1}{1}$


$\frac{0}{1}$
$\frac{1}{4}$
$\frac{1}{3}$
$\frac{1}{2}$
$\frac{2}{3}$
$\frac{3}{4}$
$\frac{1}{1}$

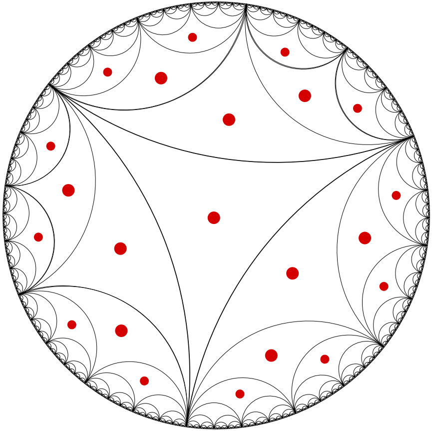
Farey sequences



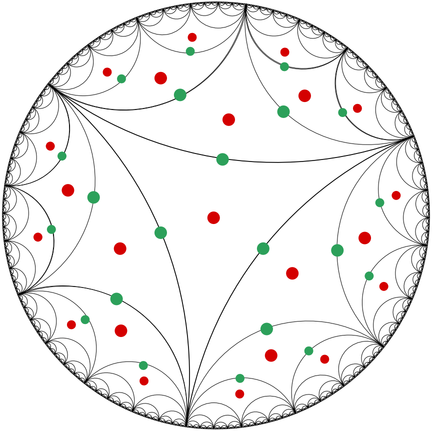
Automorphisms of the Farey graph



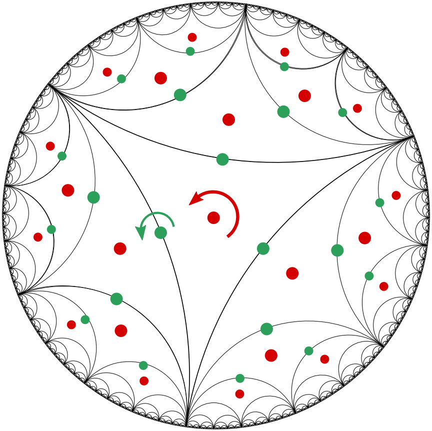
Automorphisms of the Farey graph

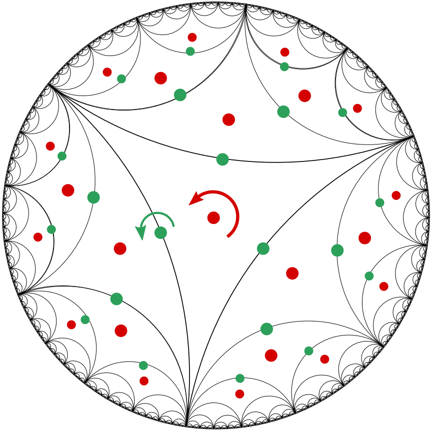


Automorphisms of the Farey graph



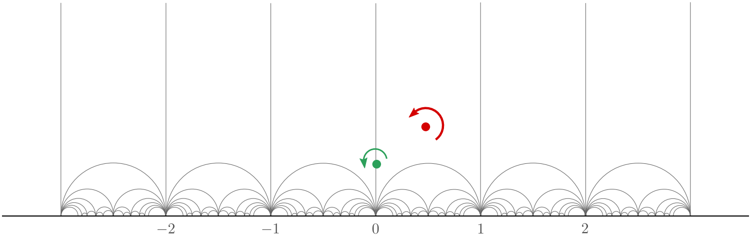
Automorphisms of the Farey graph

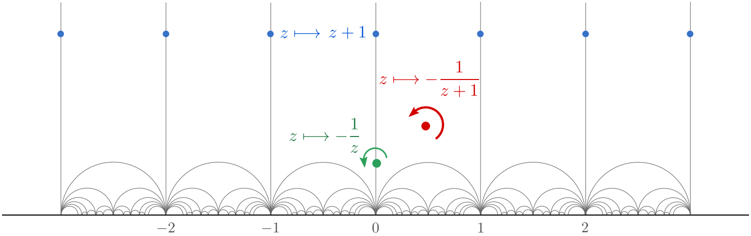


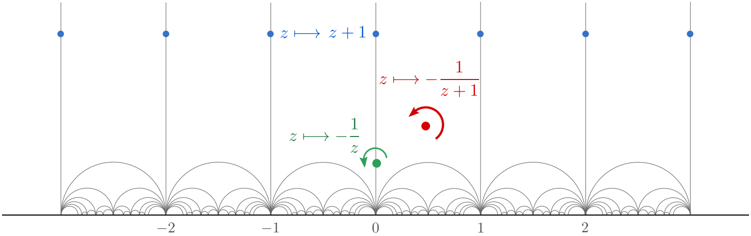


Automorphism group $\cong C_2 * C_3$

Automorphisms of the Farey graph




$z \longmapsto z+1$
$z \longmapsto -\frac{1}{z+1}$
$z \longmapsto -\frac{1}{z}$
−2
−1
0
1
2

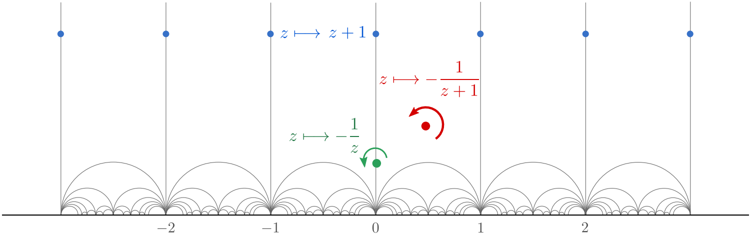


Observation The matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

generate the group $\mathsf{SL}_2(\mathbb{Z})$ (modulo $\pm I$).

Automorphisms of the Farey graph



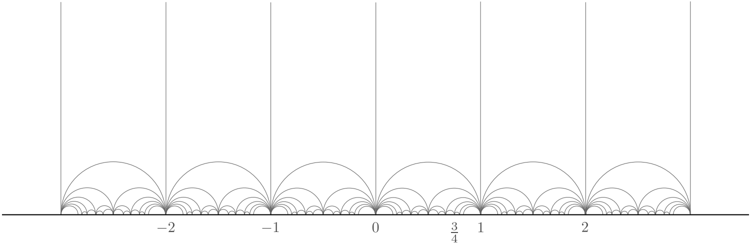
Observation   The matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

generate the group $\mathrm{SL}_2(\mathbb{Z})$ (modulo $\pm I$).

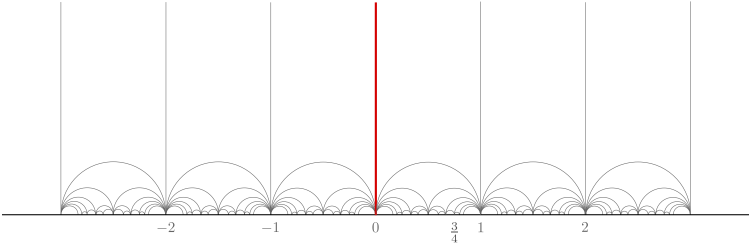
Key property $\mathrm{SL}_2(\mathbb{Z})$ is the group of orientation preserving automorphisms of the Farey graph.

It acts transitively on directed edges.



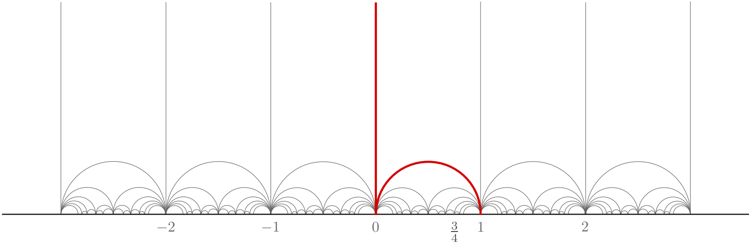
$$\frac{3}{4} = 0 - \cfrac{1}{-1 - \cfrac{1}{2 - \cfrac{1}{-1}}}$$

Paths in the Farey graph



$$\frac{3}{4} = 0 - \cfrac{1}{-1 - \cfrac{1}{2 - \cfrac{1}{-1}}}$$

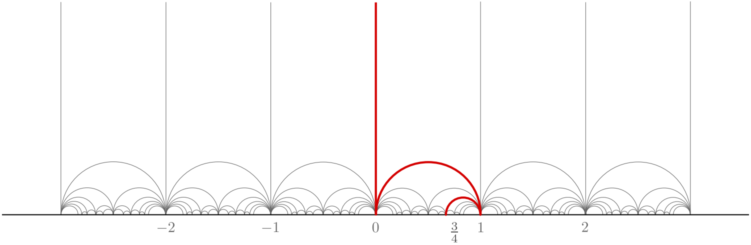
$$\frac{A_1}{B_1} = 0,$$



$$\frac{3}{4} = 0 - \cfrac{1}{-1 - \cfrac{1}{2 - \cfrac{1}{-1}}}$$

$$\frac{A_1}{B_1} = 0, \quad \frac{A_2}{B_2} = -\frac{1}{-1} = 1,$$

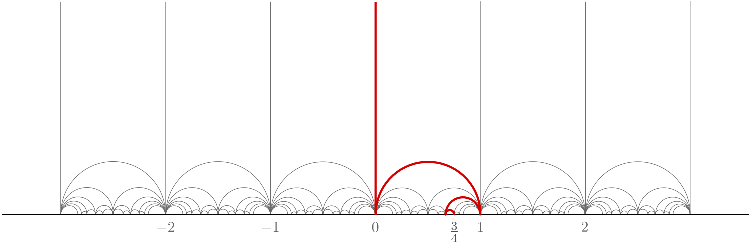
Paths in the Farey graph



$$\frac{3}{4} = 0 - \cfrac{1}{-1 - \cfrac{1}{2 - \cfrac{1}{-1}}}$$

$$\frac{A_1}{B_1} = 0, \quad \frac{A_2}{B_2} = -\frac{1}{-1} = 1, \quad \frac{A_3}{B_3} = -\cfrac{1}{-1 - \cfrac{1}{2}} = \frac{2}{3},$$

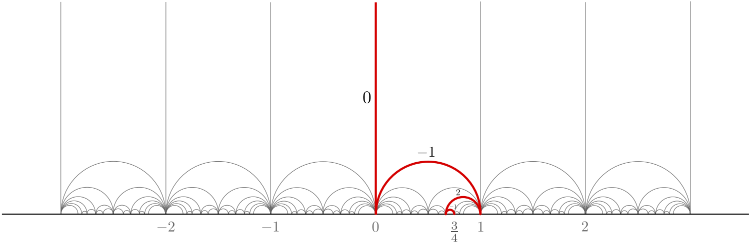
Paths in the Farey graph



$$\frac{3}{4} = 0 - \cfrac{1}{-1 - \cfrac{1}{2 - \cfrac{1}{-1}}}$$

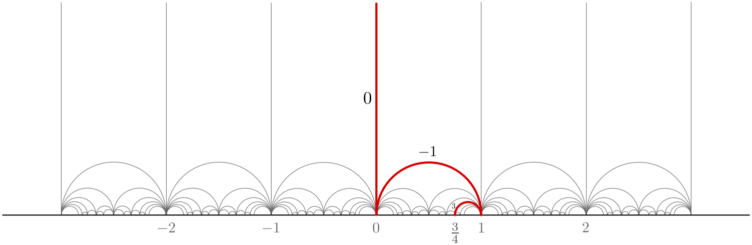
$$\frac{A_1}{B_1} = 0, \quad \frac{A_2}{B_2} = -\frac{1}{-1} = 1, \quad \frac{A_3}{B_3} = -\cfrac{1}{-1 - \cfrac{1}{2}} = \frac{2}{3}, \quad \frac{A_4}{B_4} = -\cfrac{1}{-1 - \cfrac{1}{2 - \cfrac{1}{-1}}} = \frac{3}{4}$$

Paths in the Farey graph



$$\frac{3}{4} = 0 - \cfrac{1}{-1 - \cfrac{1}{2 - \cfrac{1}{-1}}}$$

A.F. Beardon, M. Hockman, & I. Short, *Michigan Math. J.*, 2012



$$\frac{3}{4} = 0 - \cfrac{1}{-1 - \cfrac{1}{3}}$$

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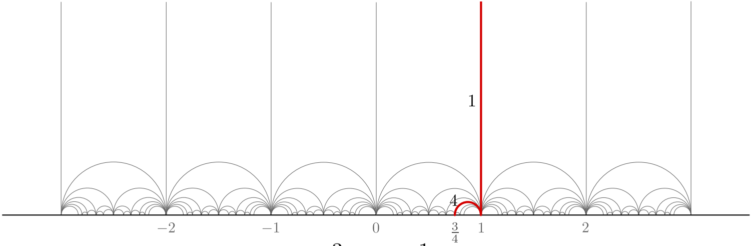
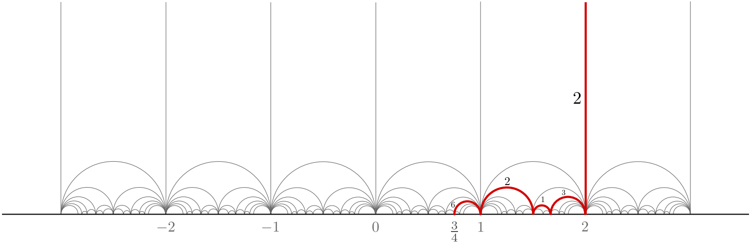


$$\frac{3}{4} = 1 - \frac{1}{4}$$

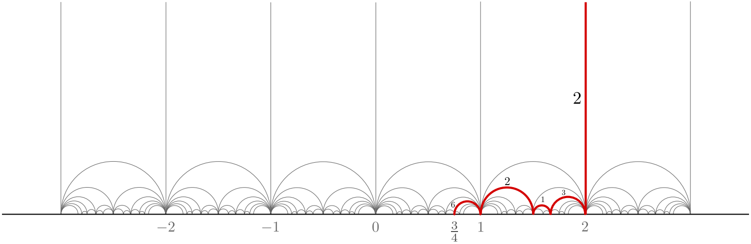
A.F. Beardon, M. Hockman, & I. Short, *Michigan Math. J.*, 2012



$$\frac{3}{4} = 2 - \cfrac{1}{3 - \cfrac{1}{1 - \cfrac{1}{2 - \cfrac{1}{6}}}}$$

A.F. Beardon, M. Hockman, & I. Short, *Michigan Math. J.*, 2012

Paths in the Farey graph

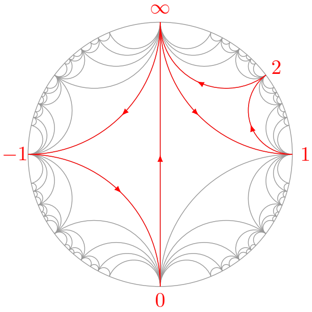


$$\frac{3}{4} = 2 - \cfrac{1}{3 - \cfrac{1}{1 - \cfrac{1}{2 - \cfrac{1}{6}}}}$$

Theorem For $n \in \mathbb{N} \cup \{\infty\}$, there is a one-to-one corresondence

$$\left\{ \begin{array}{c} \text{integer continued} \\ \text{fractions of length } n \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{paths of length } n \\ \text{beginning at } \infty \end{array} \right\}.$$

A.F. Beardon, M. Hockman, & I. Short, *Michigan Math. J.*, 2012



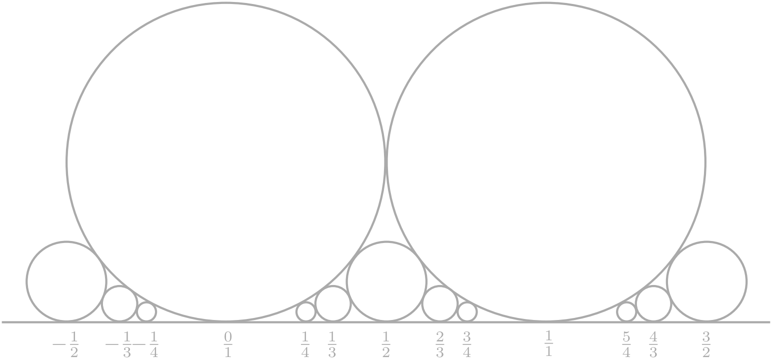
Theorem There are one-to-one correspondences

$$\mathsf{SL}_2(\mathbb{Z})\Big\backslash\left\{\begin{matrix}\text{closed paths}\\ \text{of length } n\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{continued fractions of length } n \text{ with}\\ \text{final two convergents } 0 \text{ and } \infty\end{matrix}\right\}$$

$$\longleftrightarrow \left\{[b_1, b_2, \dots, b_n] \in \mathbb{Z}^n : \begin{pmatrix} b_1 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} b_2 & -1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} b_n & -1 \\ 1 & 0 \end{pmatrix} = \pm I\right\}.$$

S. Morier-Genoud & V. Ovsienko, *Jahresber. Dtsch. Math. Ver.*, 2019

Chains of Ford circles



L.R. Ford, *Amer. Math. Monthly*, 1938

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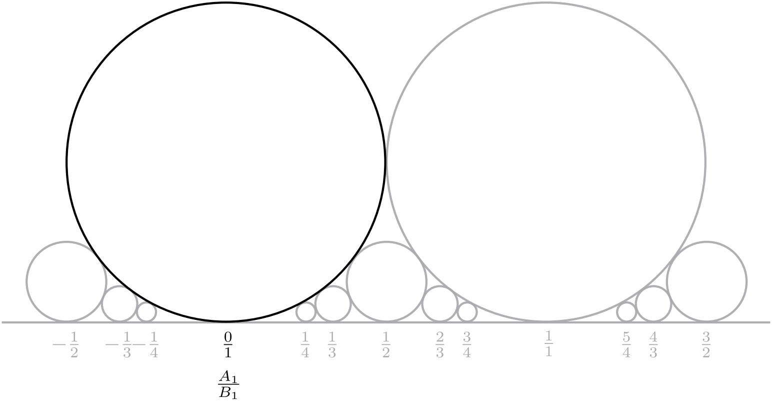

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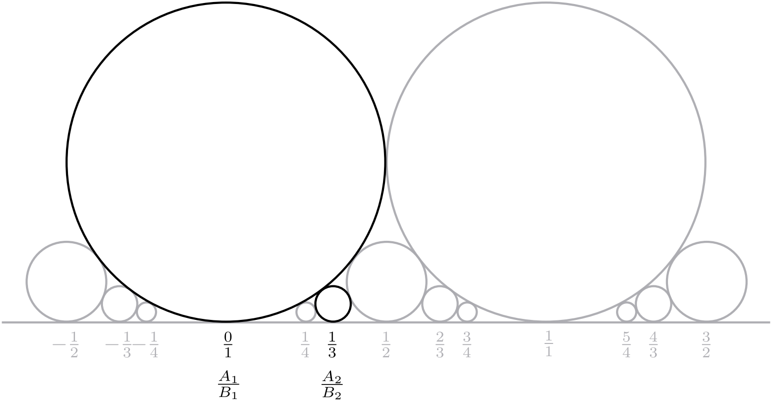

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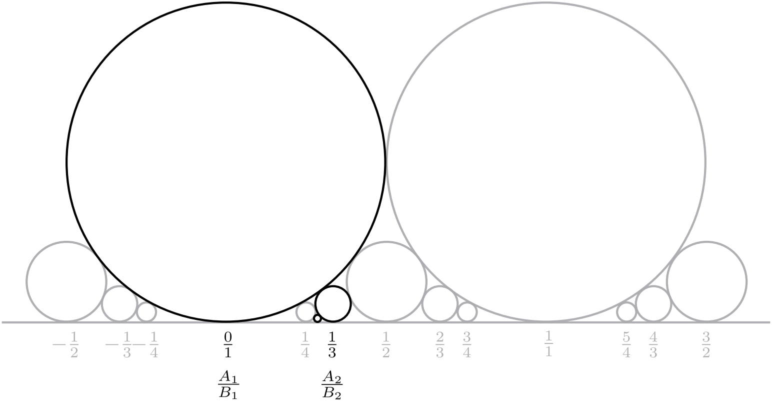

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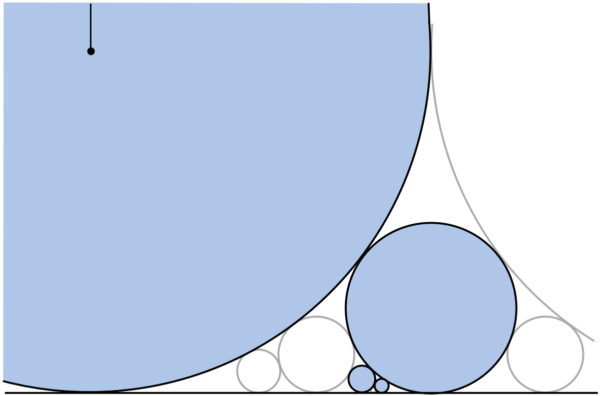

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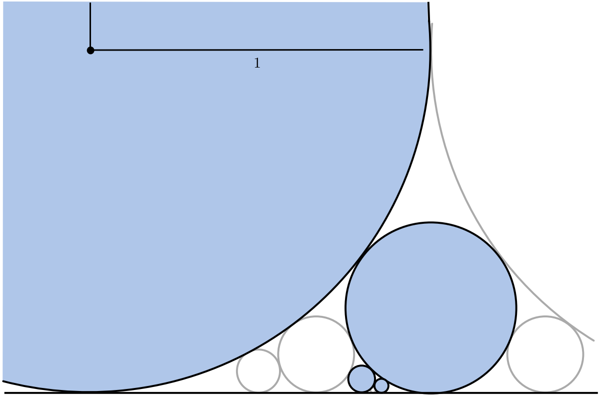
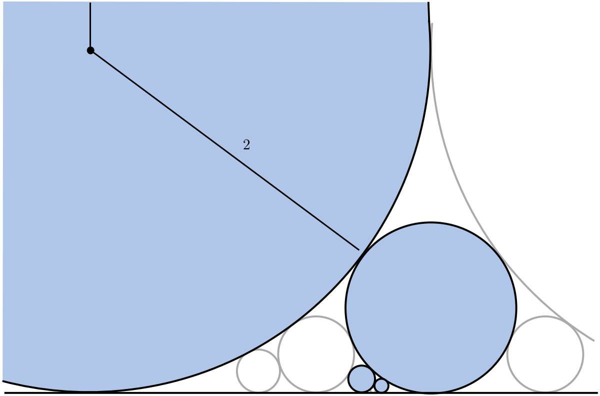


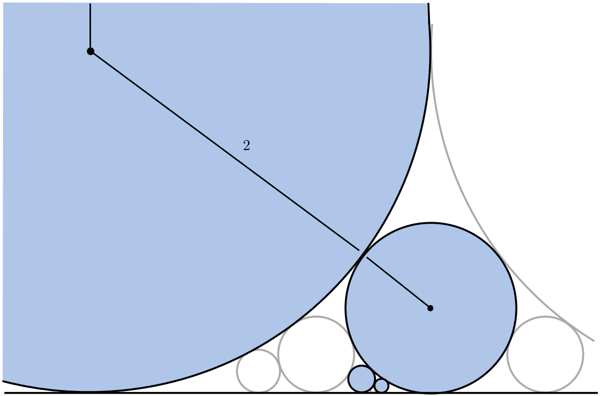
L.R. Ford, *Amer. Math. Monthly*, 1938

Chains of Ford circles



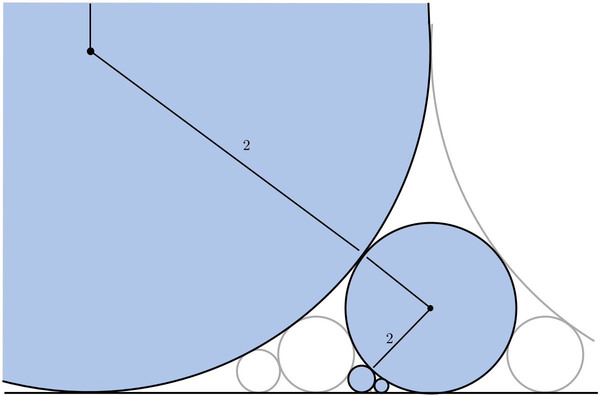
L.R. Ford, *Amer. Math. Monthly*, 1938

Chains of Ford circles



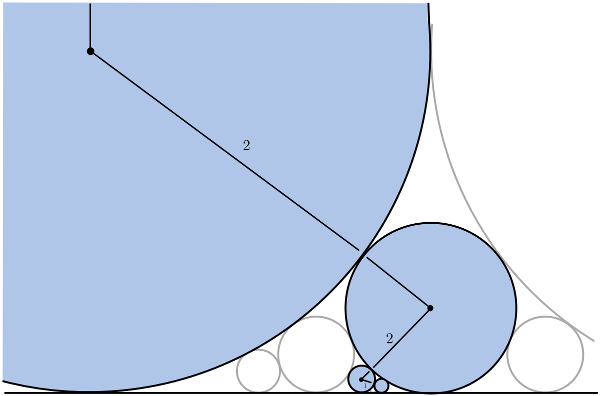
L.R. Ford, *Amer. Math. Monthly*, 1938

Chains of Ford circles



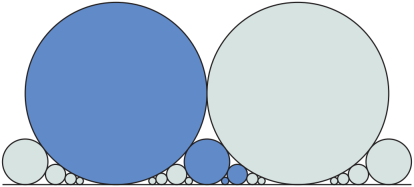
L.R. Ford, *Amer. Math. Monthly*, 1938

Chains of Ford circles



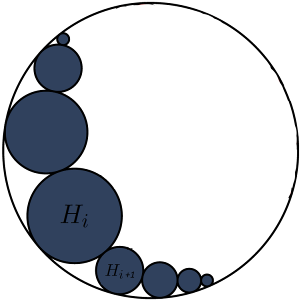
L.R. Ford, *Amer. Math. Monthly*, 1938

Chains of Ford circles



Theorem For $n \in \mathbb{N} \cup \{\infty\}$, there is a one-to-one correspondence

$$\left\{ \begin{array}{c} \text{integer continued} \\ \text{fractions of length } n \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{chains of Ford circles of} \\ \text{length } n \text{ beginning at } \infty \end{array} \right\}.$$



Theorem For $n \in \mathbb{N} \cup \{\infty\}$, there is a one-to-one correspondence

$$\left\{ \begin{array}{c} \text{real continued fractions} \\ \text{of length } n \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{chains of horocycles of} \\ \text{length } n \text{ beginning at } \infty \end{array} \right\}.$$

A.F. Beardon and I. Short, *Amer. Math. Monthly*, 2014

Coxeter's frieze patterns

$$
\begin{array}{ccccccccccccccccccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
 & 1 & & 2 & & 2 & & 3 & & 1 & & 2 & & 4 & & 1 & & 2 & & 2 & & 3 & & 1 & & & \\
 & & 1 & & 3 & & 5 & & 2 & & 1 & & 7 & & 3 & & 1 & & 3 & & 5 & & 2 & & 1 & \\
\cdots & 2 & & 1 & & 7 & & 3 & & 1 & & 3 & & 5 & & 2 & & 1 & & 7 & & 3 & & 1 & & & \cdots \\
 & & 1 & & 2 & & 4 & & 1 & & 2 & & 2 & & 3 & & 1 & & 2 & & 4 & & 1 & & 2 & \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & & \\
 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 &
\end{array}
$$

H.S.M. Coxeter, *Acta Arith.*, 1971

$$\cdots\quad
\begin{array}{cccccccccccccccccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
& 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
1 & & 2 & & 2 & & 3 & & 1 & & 2 & & 4 & & 1 & & 2 & & 2 & & 3 & & 1 & \\
& 1 & & 3 & & 5 & & 2 & & 1 & & 7 & & 3 & & 1 & & 3 & & 5 & & 2 & & 1 \\
2 & & 1 & & 7 & & 3 & & 1 & & 3 & & 5 & & 2 & & 1 & & 7 & & 3 & & 1 & \\
& 1 & & 2 & & 4 & & 1 & & 2 & & 2 & & 3 & & 1 & & 2 & & 4 & & 1 & & 2 \\
1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
& 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0
\end{array}
\quad\cdots$$

$$\begin{array}{ccc}
& b & \\
a & & d \\
& c &
\end{array}
\qquad\qquad ad - bc = 1$$

H.S.M. Coxeter, *Acta Arith.*, 1971

$$\begin{array}{ccccccccccccc} & 0 & & 0 & & 0 & & 0 & & 0 & & & \\ & & 1 & & 1 & & 1 & & 1 & & 1 & & \\ \cdots & 2 & & 1 & & 2 & & 1 & & 2 & & \cdots & \\ & & 1 & & 1 & & 1 & & 1 & & 1 & & \\ & 0 & & 0 & & 0 & & 0 & & 0 & & & \end{array}$$

$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$$

H.S.M. Coxeter, *Acta Arith.*, 1971

$$\begin{array}{ccccccccccccc} & 0 & & 0 & & 0 & & 0 & & 0 & & & \\ & & 1 & & 1 & & 1 & & 1 & & 1 & & \\ \cdots & 2 & & 1 & & 2 & & 1 & & 2 & & \cdots & \\ & & 1 & & 1 & & 1 & & 1 & & 1 & & \\ & 0 & & 0 & & 0 & & 0 & & 0 & & & \end{array} \qquad \begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$$

Definition An infinite strip of integers of this type is called a *positive integer frieze*.

H.S.M. Coxeter, *Acta Arith.*, 1971

Coxeter's friezes

$$
\begin{array}{ccccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & & & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & & \\
\cdots & 2 & & 1 & & 2 & & 1 & & 2 & & \cdots & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & &
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & b & \\
a & & d \\
 & c &
\end{array}
$$

Definition An infinite strip of integers of this type is called a *positive integer frieze*.

Theorem Every positive integer frieze is periodic.

H.S.M. Coxeter, *Acta Arith.*, 1971

$$\begin{array}{ccccccccccccc} & 0 & & 0 & & 0 & & 0 & & 0 & & & \\ & & 1 & & 1 & & 1 & & 1 & & 1 & & \\ \cdots & 2 & & 1 & & 2 & & 1 & & 2 & & \cdots & \\ & & 1 & & 1 & & 1 & & 1 & & 1 & & \\ & 0 & & 0 & & 0 & & 0 & & 0 & & & \end{array} \qquad \begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$$

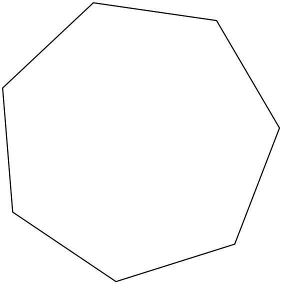
Definition An infinite strip of integers of this type is called a *positive integer frieze*.

Theorem Every positive integer frieze is periodic.

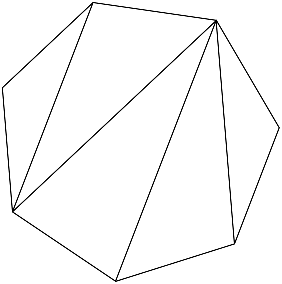
Observation Each positive integer frieze is determined by its *quiddity cycle*, the periodic part of its third row.

H.S.M. Coxeter, *Acta Arith.*, 1971

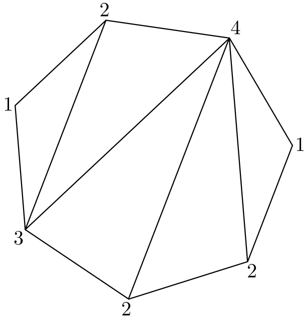
Triangulated polygons



Triangulated polygons



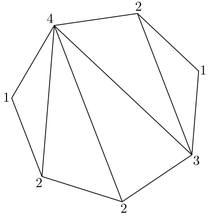
Triangulated polygons



Conway's insight

$$
\cdots\quad
\begin{array}{cccccccccccccccccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
& 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
1 & & 2 & & 2 & & 3 & & 1 & & 2 & & 4 & & 1 & & 2 & & 2 & & 3 & & 1 & \\
& 1 & & 3 & & 5 & & 2 & & 1 & & 7 & & 3 & & 1 & & 3 & & 5 & & 2 & & 1 \\
2 & & 1 & & 7 & & 3 & & 1 & & 3 & & 5 & & 2 & & 1 & & 7 & & 3 & & 1 & \\
& 1 & & 2 & & 4 & & 1 & & 2 & & 2 & & 3 & & 1 & & 2 & & 4 & & 1 & & 2 \\
1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
& 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0
\end{array}
\quad\cdots
$$

width $= 7$
period $= 7$



Theorem There is a one-to-one correspondence

$$\left\{\begin{array}{c}\text{positive integer friezes}\\ \text{of width } n\end{array}\right\} \quad\longleftrightarrow\quad \{\text{triangulated } n\text{-gons}\}.$$

J.H. Conway & H.S.M. Coxeter, *Math. Gaz.*, 1973

SL_2-tilings

SL_2-tilings

$$\begin{array}{ccccccccccc} 0 & & 0 & & 0 & & 0 & & 0 & & \\ & 1 & & 1 & & 1 & & 1 & & 1 \\ 2 & & 1 & & 2 & & 1 & & 2 & & \\ & 1 & & 1 & & 1 & & 1 & & 1 \\ 0 & & 0 & & 0 & & 0 & & 0 & & \end{array} \xrightarrow{\text{rotate } 45^\circ} \begin{array}{cccccccc} 0 & 1 & 1 & 1 & 0 & & & \\ & 0 & 1 & 2 & 1 & 0 & & \\ & & 0 & 1 & 1 & 1 & 0 & \\ & & & 0 & 1 & 2 & 1 & 0 \end{array}$$

$$
\begin{array}{cccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & \\
 & 1 & & 1 & & 1 & & 1 & & 1 \\
2 & & 1 & & 2 & & 1 & & 2 & \\
 & 1 & & 1 & & 1 & & 1 & & 1 \\
0 & & 0 & & 0 & & 0 & & 0 &
\end{array}
\xrightarrow{\text{rotate } 45^\circ}
\begin{array}{cccccccc}
0 & 1 & 1 & 1 & 0 & & & \\
 & 0 & 1 & 2 & 1 & 0 & & \\
 & & 0 & 1 & 1 & 1 & 0 & \\
 & & & 0 & 1 & 2 & 1 & 0
\end{array}
$$

$$
\begin{array}{cccccccccccccccccc}
 & & & & & & & & \vdots & & & & & & & & & \\
 & \color{red}0 & 1 & 1 & 1 & \color{red}0 & -1 & -1 & -1 & \color{red}0 & 1 & 1 & 1 & \color{red}0 & & & & \\
\cdots & & \color{red}0 & 1 & 2 & 1 & \color{red}0 & -1 & -2 & -1 & \color{red}0 & 1 & 2 & 1 & \color{red}0 & & & \cdots \\
 & & & \color{red}0 & 1 & 1 & 1 & \color{red}0 & -1 & -1 & -1 & \color{red}0 & 1 & 1 & 1 & \color{red}0 & & \\
 & & & & \color{red}0 & 1 & 2 & 1 & \color{red}0 & -1 & -2 & -1 & \color{red}0 & 1 & 2 & 1 & \color{red}0 & \\
 & & & & & & & & & \vdots & & & & & & & &
\end{array}
$$

Definition Let R be a commutative ring with multiplicative identity 1, and let

$$\mathsf{SL}_2(R) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in R, ad - bc = 1 \right\}.$$

Definition Let R be a commutative ring with multiplicative identity 1, and let

$$\mathrm{SL}_2(R) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in R, ad - bc = 1 \right\}.$$

Definition An *SL_2-tiling* over R is a bi-infinite array of elements of R such that any two-by-two submatrix belongs to $\mathrm{SL}_2(R)$.

$$\begin{array}{ccccccc} & & & \vdots & & & \\ & 5 & 9 & 4 & 7 & 17 & \\ & 1 & 2 & 1 & 2 & 5 & \\ \cdots & 2 & 5 & 3 & 7 & 18 & \cdots \\ & 1 & 3 & 2 & 5 & 13 & \\ & 3 & 10 & 7 & 18 & 47 & \\ & & & \vdots & & & \end{array}$$

$$\begin{array}{ccccccc} & & & \vdots & & & \\ & 13 & 8 & 3 & 4 & 5 & \\ & 8 & 5 & 2 & 3 & 4 & \\ \cdots & 3 & 2 & 1 & 2 & 3 & \cdots \\ & 4 & 3 & 2 & 5 & 8 & \\ & 5 & 4 & 3 & 8 & 13 & \\ & & & \vdots & & & \end{array}$$

Definition An SL_2-tiling is *tame* if the determinant of each three-by-three submatrix is 0.

Definition An SL_2-tiling is *tame* if the determinant of each three-by-three submatrix is 0.

Observation
$$e\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b \\ d & e \end{vmatrix}\begin{vmatrix} e & f \\ h & i \end{vmatrix} - \begin{vmatrix} b & c \\ e & f \end{vmatrix}\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Tame SL_2-tilings and Dodgson's determinants

Definition An SL_2-tiling is *tame* if the determinant of each three-by-three submatrix is 0.

Observation $$e\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b \\ d & e \end{vmatrix}\begin{vmatrix} e & f \\ h & i \end{vmatrix} - \begin{vmatrix} b & c \\ e & f \end{vmatrix}\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Theorem Positive integer SL_2-tilings are tame.

Observation Let $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ be a 3-by-3 submatrix of a tame SL_2-tiling.

Then

$$d + f = \Delta e,$$

where $\Delta = af - cd = di - fg$.

$$\begin{array}{ccccccccc} & & & & \vdots & & & & \\ & 65 & 47 & 29 & 11 & 26 & 41 & 56 & \\ & 47 & 34 & 21 & 8 & 19 & 30 & 41 & \\ & 29 & 21 & 13 & 5 & 12 & 19 & 26 & \\ \cdots & 11 & 8 & 5 & 2 & 5 & 8 & 11 & \cdots \\ & 26 & 19 & 12 & 5 & 13 & 21 & 29 & \\ & 41 & 30 & 19 & 8 & 21 & 34 & 47 & \\ & 56 & 41 & 26 & 11 & 29 & 47 & 65 & \\ & & & & \vdots & & & & \end{array}$$

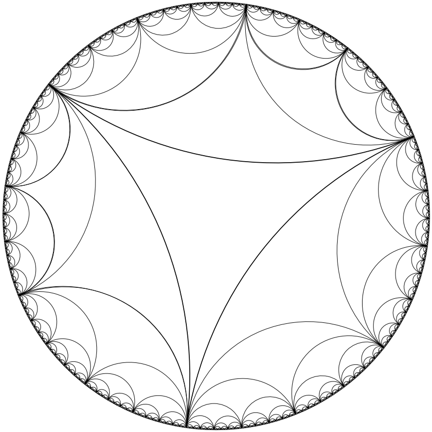
Definition A *wild* SL_2-tiling is an SL_2-tiling that is not tame.

Informally speaking, wild integer SL_2-tilings comprise tame blocks demarcated by wild zeros.

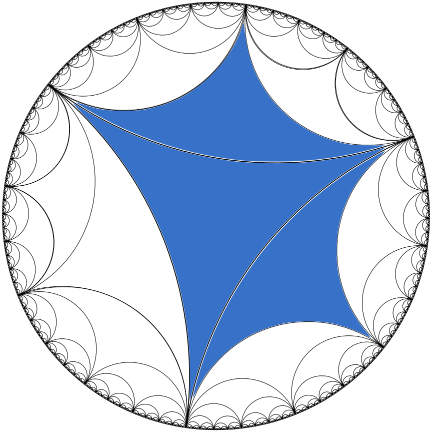
$$
\begin{array}{ccccccccccc}
 & & & & & \vdots & & & & & \\
 & * & * & * & * & * & * & * & 1 & * & \\
 & * & * & * & * & * & * & -1 & 0 & 1 & \\
 & * & * & * & * & * & * & * & -1 & * & \\
 & * & * & * & * & -1 & * & * & * & * & \\
\cdots & * & * & * & 1 & 0 & -1 & * & * & * & \cdots \\
 & * & * & * & * & 1 & * & * & * & * & \\
 & 1 & * & * & * & * & * & * & * & * & \\
 & 0 & 1 & * & * & * & * & * & * & * & \\
 & -1 & * & * & * & * & * & * & * & * & \\
 & & & & & \vdots & & & & &
\end{array}
$$

Classifying SL_2-tilings using the Farey graph

Triangulated polygons in the Farey graph



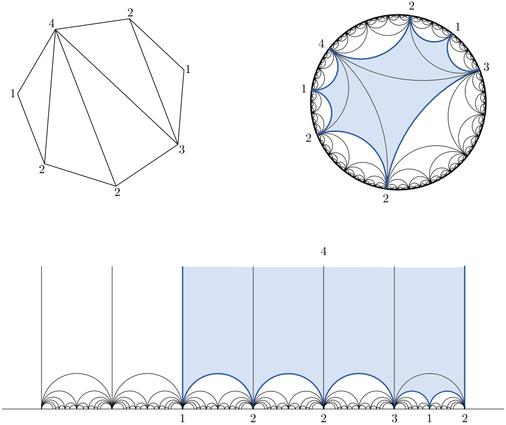
Triangulated polygons in the Farey graph



Triangulated polygons in the Farey graph

Theorem There is a one-to-one correspondence

$$\left\{\begin{array}{c}\text{positive integer}\\ \text{friezes of width } n\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{simple clockwise closed}\\ \text{paths of length } n\end{array}\right\}.$$



J.H. Conway & H.S.M. Coxeter, *Math. Gaz.*, 1973
S. Morier-Genoud, V. Ovsienko & S. Tabachnikov, *Enseign. Math.*, 2015

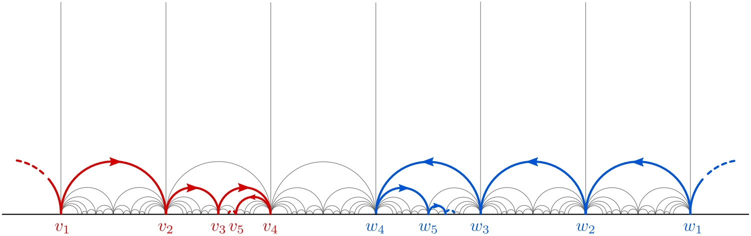
Paths on a tame SL_2-tiling

61	50	39	28	17	6	7	8	9	10	11
50	41	32	23	14	5	6	7	8	9	10
39	32	25	18	11	4	5	6	7	8	9
28	23	18	13	8	3	4	5	6	7	8
17	14	11	8	5	2	3	4	5	6	7
6	5	4	3	2	1	2	3	4	5	6
7	6	5	4	3	2	5	8	11	14	17
8	7	6	5	4	3	8	13	18	23	28
9	8	7	6	5	4	11	18	25	32	39
10	9	8	7	6	5	14	23	32	41	50
11	10	9	8	7	6	17	28	39	50	61

Classifying tame SL_2-tilings

Theorem There is a one-to-one correspondence

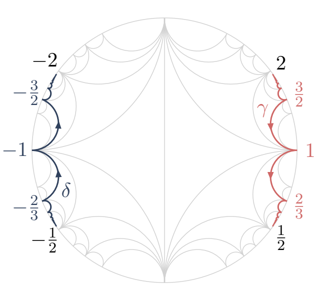
$$\mathrm{SL}_2(\mathbb{Z})\backslash\left\{\begin{array}{c}\text{pairs of bi-infinite}\\ \text{paths in the Farey graph}\end{array}\right\} \quad\longleftrightarrow\quad \{\pm 1\}\backslash\left\{\begin{array}{c}\text{tame } \mathrm{SL}_2\text{-tilings}\\ \text{over } \mathbb{Z}\end{array}\right\}.$$

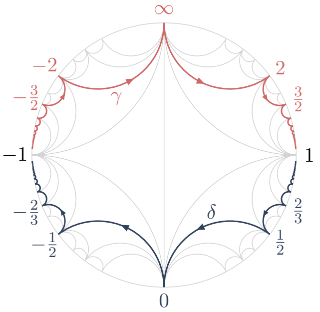


Correspondence $\quad v_i = \dfrac{a_i}{b_i},\ w_j = \dfrac{c_j}{d_j} \quad\longmapsto\quad m_{i,j} = a_i d_j - b_i c_j$

I. Short, *Trans. Amer. Math. Soc.*, 2023

F. Bergeron & C. Reutenauer, *Illinois J. Math.*, 2010





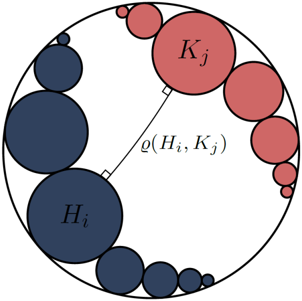
				⋮				
	65	47	29	11	26	41	56	
	47	34	21	8	19	30	41	
	29	21	13	5	12	19	26	
⋯	11	8	5	2	5	8	11	⋯
	26	19	12	5	13	21	29	
	41	30	19	8	21	34	47	
	56	41	26	11	29	47	65	
				⋮				

				⋮				
	25	18	11	4	5	6	7	
	18	13	8	3	4	5	6	
	11	8	5	2	3	4	5	
⋯	4	3	2	1	2	3	4	⋯
	5	4	3	2	5	8	11	
	6	5	4	3	8	13	18	
	7	6	5	4	11	18	25	
				⋮				

C. Bessenrodt, T. Holm, & P. Jørgensen, *Adv. Math.*, 2017

Ford circles and SL_2-tilings

				$\vdots$				
	65	47	29	11	26	41	56	
	47	34	21	8	19	30	41	
	29	21	13	5	12	19	26	
$\cdots$	11	8	5	2	5	8	11	$\cdots$
	26	19	12	5	13	21	29	
	41	30	19	8	21	34	47	
	56	41	26	11	29	47	65	
				$\vdots$				

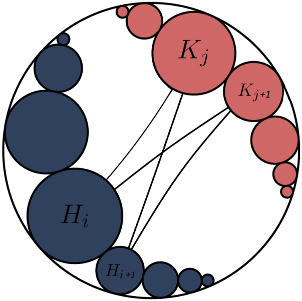


$$m_{i,j} = \exp \tfrac{1}{2}\varrho(H_i, K_j)$$

R.C. Penner, *Comm. Math. Phys.*, 1987
A. Felikson, O. Karpenkov, K. Serhiyenko, P. Tumarkin, *arXiv:2306.17118*, 2023

Ford circles and SL_2-tilings

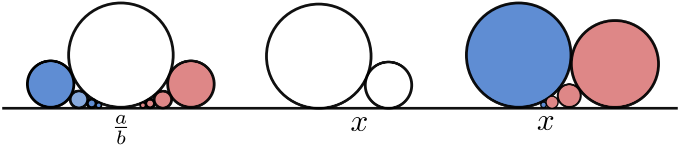
				$\vdots$				
	65	47	29	11	26	41	56	
	47	34	21	8	19	30	41	
	29	21	13	5	12	19	26	
$\cdots$	11	8	5	2	5	8	11	$\cdots$
	26	19	12	5	13	21	29	
	41	30	19	8	21	34	47	
	56	41	26	11	29	47	65	
				$\vdots$				



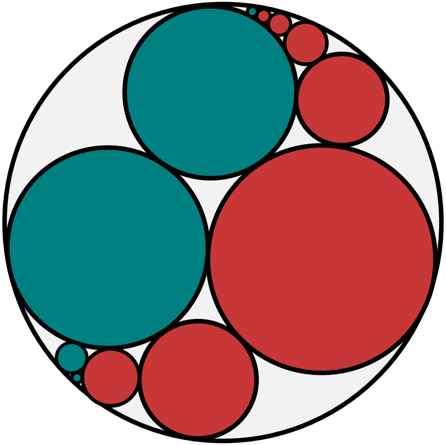
$$m_{i,j} = \exp \tfrac{1}{2}\varrho(H_i, K_j)$$

R.C. Penner, *Comm. Math. Phys.*, 1987
A. Felikson, O. Karpenkov, K. Serhiyenko, P. Tumarkin, *arXiv:2306.17118*, 2023

Converging chains

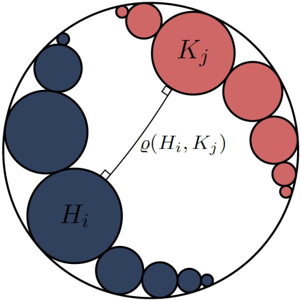


Converging chains with lots of 1's



Theorem There is a one-to-one correspondence

$$\mathrm{SL}_2(\mathbb{R})\Big\backslash\left\{\begin{matrix}\text{pairs of chains of}\\ \text{horocycles}\end{matrix}\right\} \quad\longleftrightarrow\quad (\mathbb{R}^\times \times \mathbb{R}^\times)\Big\backslash\left\{\begin{matrix}\text{tame SL}_2\text{-tilings}\\ \text{over } \mathbb{R}\end{matrix}\right\}.$$



I. Short, M. van Son, & A. Zabolotskii, *arXiv:2312.12953*, 2024

Infinite friezes

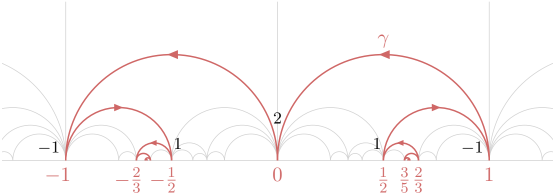
$$
\begin{array}{cccccccccccccccccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
 & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & & \\
 & & 3 & & 3 & & 3 & & 3 & & 3 & & 3 & & 3 & & 3 & & 3 & & 3 & & 3 & & 3 & \\
\cdots & 4 & & 4 & & 4 & & 4 & & 4 & & 4 & & 4 & & 4 & & 4 & & 4 & & 4 & & 4 & & \cdots \\
 & & 5 & & 5 & & 5 & & 5 & & 5 & & 5 & & 5 & & 5 & & 5 & & 5 & & 5 & & 5 & \\
 & 6 & & 6 & & 6 & & 6 & & 6 & & 6 & & 6 & & 6 & & 6 & & 6 & & 6 & & 6 & & \\
 & & 7 & & 7 & & 7 & & 7 & & 7 & & 7 & & 7 & & 7 & & 7 & & 7 & & 7 & & 7 & \\
 & & & & & & & \vdots & & & & & & & & & & & \vdots & & & & & & &
\end{array}
$$

K. Baur, M.J. Parsons, & M. Tschabold, *European J. Combin.*, 2016

Classifying tame infinite friezes

Theorem There is a one-to-one correspondence

$$\mathsf{SL}_2(\mathbb{Z})\Big\backslash\left\{\begin{array}{c}\text{bi-infinite paths in the}\\ \text{Farey graph}\end{array}\right\} \longleftrightarrow \{\pm 1\}\Big\backslash\left\{\begin{array}{c}\text{tame infinite friezes}\\ \text{over } \mathbb{Z}\end{array}\right\}.$$

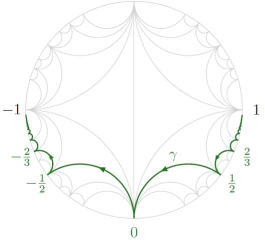


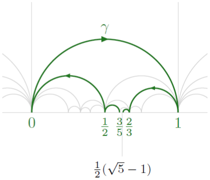
$$\gamma = \langle \dots, \tfrac{3}{5}, \tfrac{2}{3}, \tfrac{1}{2}, 1, 0, -1, -\tfrac{1}{2}, -\tfrac{2}{3}, -\tfrac{3}{5}, \dots \rangle$$

$$\begin{array}{ccccccccc}
 & & & & \vdots & & & & \\
 & 0 & -1 & -1 & 2 & 5 & -7 & -12 & \\
 & 1 & 0 & -1 & 1 & 3 & -4 & -7 & \\
 & 1 & 1 & 0 & -1 & -2 & 3 & 5 & \\
\cdots & -2 & -1 & 1 & 0 & -1 & 1 & 2 & \cdots \\
 & -5 & -3 & 2 & 1 & 0 & -1 & -1 & \\
 & 7 & 4 & -3 & -1 & 1 & 0 & -1 & \\
 & 12 & 7 & -5 & -2 & 1 & 1 & 0 & \\
 & & & & \vdots & & & &
\end{array}$$

Theorem There is a one-to-one correspondence

$$\mathsf{SL}_2(\mathbb{Z})\Big\backslash\left\{\begin{matrix}\text{simple clockwise bi-infinite}\\ \text{paths in the Farey graph}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{positive tame infinite}\\ \text{friezes over } \mathbb{Z}\end{matrix}\right\}.$$



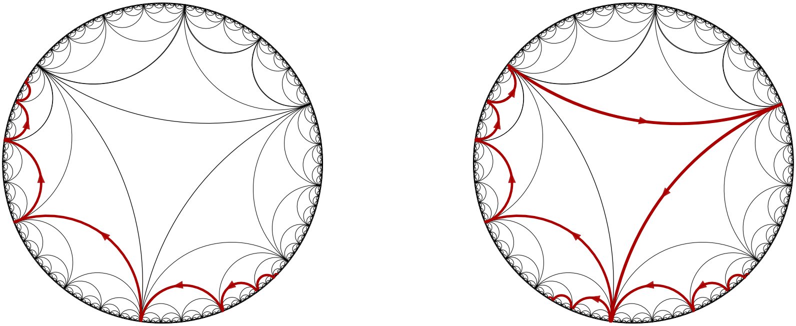



```
    0     0     0     0     0     0     0
       1     1     1     1     1     1     1
…   2     2     2     4     2     2     2    …
       3     3     7     7     3     3     3
    4     4    10    12    10     4     4
       5    13    17    17    13     5     5
                  ⋮
```

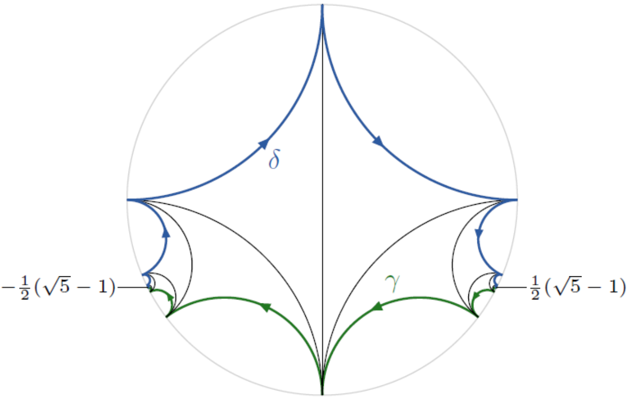
```
    0     0     0     0     0     0     0
       1     1     1     1     1     1     1
…   3     3     3     1     2     3     3    …
       8     8     2     1     5     8     8
   21    21     5     1     2    13    21
      55    13     2     1     5    34    55
                     ⋮
```

K. Baur, M.J. Parsons, & M. Tschabold, *European J. Combin.*, 2016

Theorem A bi-infinite sequence of positive integers is the quiddity sequence of a *positive* infinite frieze if and only if it does not contain a Conway–Coxeter sequence* as a subsequence.



I. Short, *Trans. Amer. Math. Soc.*, 2023



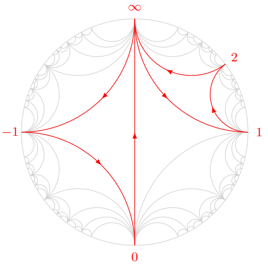
K. Baur, I. Canakci, K.M. Jacobsen, M.C. Kulkarni, & G. Todorov, *J. Alg. and its Appl.*, To appear

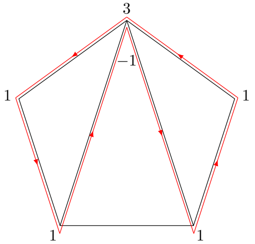
Finite friezes

Theorem There is a one-to-one correspondence

$$\left\{\begin{matrix}\text{closed paths of length } n \\ \text{in triangulated polygons}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{tame friezes over } \mathbb{Z} \\ \text{of width } n\end{matrix}\right\}.$$

	0		0		0		0		0		0		0		0		
		1		1		1		1		1		1		1		1	
	1		1		-1		1		1		3		1		1		
$\cdots$		2		0		-2		-2		0		2		2		0	$\cdots$
	-1		-1		-3		-1		-1		1		-1		-1		
		-1		-1		-1		-1		-1		-1		-1		-1	
	0		0		0		0		0		0		0		0		

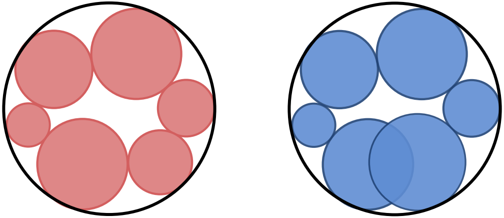




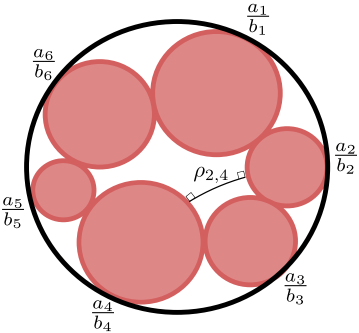
I. Short, *Trans. Amer. Math. Soc.*, 2023

Theorem* There is a one-to-one correspondence

$$\{\,\text{regular positive real friezes}\,\} \quad \longleftrightarrow \quad \{\,\text{bracelets of horocycles}\,\}.$$

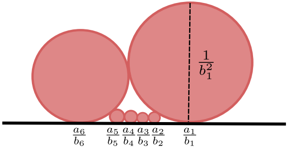


Bracelet measurements



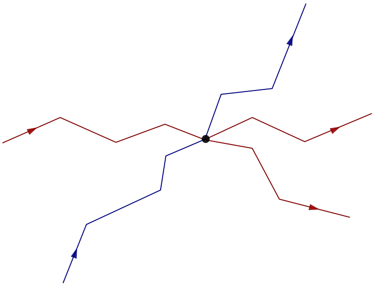
$$a_{i-1}b_i - b_{i-1}a_i = 1$$

$$m_{i,j} = a_i b_j - b_i a_j = \exp \tfrac{1}{2}\rho_{i,j}$$



Wild SL_2-tilings

$$
\begin{array}{ccccccccccc}
 & & & & & \vdots & & & & & \\
 & * & * & * & * & * & * & * & * & * & \\
 & * & * & * & * & * & * & * & * & * & \\
 & * & * & * & * & * & * & * & * & * & \\
 & * & * & * & * & -1 & * & * & * & * & \\
\cdots & * & * & * & 1 & 0 & -1 & * & * & * & \cdots \\
 & * & * & * & * & 1 & * & * & * & * & \\
 & * & * & * & * & * & * & * & * & * & \\
 & * & * & * & * & * & * & * & * & * & \\
 & * & * & * & * & * & * & * & * & * & \\
 & & & & & \vdots & & & & &
\end{array}
$$



Wild tilings and twisted paths

$$
\begin{array}{ccccccccccc}
 & & & & & \vdots & & & & & \\
 & * & * & * & * & * & * & * & 1 & * & \\
 & * & * & * & * & * & * & -1 & 0 & 1 & \\
 & * & * & * & * & * & * & * & -1 & * & \\
 & * & * & * & * & -1 & * & * & * & * & \\
\cdots & * & * & * & 1 & 0 & -1 & * & * & * & \cdots \\
 & * & * & * & * & 1 & * & * & * & * & \\
 & 1 & * & * & * & * & * & * & * & * & \\
 & 0 & 1 & * & * & * & * & * & * & * & \\
 & -1 & * & * & * & * & * & * & * & * & \\
 & & & & & \vdots & & & & &
\end{array}
$$