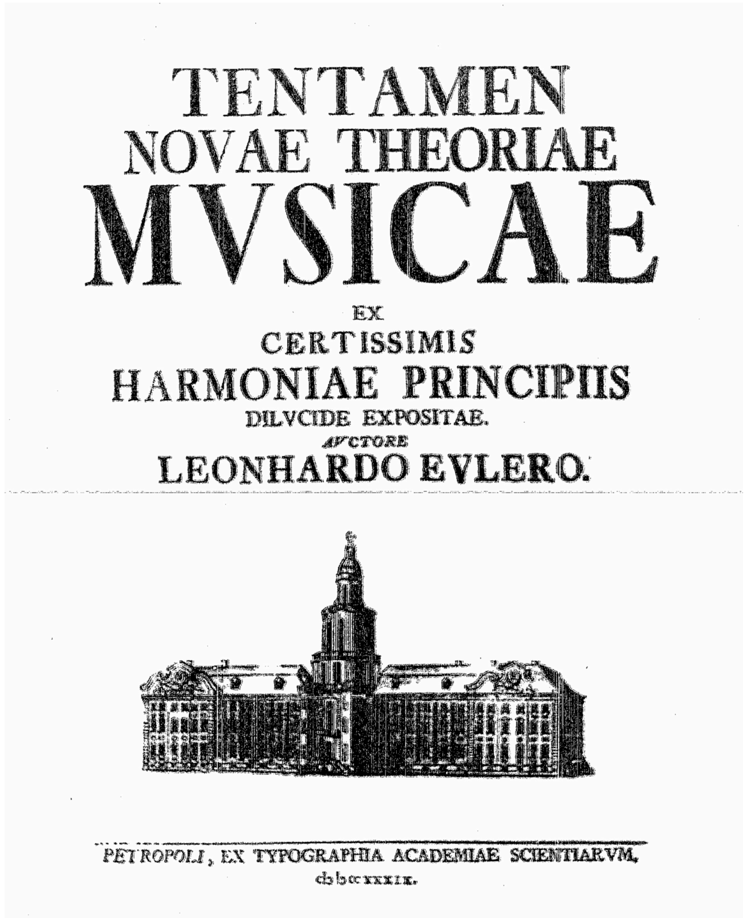
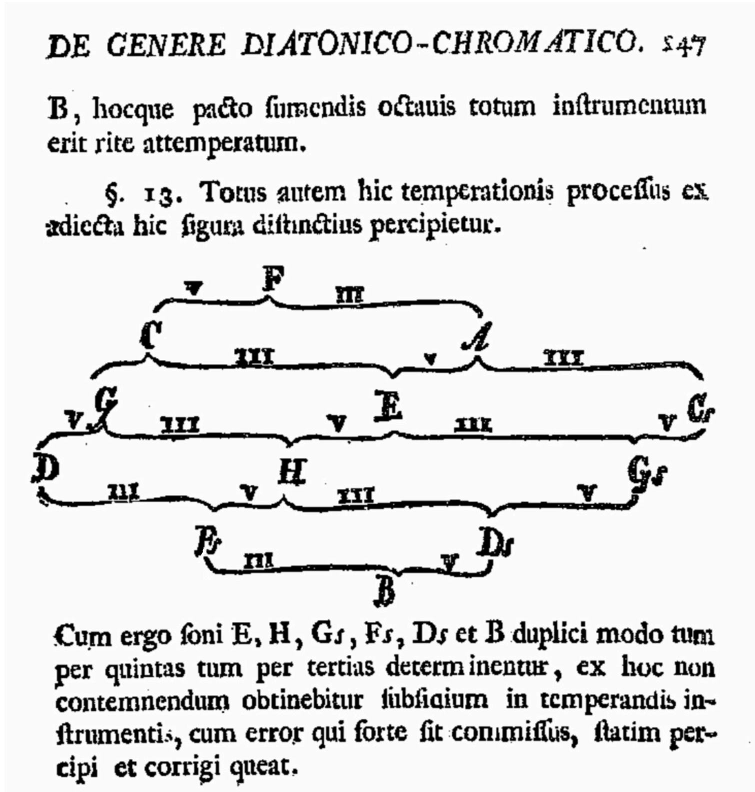
March 26, 2024
Durham

Generalising Euler's Tonnetz

Konstanze Rietsch


TENTAMEN
NOVAE THEORIAE
MVSICAE
EX
CERTISSIMIS
HARMONIAE PRINCIPIIS
DILVCIDE EXPOSITAE.
AVCTORE
LEONHARDO EVLERO.

PETROPOLI, EX TYPOGRAPHIA ACADEMIAE SCIENTIARVM.
cIↄIↄccxxxix.


DE GENERE DIATONICO-CHROMATICO. 547

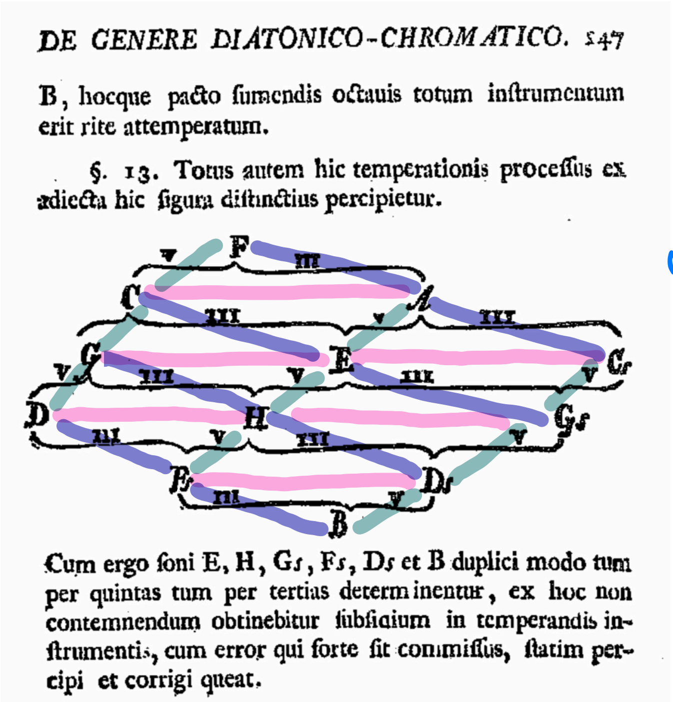
B, hocque pacto ſumendis octauis totum inſtrumentum erit rite attemperatum.

§. 13. Totus autem hic temperationis proceſſus ex adiecta hic figura diſtinctius percipietur.

Cum ergo ſoni E, H, Gs, Fs, Ds et B duplici modo tum per quintas tum per tertias determinentur, ex hoc non contemnendum obtinebitur ſubſidium in temperandis inſtrumentis, cum error qui forte ſit commiſſus, ſtatim percipi et corrigi queat.

BACKGROUND

Euler's 1739 book on a new music theory has the following planar arrangement of the 12 notes of the chromatic scale:



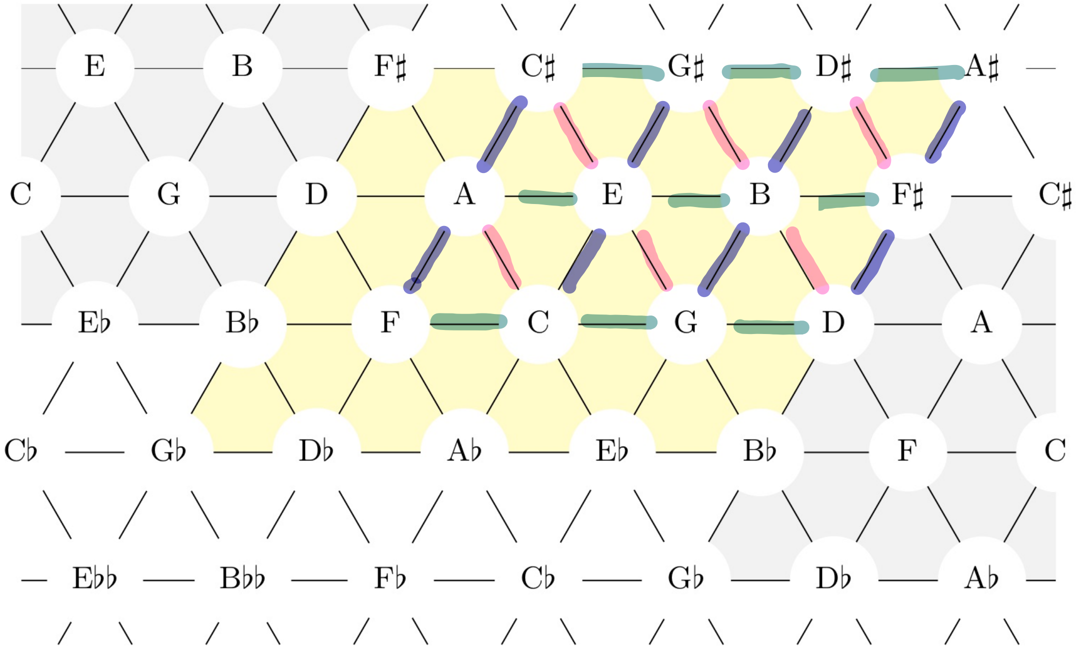
DE GENERE DIATONICO-CHROMATICO. 547

B, hocque pacto sumendis octauis totum instrumentum erit rite attemperatum.

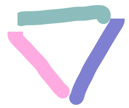
§. 13. Totus autem hic temperationis processus ex adiecta hic figura distinctius percipietur.

Cum ergo soni E, H, Gs, Fs, Ds et B duplici modo tum per quintas tum per tertias determinentur, ex hoc non contemnendum obtinebitur subsidium in temperandis instrumentis, cum error qui forte sit commissus, statim percipi et corrigi queat.

rotate



Continued infinitely in its more modern depiction ↑ it periodically tiles the plane (or tiles a torus, after identifying octaves).

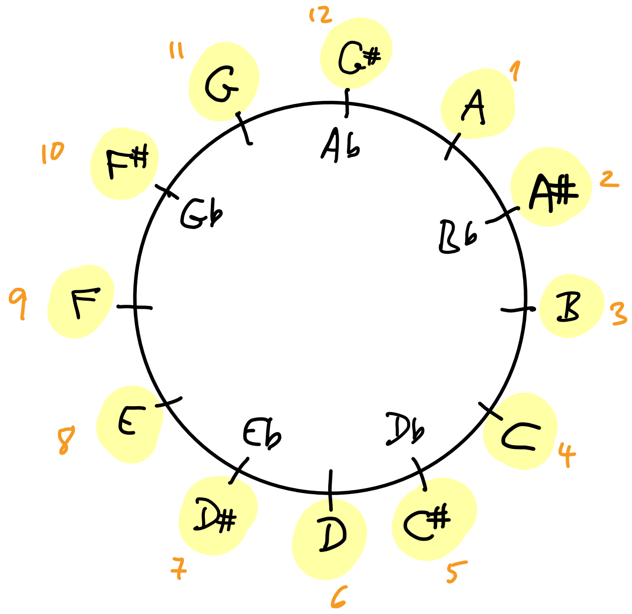
The triangles encode all major  and minor  triads.

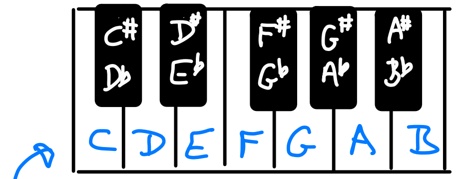
The tonnetz is foundational to "NeoRiemannian music theory": Hugo Riemann 1880
Richard Cohn 1990's

Music basics

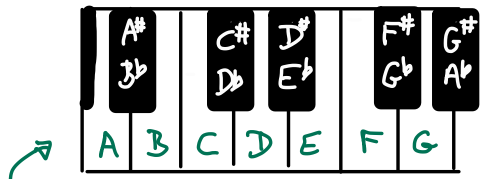
equal temperament / identifying octaves

corresponds to: $\mathbb{Z}/12\mathbb{Z}$





major scale starting on C



minor scale starting on A

Generalised Tonnetz

Let S be a triangulated surface. More generally, we may replace triangles by polygons.

Notation:

- $\mathcal{V} = \mathcal{V}_0 \sqcup \mathcal{V}_1 \sqcup \mathcal{V}_2$ set of vertices, edges and faces
- $\mathcal{V}$ is a graded poset with partial order $\leq$ of inclusion.
- Write $\tau \prec \sigma$ if σ covers τ (so $\tau \leq \sigma$ and $\dim(\sigma) = \dim(\tau) + 1$)

Def: A generalised tonnetz is a map:

$$T : \mathcal{V} \longrightarrow \text{Multisets in } \mathbb{Z}/12\mathbb{Z} \quad (\text{or } \mathcal{N} = \{A, A^\#, B, C, \dots, G, G^\#\})$$

satisfying the following coherence conditions:

- <u>upwards coherence</u>: for any $\tau \in \mathcal{V}_0 \cup \mathcal{V}_1$ there is a bijection:

  $$d_\tau : \{\sigma \mid \sigma \succ \tau\} \longrightarrow T(\tau)$$

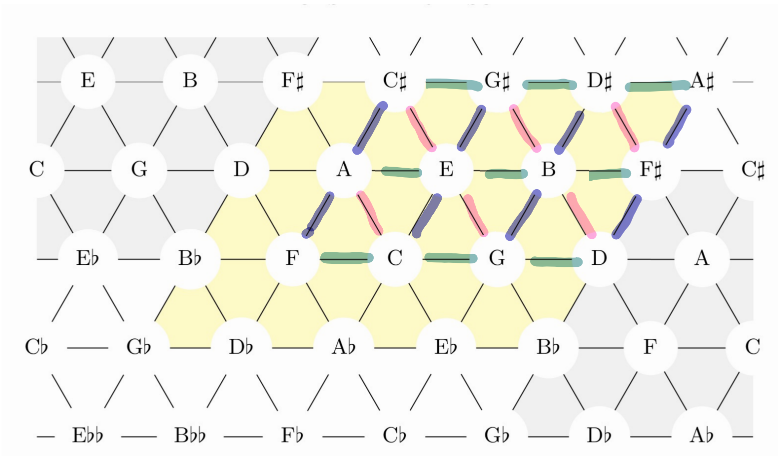
  such that $d_\tau(\sigma) \in T(\sigma)$ for any σ.

- <u>downwards coherence</u>: for any $\tau \in \mathcal{V}_1 \cup \mathcal{V}_2$ there is a bijection:

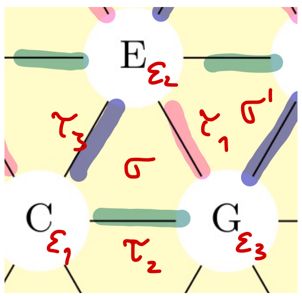
  $$\partial_\tau : \{\varepsilon \mid \varepsilon \prec \tau\} \longrightarrow T(\tau)$$

  such that $\partial_\tau(\varepsilon) \in T(\varepsilon)$ for any ε.

EULER EXAMPLE:



$\rightsquigarrow$

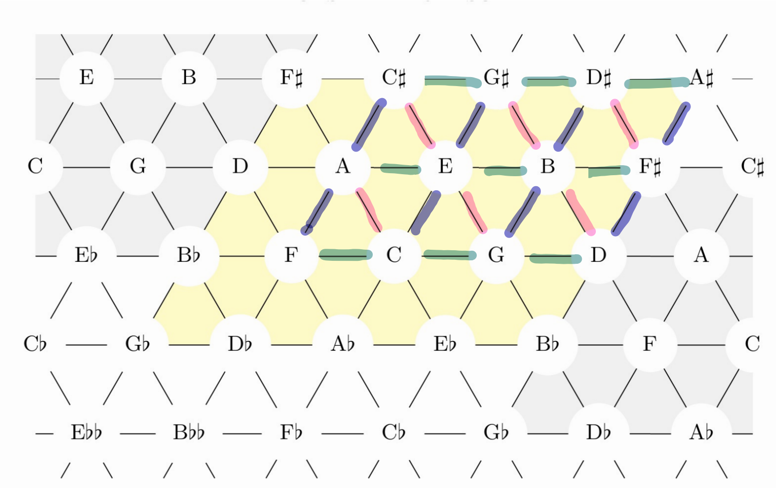


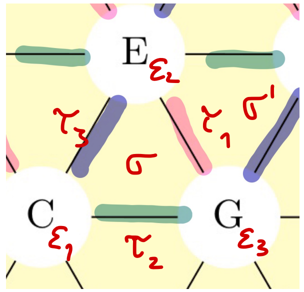
$$\boxed{\begin{array}{ll} T(\varepsilon_1) = \{C,C,C,C,C,C\} & T(\tau_1) = \{E, G\} \\ T(\varepsilon_2) = \{E,E,E,E,E,E\} & T(\tau_2) = \{C, G\} \\ T(\varepsilon_3) = \{G,G,G,G,G,G\} & T(\tau_3) = \{C, E\} \\ \multicolumn{2}{c}{T(\sigma) = \{C, E, G\}} \end{array}}$$

eg: $\partial_{\tau_1}(\varepsilon_2) = E$ $\quad$ $d_{\tau_1}(\sigma) = G$

$\partial_{\tau_1}(\varepsilon_3) = G$ $\quad$ $d_{\tau_1}(\sigma') = E$

EULER EXAMPLE :





$T(\varepsilon_1) = \{C,C,C,C,C,C\}$ $T(\tau_1) = \{E,G\}$

$T(\varepsilon_2) = \{E,E,E,E,E,E\}$ $T(\tau_2) = \{C,G\}$

$T(\varepsilon_3) = \{G,G,G,G,G,G\}$ $T(\tau_3) = \{C,E\}$

$T(\sigma) = \{C,E,G\}$

eg: $\partial_{\tau_1}(\varepsilon_2) = E$ $d_{\tau_1}(\sigma) = G$

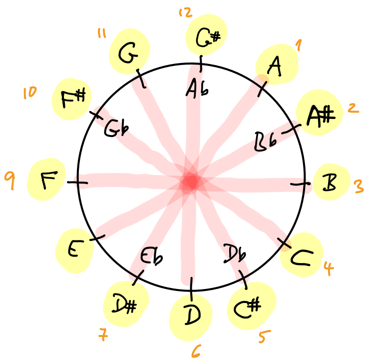
$\partial_{\tau_1}(\varepsilon_3) = G$ $d_{\tau_1}(\sigma') = E$

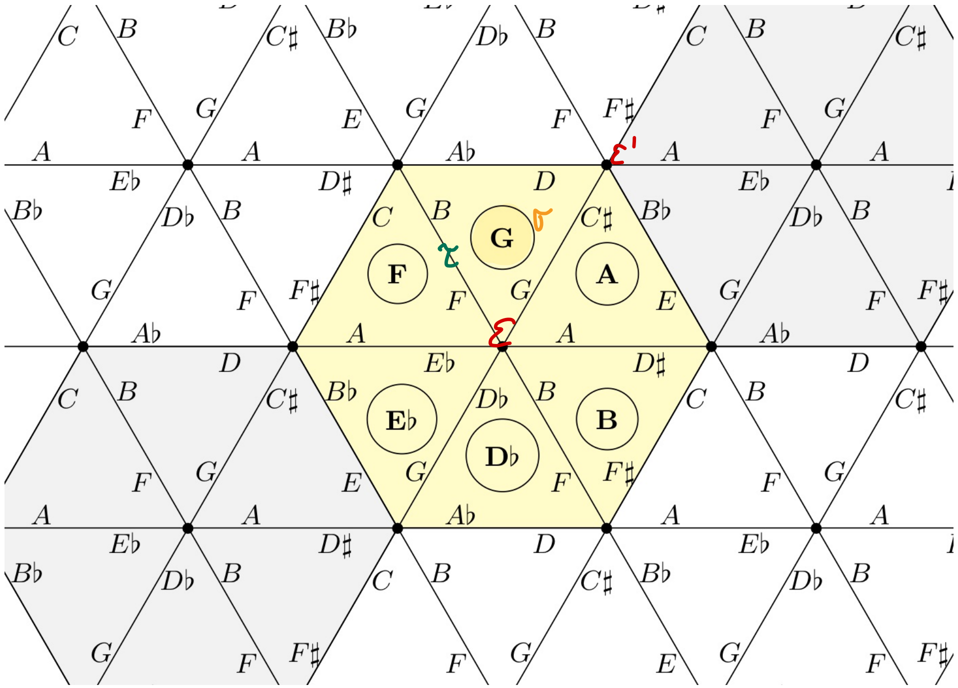
$\partial_{\tau_1} : \{\varepsilon_2, \varepsilon_3\} \to \{E,G\}$, $d_{\tau_1} : \{\sigma, \sigma'\} \to \{E,G\}$

from definition $\Rightarrow$ $|T(\sigma)| = 3$ So: face ⇝ triad (in triangulated surface case)

$|T(\tau)| = 2$ edge ⇝ dyad

TRITONE tonnetz :





$T(\sigma) = \{G, B, D\}$

G-major triad

$T(\tau) = \{F, B\}$

tritone (major 4ths)

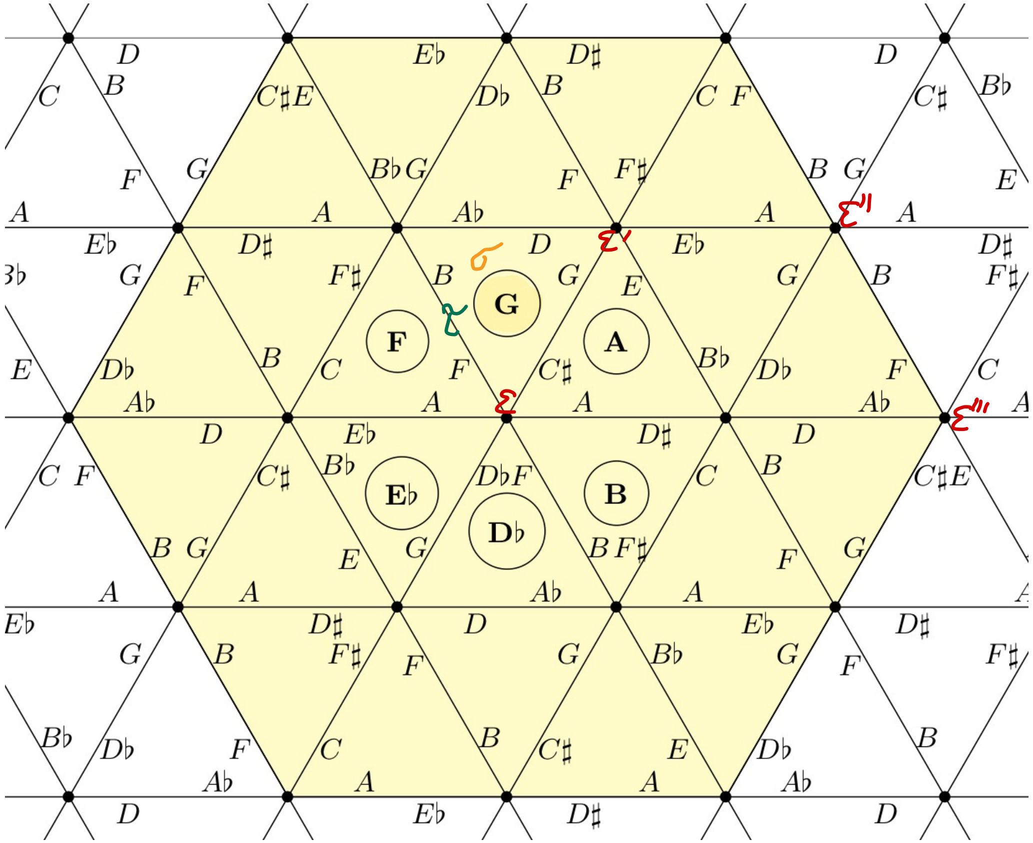
$T(\varepsilon) = \{A, B, D\flat, E\flat, F, G\}$

wholetone scale

$T(\varepsilon') = \{F, A, C\#\} \cup \{F\#, B\flat, D\}$

augmented triads

TRITONE tonnetz II :



$T(\sigma) = \{G, B, D\}$

G-major triad

$T(\tau) = \{F, B\}$

tritone (major 4th)

$T(\varepsilon) = \{A, A, C\#, D\flat, F, F\}$

augmented triad

$T(\varepsilon') = \{D, E\flat, E, F, F\#, G\}$

half a chromatic scale

$T(\varepsilon'') = \{G, G, A, A, B, B\}$

start of major scale

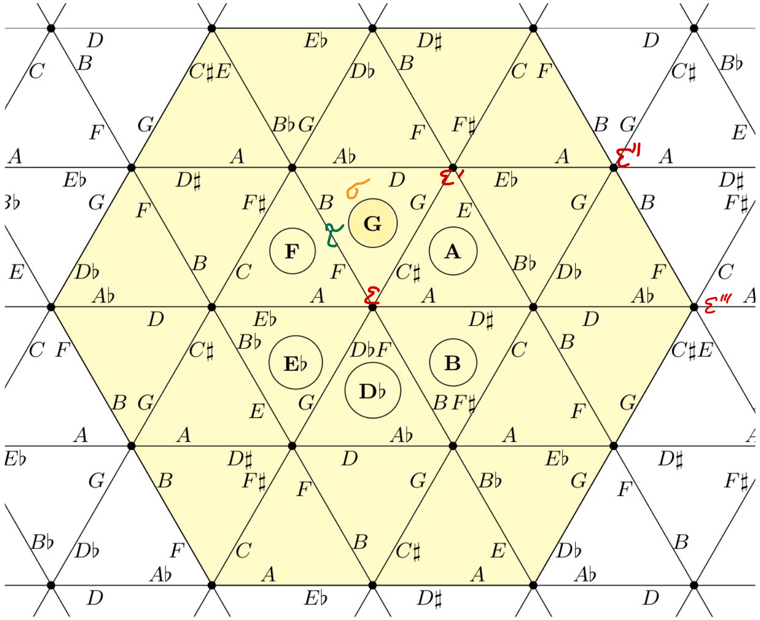
$T(\varepsilon''') = \{F, A, C\#\} \cup \{E, A\flat, C\}$

augmented triads

Same triads and dyads but different vertex sets & different symmetries!

Both examples live on a torus, but in one case the triangulation has 6 triangles, in the other it has 24.

TRITONE tonnetz II :



$T(\sigma) = \{G, B, D\}$

G-major triad

$T(\tau) = \{F, B\}$

tritone (major 4^{th})

$T(\varepsilon) = \{A, A, C\#, D\flat, F, F\}$

augmented triad

$T(\varepsilon') = \{D, E\flat, E, F, F\#, G\}$

half a chromatic scale

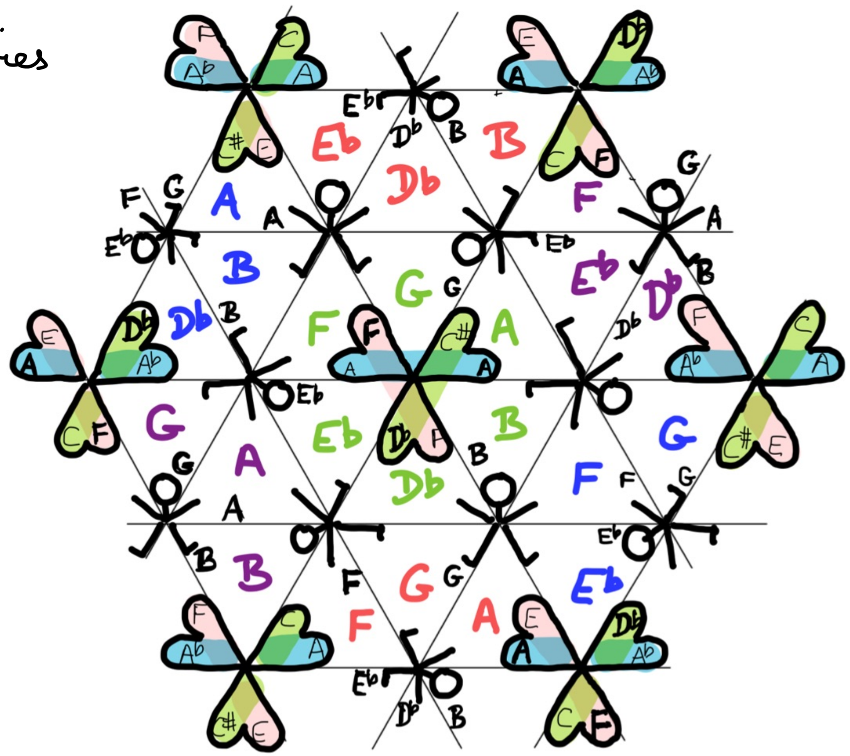
$T(\varepsilon'') = \{G, G, A, A, B, B\}$

start of major scale

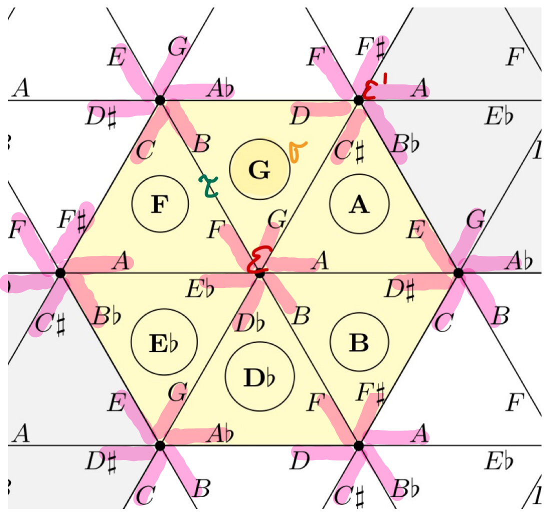
$T(\varepsilon''') = \{F, A, C\#\} \cup \{E, A\flat, C\}$

augmented triads

Symmetries



vs.



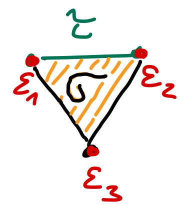
Let $(S, \mathcal{V})$ be fixed.

- For any *vertex labeling* $L : \mathcal{V}_0 \longrightarrow \mathcal{N}$,

$$T(\varepsilon) = \{\overbrace{L(\varepsilon), L(\varepsilon), \dots, L(\varepsilon)}^{\text{valences}}\}$$

$$T(\tau) = \{L(\varepsilon_1), L(\varepsilon_2)\}$$

$$T(\sigma) = \{L(\varepsilon_1), L(\varepsilon_2), L(\varepsilon_3)\}$$



  defines a tonnetz that we call the **vertex tonnetz** associated to L.

- For any *edge labeling* $M : \mathcal{V}_1 \longrightarrow \mathcal{N}$ there is a *unique* tonnetz T such that $T(\tau) = \{M(\tau), M(\tau)\}$. We call such a tonnetz an **edge tonnetz**.

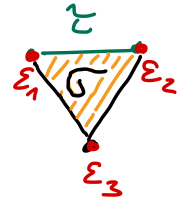
Let $(S,\mathcal{V})$ be fixed.

- For any vertex labeling $L:\mathcal{V}_0\to\mathcal{N}$,

$$T(\varepsilon)=\{\overbrace{L(\varepsilon),L(\varepsilon),\dots,L(\varepsilon)}^{\text{valencies}}\}$$

$$T(\tau)=\{L(\varepsilon_1),L(\varepsilon_2)\}$$

$$T(\sigma)=\{L(\varepsilon_1),L(\varepsilon_2),L(\varepsilon_3)\}$$

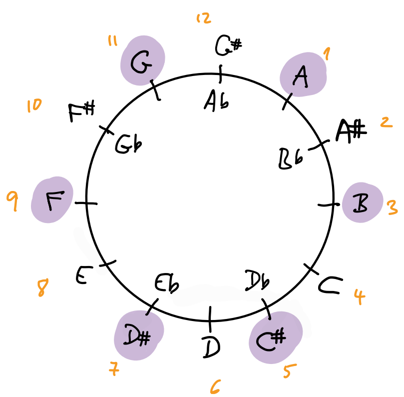


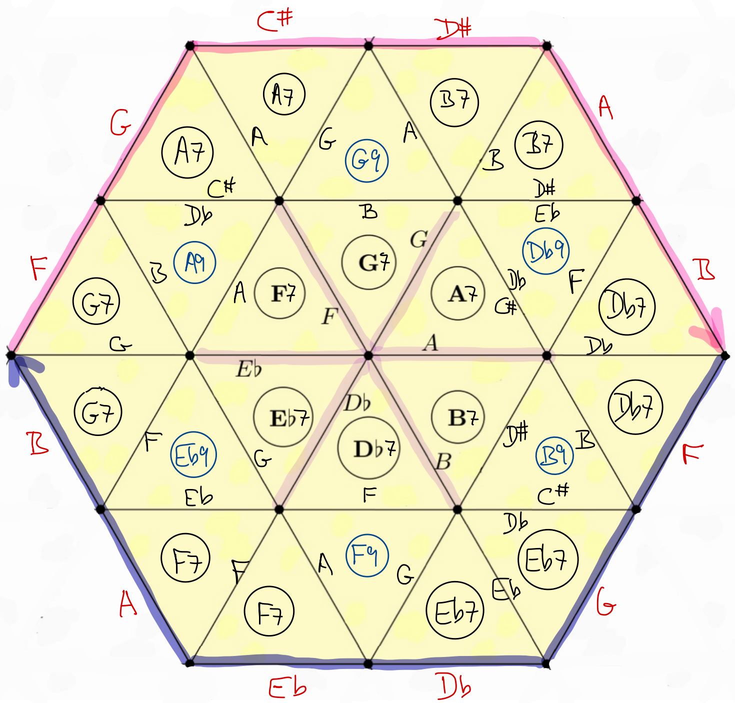
defines a tonnetz that we call the vertex tonnetz associated to L.

- For any edge labeling $M:\mathcal{V}_1\to\mathcal{N}$ there is a unique tonnetz T such that $T(\tau)=\{M(\tau),M(\tau)\}$. We call such a tonnetz an edge tonnetz.

Edge tonnetz example

- the surface is the projective plane!
- roots of the chords are from wholetone scale:

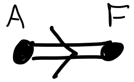


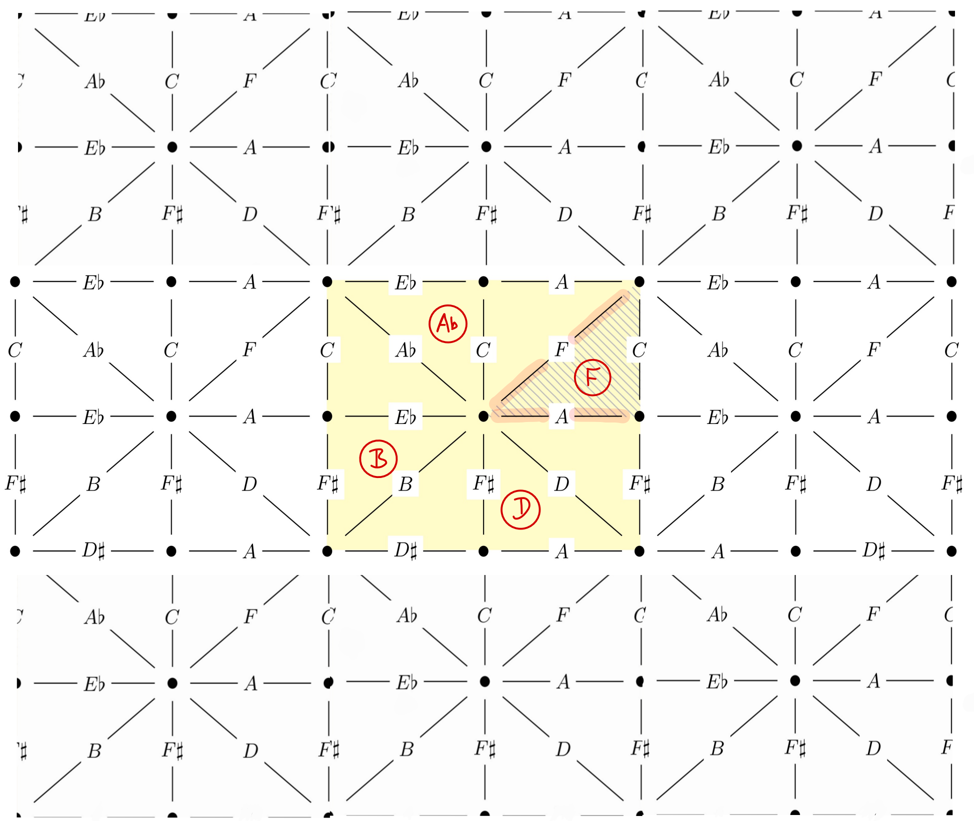


- the triads are made up of essential notes from dominant 7^{th} and major 9^{th} chords.

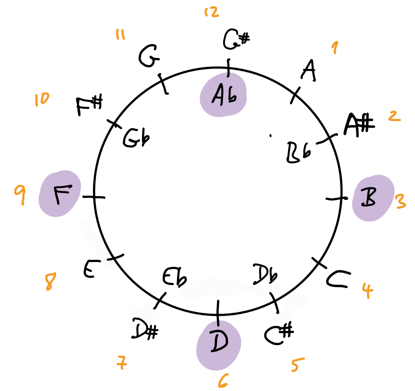
Major/minor edge tonnetz examples of type B_2, C_2, G_2

B_2



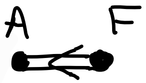


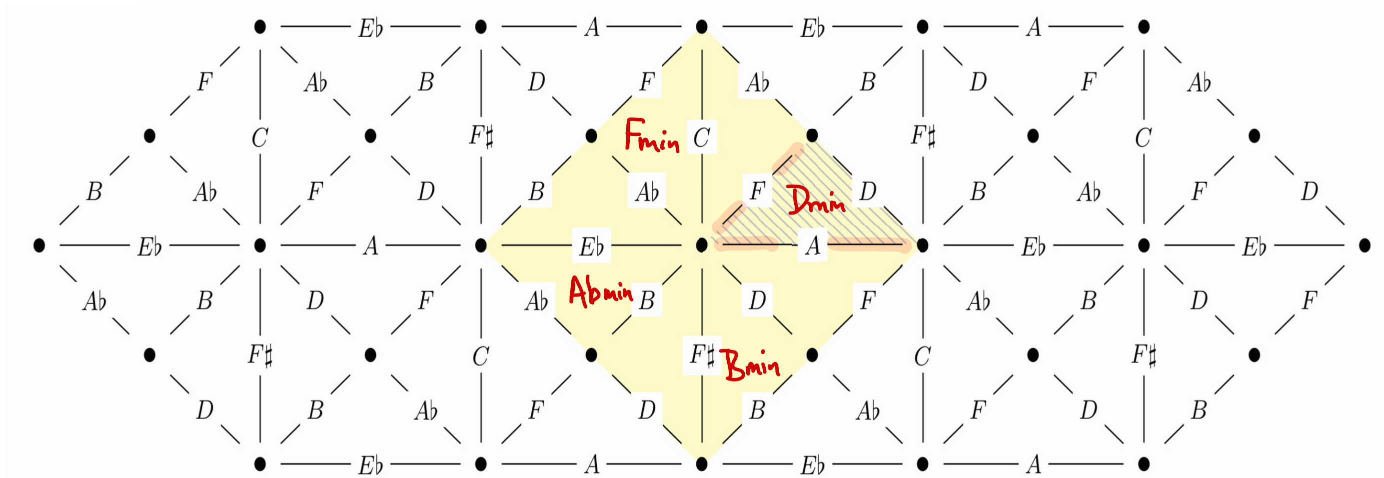
- four major triads, minor thirds apart.
- roots of the chords are from the diminished 7^{th}



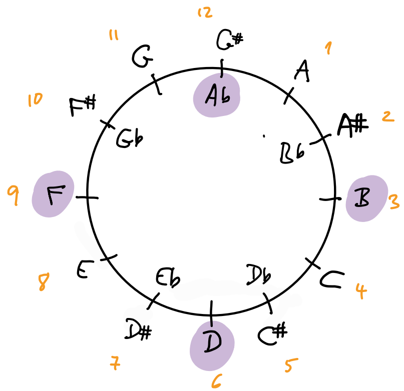
- periodic tiling of the plane giving a tonnetz on a torus.

We can transpose up or down by a tone and get two other versions, in total containing all 12 major triads.

C_2 



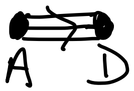
- four minor triads, minor thirds apart
- roots of the chords are again

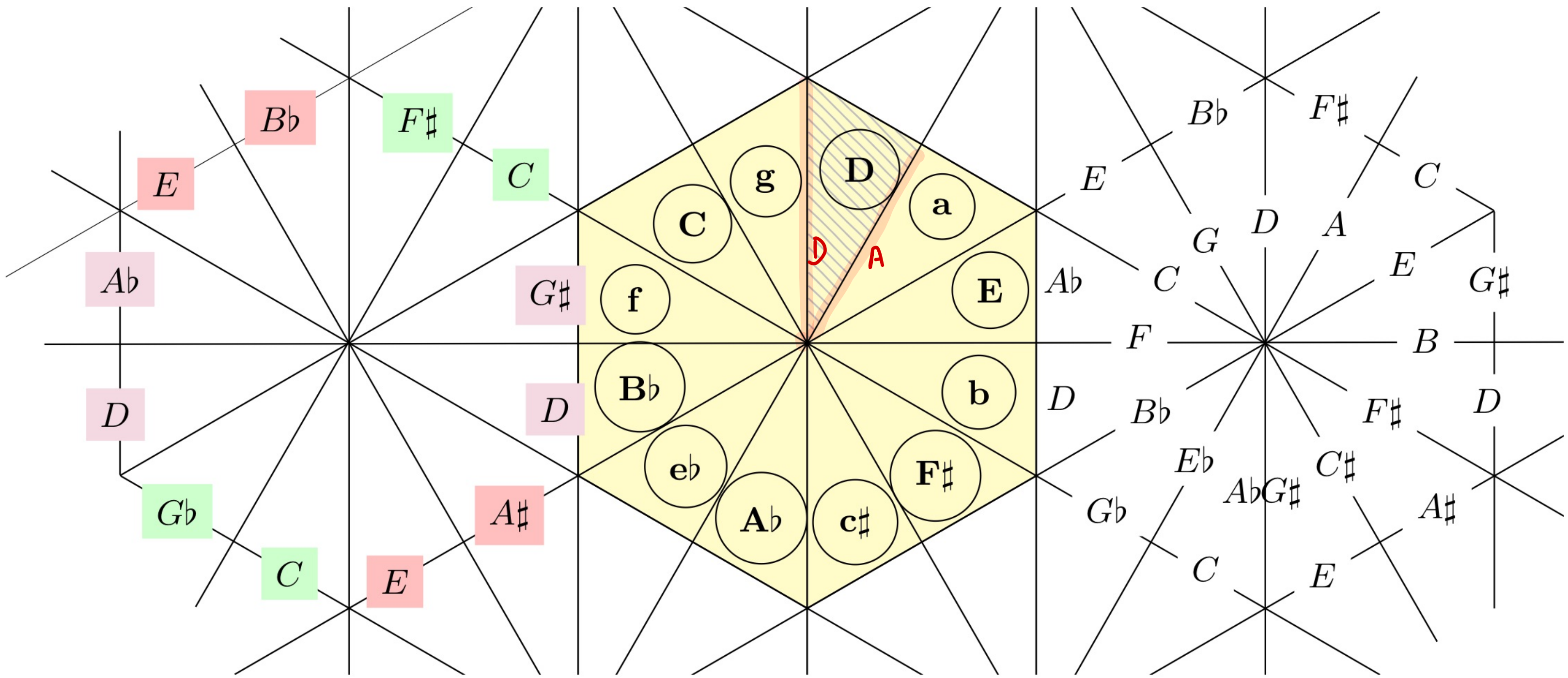


- periodic tiling of the plane giving a tonnetz on a torus.

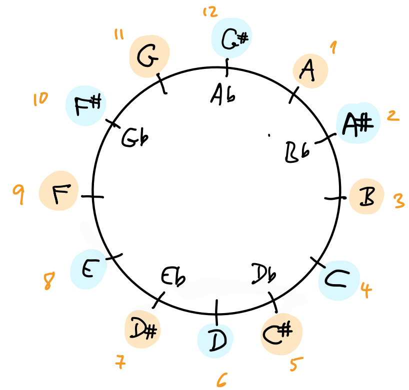
We can transpose up or down by a tone and get two other versions in total containing all 12 minor triads.

G_2

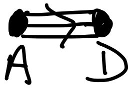


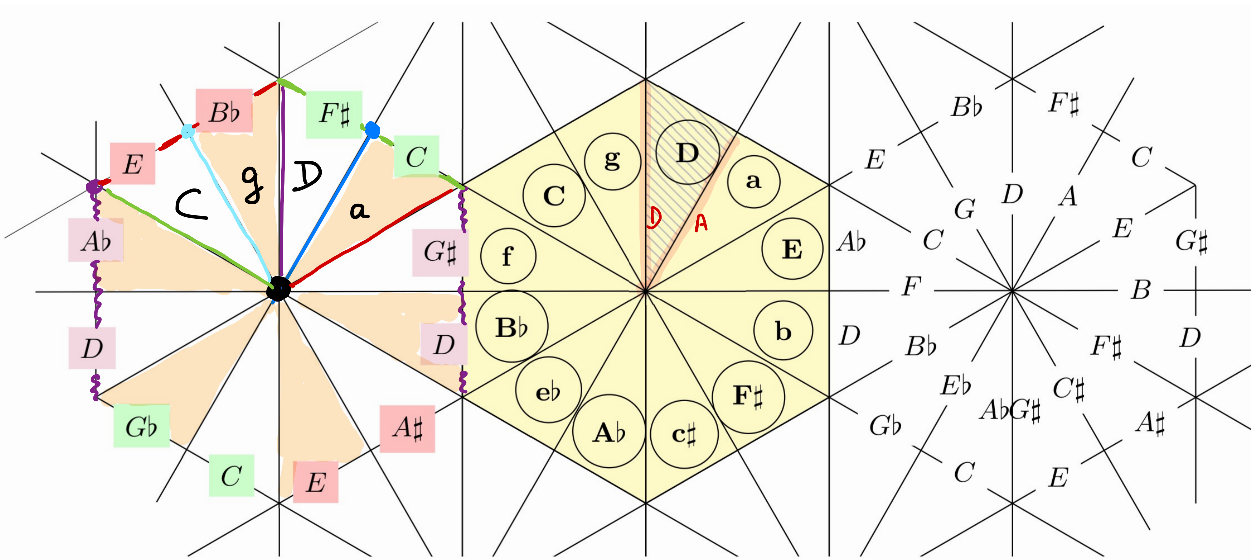


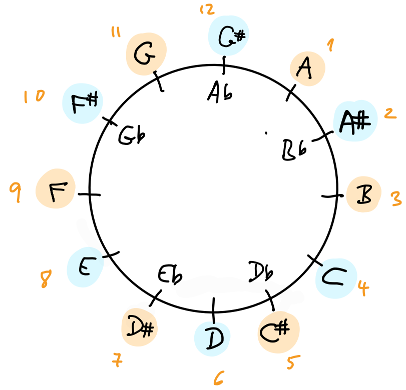
- six major and six minor triads



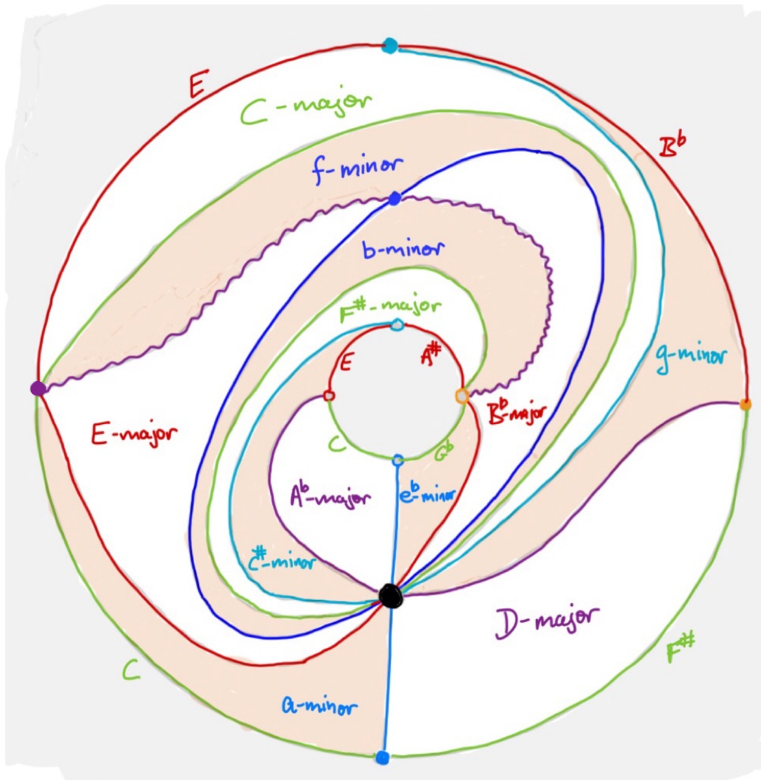
G_2


A
D

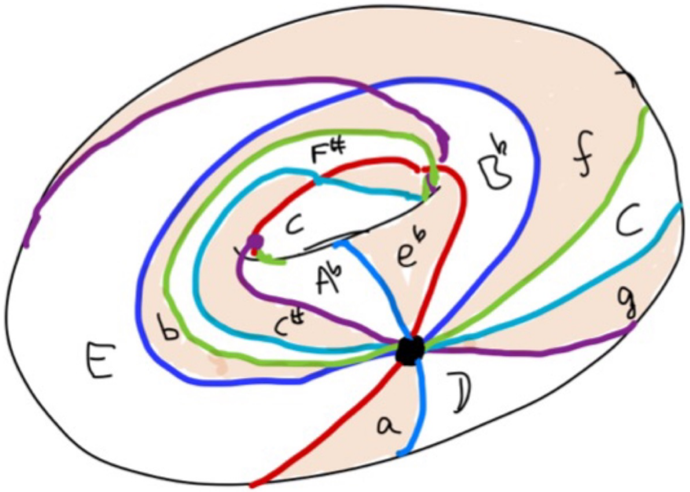

E
B♭
F♯
C
A♭
G♯
D
G♭
A♯
g
D
a
C
f
B♭
e♭
A♭
c♯
F♯
b
E

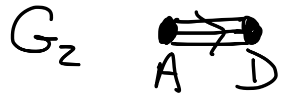

G♯
A♭
A
A♯
B♭
B
C
D♭
C♯
D
D♯
E♭
E
F
F♯
G♭
G

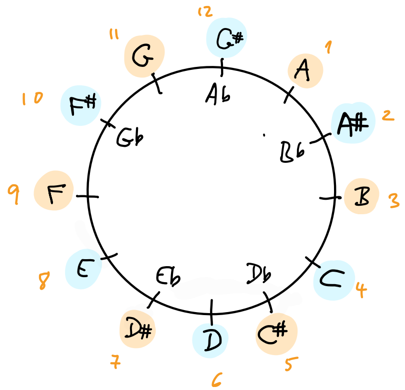
glue

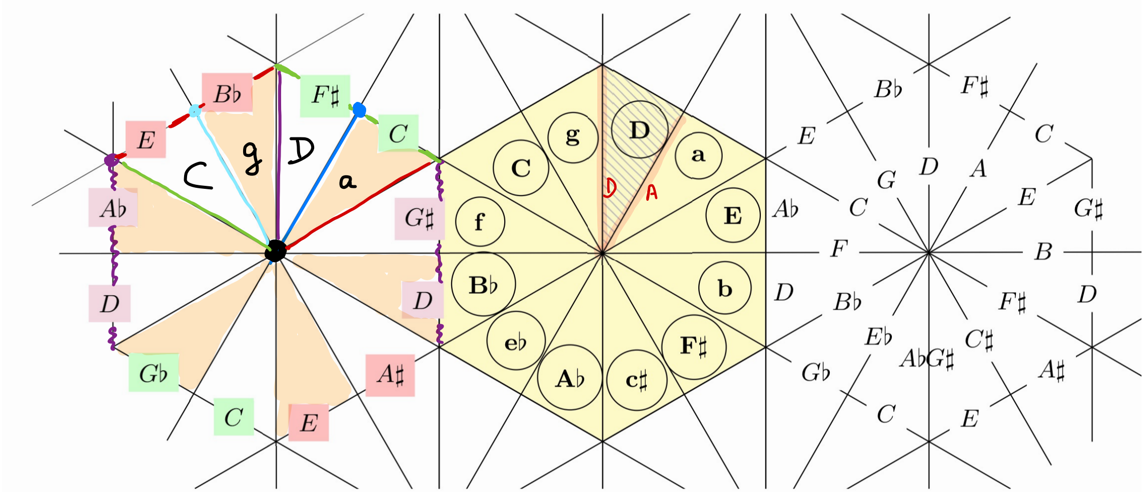

E
C-major
f-minor
b-minor
F♯-major
B♭
g-minor
E-major
B♭-major
A♭-major
e♭-minor
c♯-minor
D-major
a-minor
C
F♯

glue


F♯
B♭
f
C
g
E
b
c
e♭
A♭
c♯
a
D

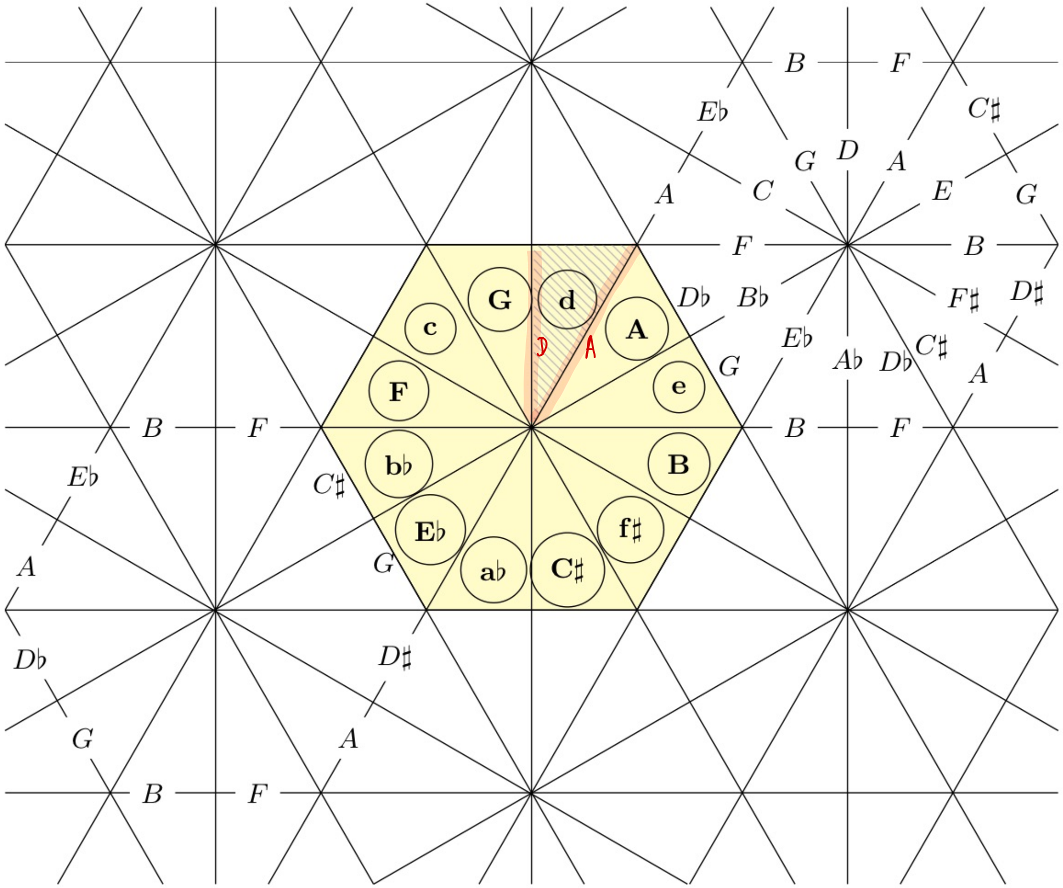


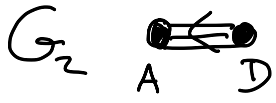


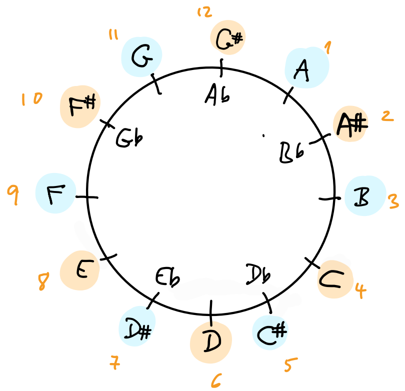


"Langlands dual" G_2 tonnetz

swap major & minor

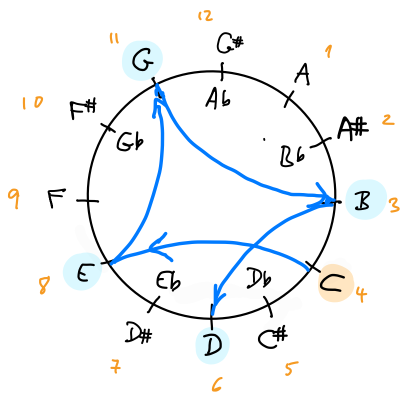






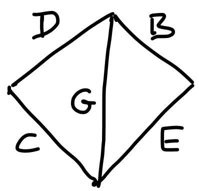
An edge tonnetz on a sphere encoding all major 9 chords

A major 9th chord has 5 notes : major triad + major 7 + major 9



Example: $C\Delta 9$ = C - E - G - B - D

in an edge tonnetz this can arise as edge labels of a quadrilateral, eg:



An edge tonnetz on a sphere encoding all major 9 chords

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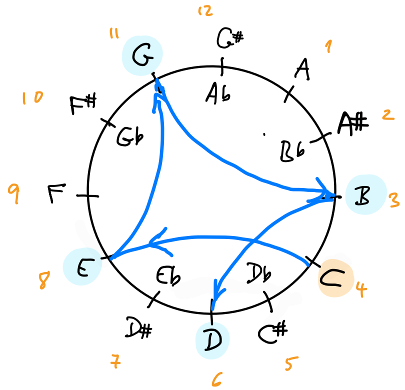
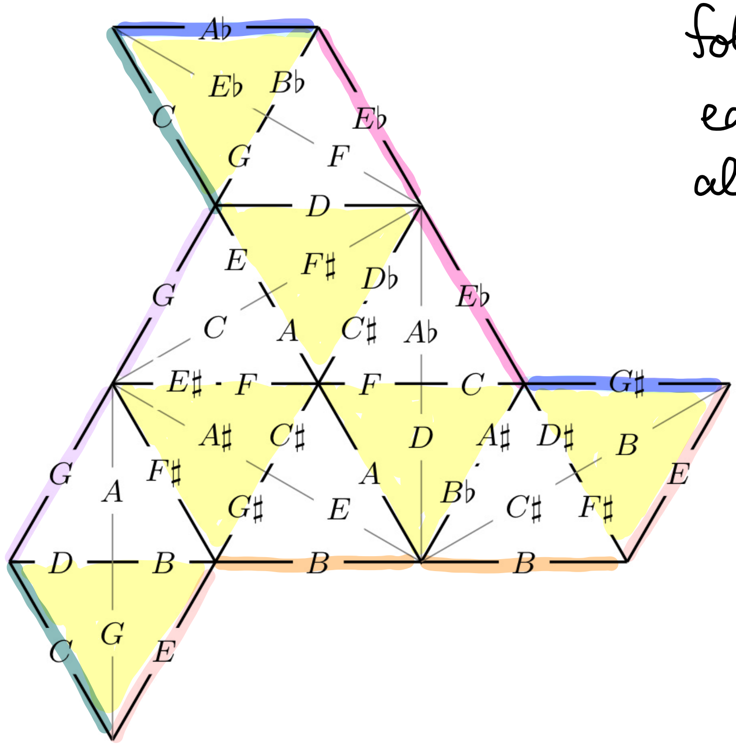
Example: $C\triangle 9$ = C - E - G - B - D



in an edge tonnetz this can arise as edge labels of a quadrilateral.



fold along grey edges and identify along boundary

tetrahedron

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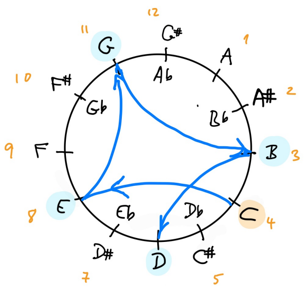
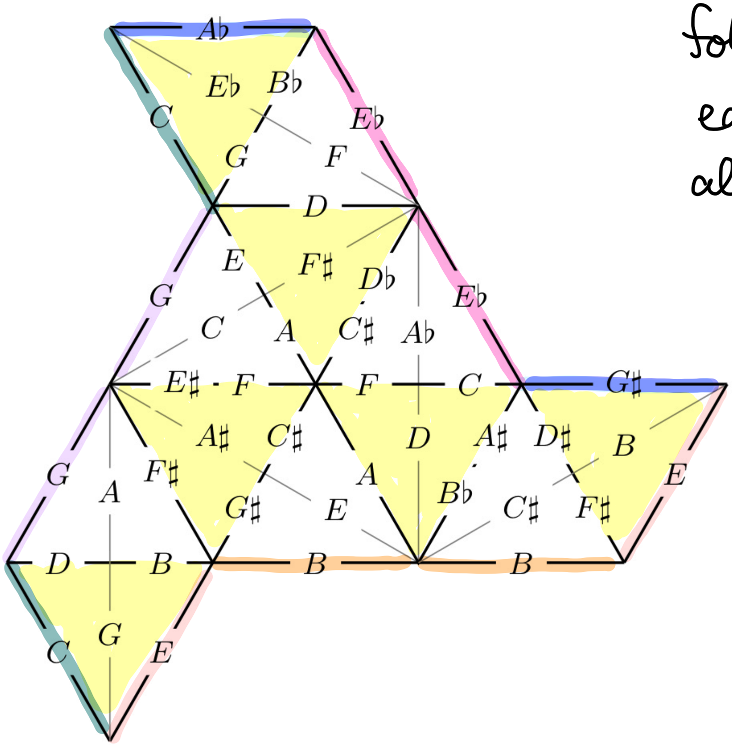
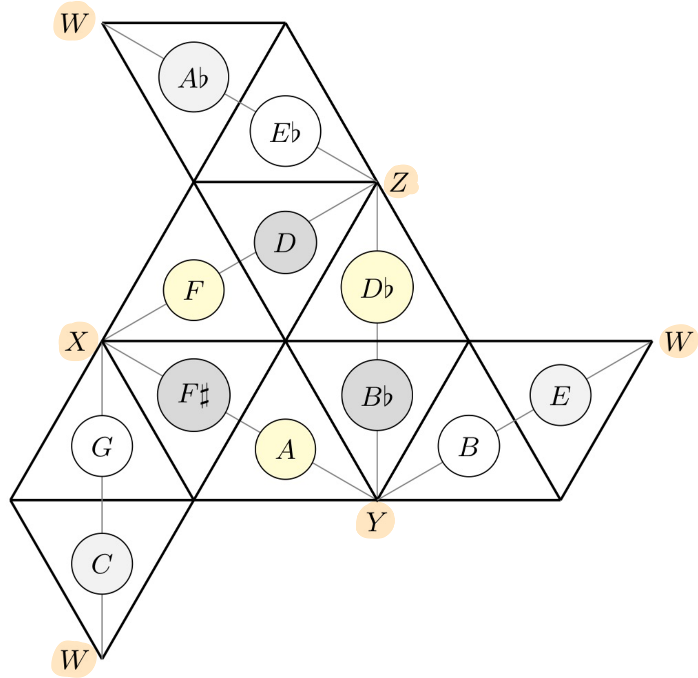
Example: $C\Delta 9$ = C − E − G − B − D





fold along grey edges and identify along boundary

⇝ tetrahedron

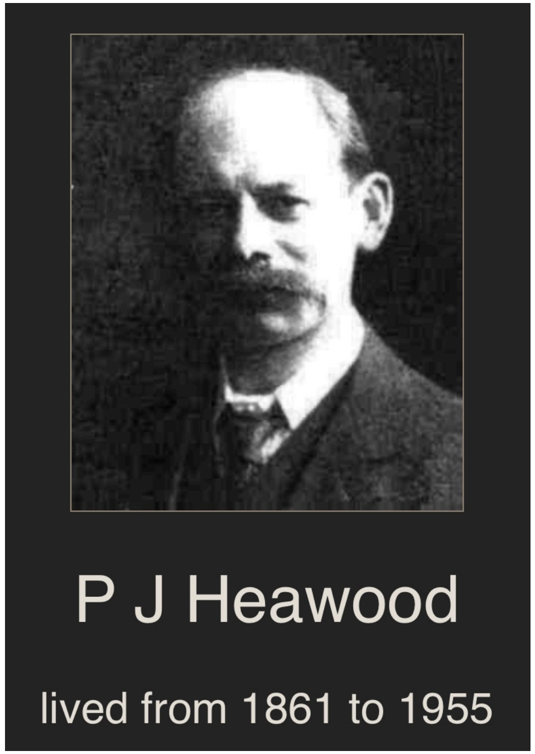


Every major 9th chord appears precisely once.

↳ 120° rotational symmetry transposes up by a major 3rd.

Fano, Heawood and diatonic 7^{th} chords

Let $\gamma(g)$ be the minimum number of colours needed to colour a "map" on a genus g orientable surface.



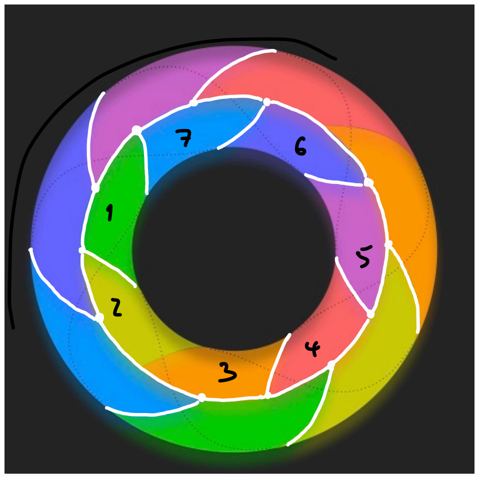
1890 : Heawood Thm : for $g \geq 1$

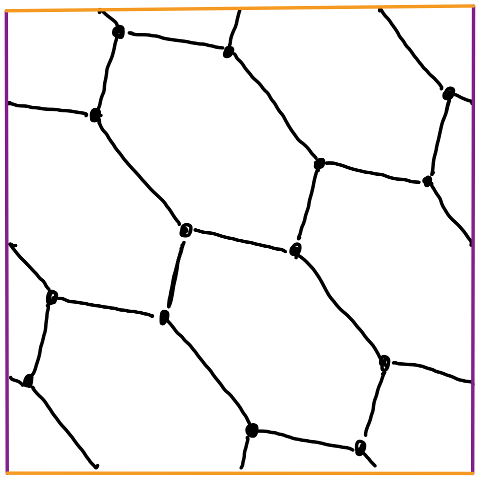
$$\gamma(g) \leq \left\lfloor \frac{7+\sqrt{1+48g}}{2} \right\rfloor$$

Heawood conjectured $=$ and proved it for $g=1$ ("7 colour theorem" on a torus), using the "Heawood graph".

- 1887 – 1939 at Durham University
- OBE in 1939 for his work as Secretary of the Durham Castle Restoration Fund.

(Maths History . St. Andrews)





Wiki

Fano, Heawood and diatonic 7^{th} chords

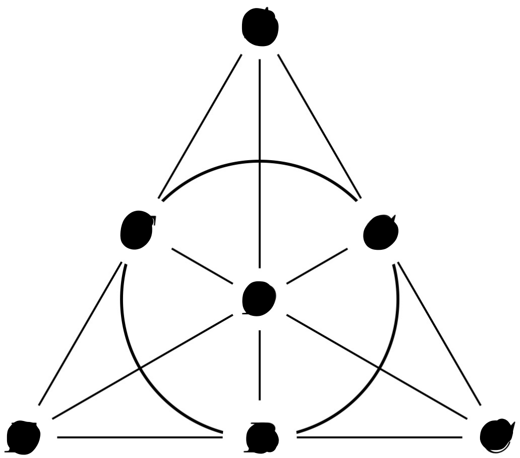
1892 "Sui postulati fundamentali della geometria proiettiva" Giorn. Mat 30

Gino Fano
1871 – 1952
(Maths History . St. Andrews)

Fano plane : $\mathbb{P}^2(\mathbb{F}_2) \triangleq \mathbb{F}_2^3 \setminus \{(0,0,0)\}$

- 7 points , 7 lines
- 3 points on each line
- one line through any two points

usual depiction:



Fano, Heawood and diatonic 7th chords

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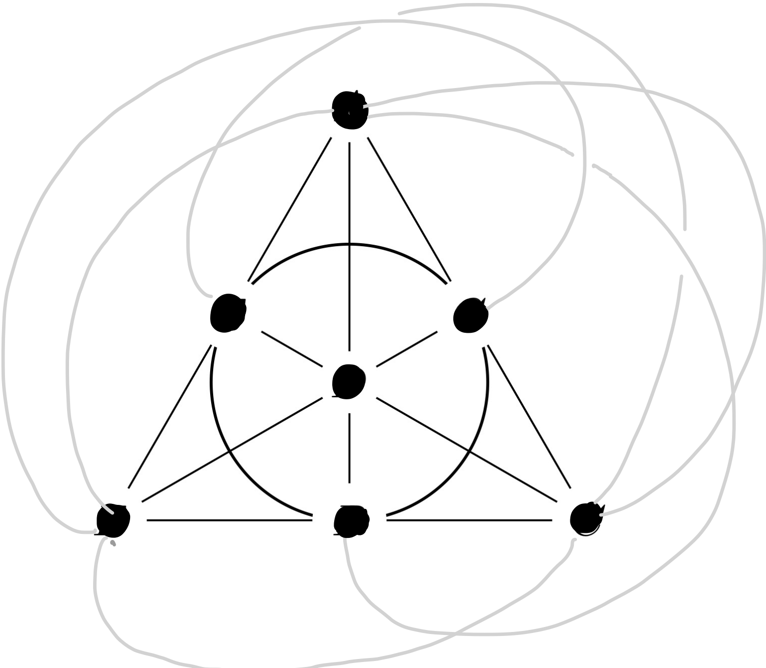
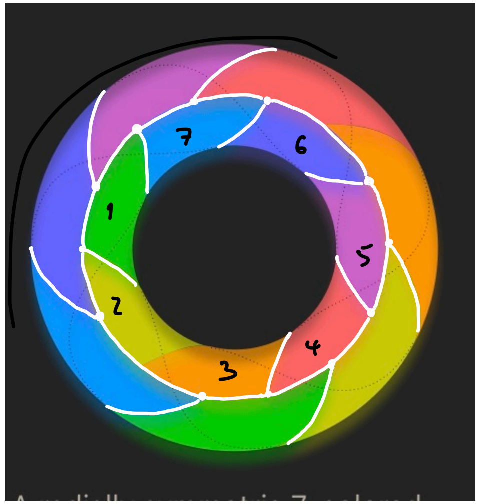
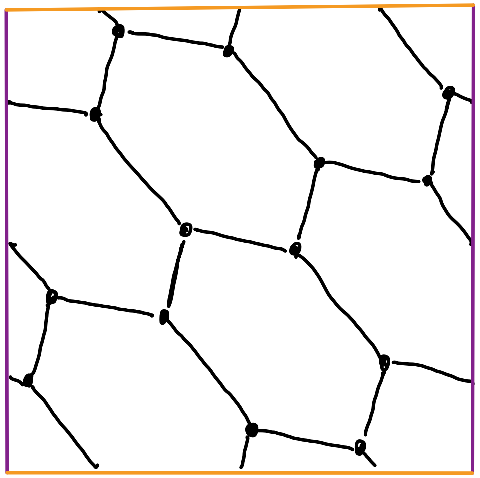


each line segment should be completed to a closed curve

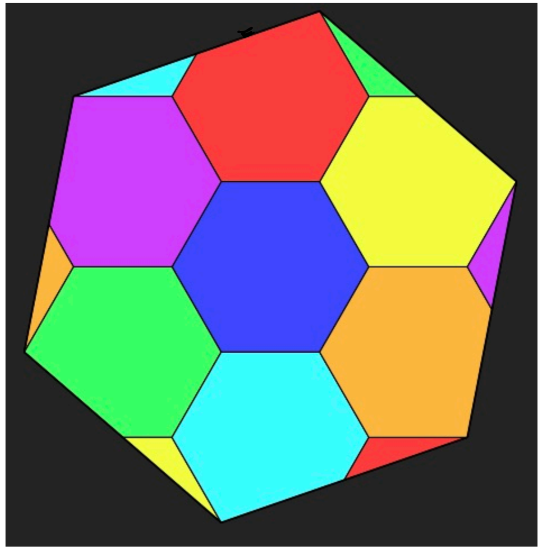
$\Longrightarrow$ complete graph on 7 vertices

Heawood's graph on the torus - revisited

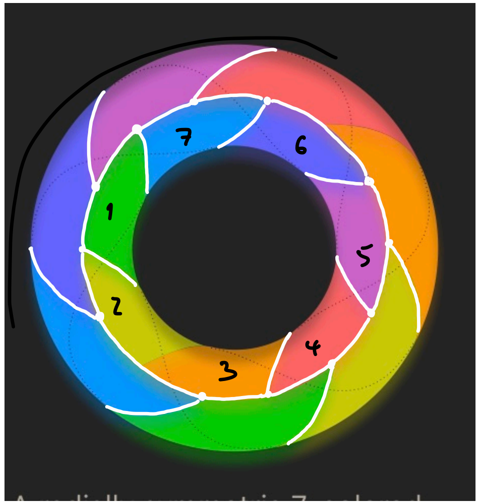


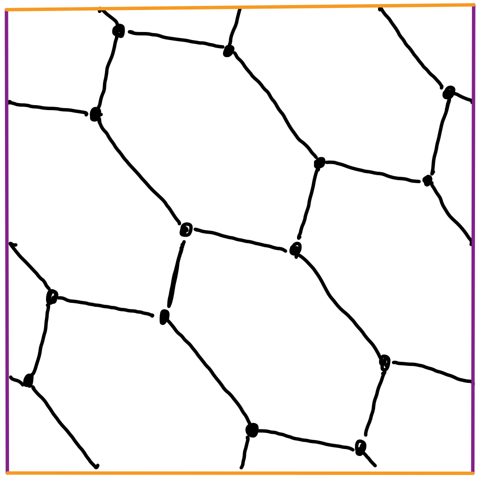


or:

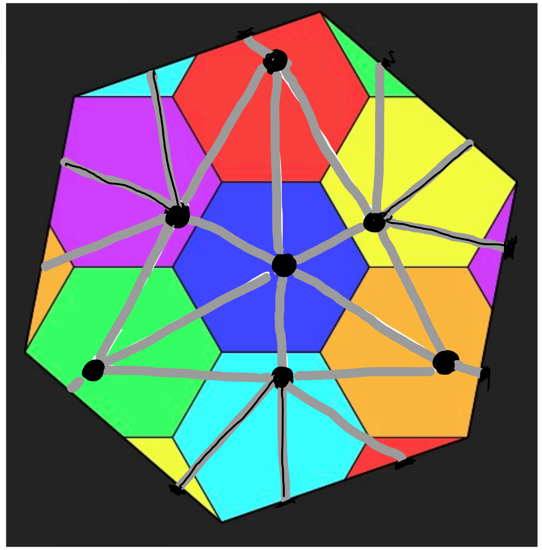


Heawood's graph on the torus – revisited



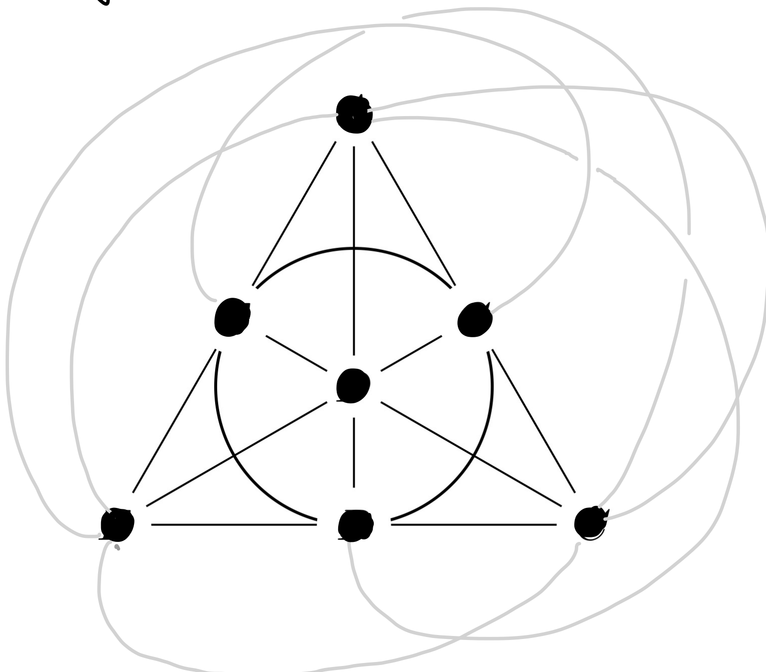


or



The Fano plane is dual graph to Heawood's graph on the torus

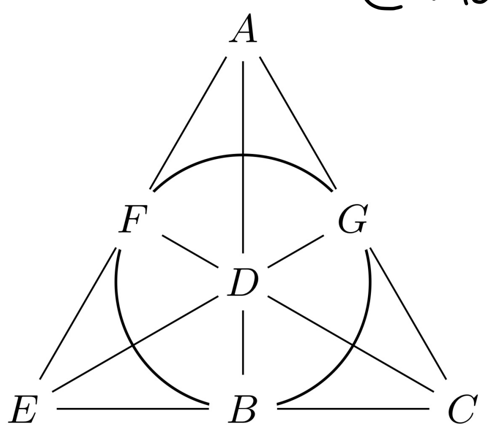
This gives an embedding of the 'Fano plane' graph in a torus.



Fano, Heawood and diatonic 7^{th} chords

Label the vertices of the Fano plane with pitch classes of a major scale as follows:

C-major



Every line is a diatonic 7^{th} chord (with 5^{th} omitted)

C - E - ~~G~~ - B   C△   major 7^{th}

D - F - ~~A~~ - C   Dm7   minor 7^{th}

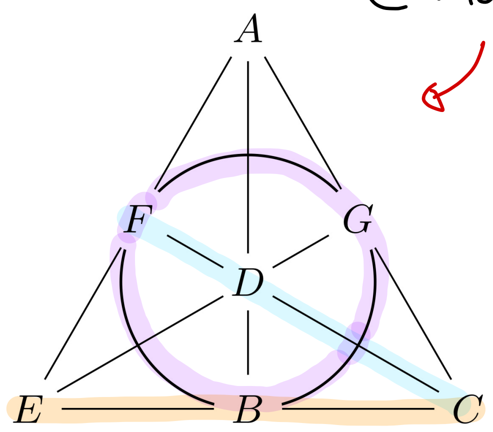
⋮   ⋮   ⋮

G - B - ~~D~~ - F   G7   dominant 7^{th}

Fano, Heawood and diatonic 7^{th} chords

Label the vertices of the Fano plane with pitch classes of a major scale as follows:

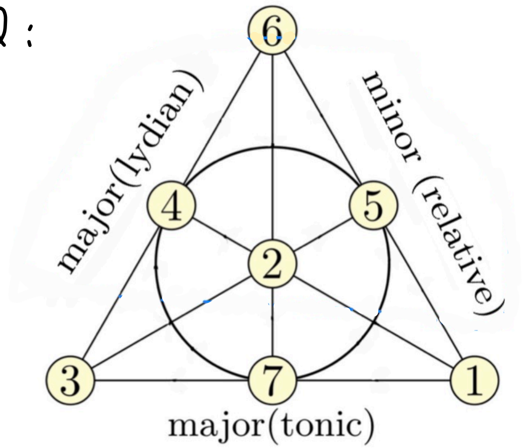
C-major



- Every line is a diatonic 7^{th} chord (with 5^{th} omitted)

C-E-~~G~~-B	C△	major 7^{th}
D-F-~~A~~-C	Dm7	minor 7^{th}
⋮	⋮	⋮
G-B-~~D~~-F	G7	dominant 7^{th}

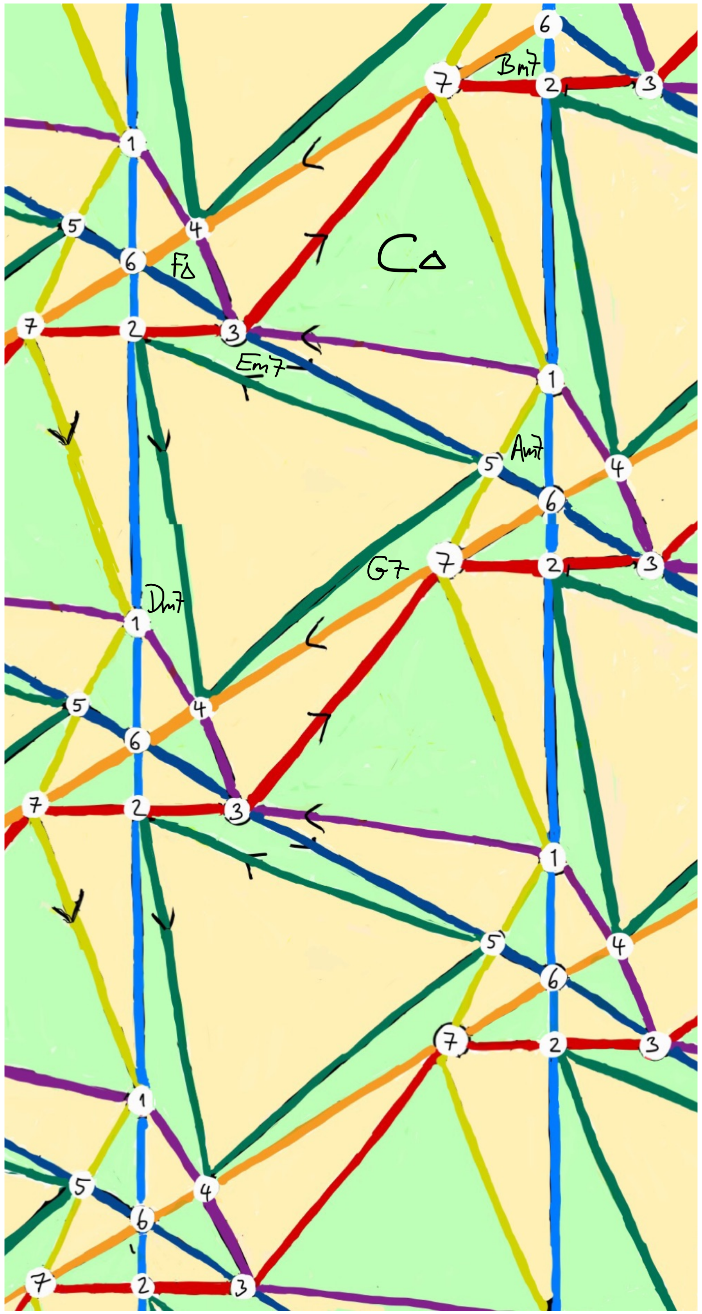
General:



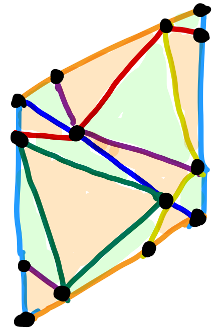
projective geometry

- any two notes of the major scale belong to a unique diatonic 7^{th} chord (in shell voicing)
- any two diatonic 7^{th} chords (in shell voicing) have a unique pitch class in common.

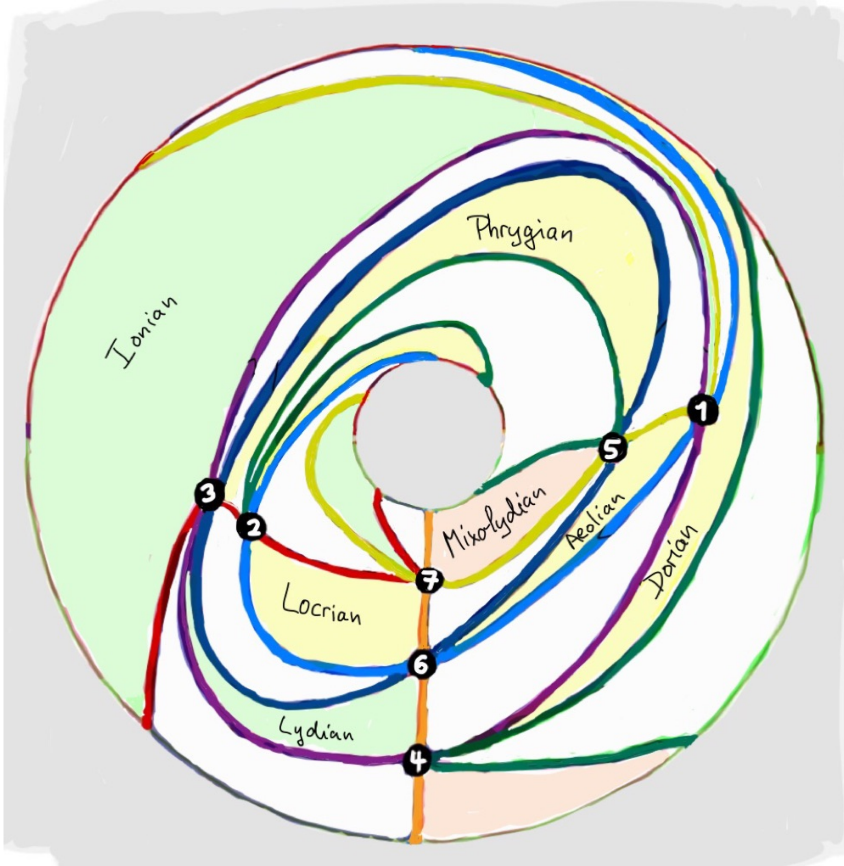
Diatonic vertex tonnetz on a torus



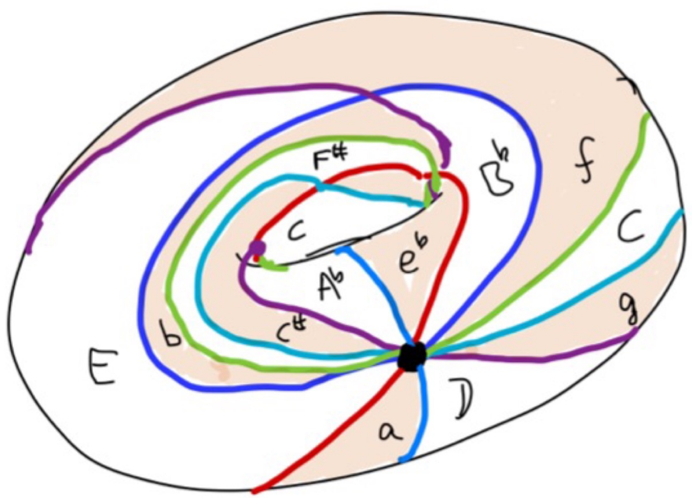
- periodic picture for Fano-plane embedded in torus

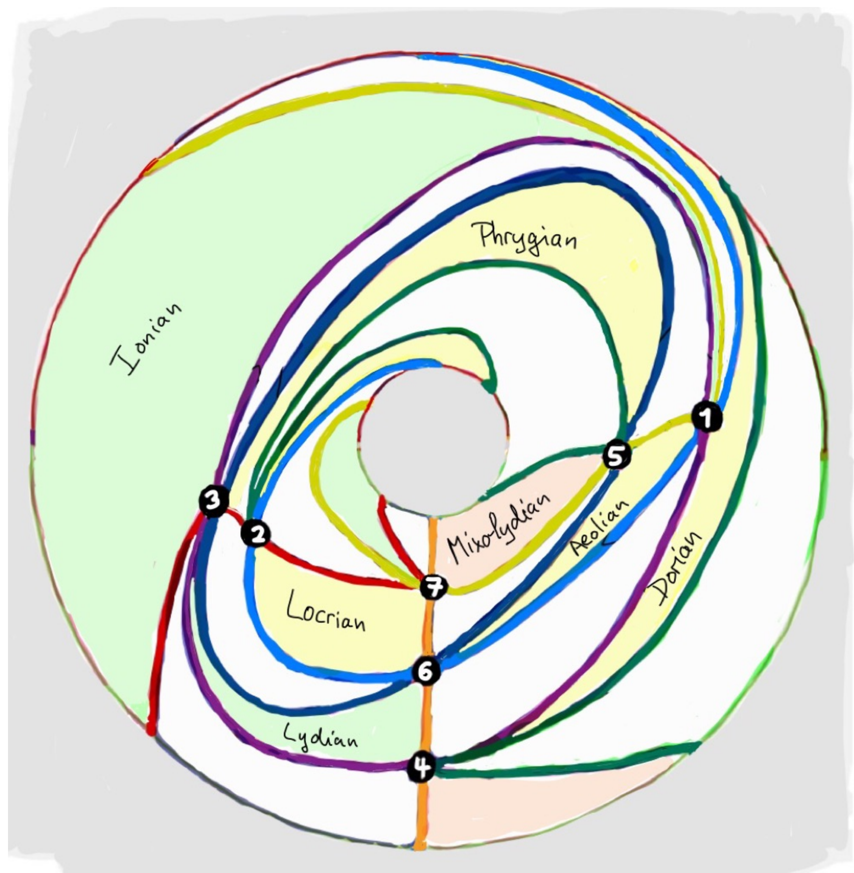


- vertex labeling (1 ⟷ C, 2 ⟷ D, . . .)
- 7 green triangles ⟷ diatonic 7th chords.
- remaining triangles 'transition' between them.



Thank You!


F#
c
B^b
f
C
e^b
A^b
$c^\#$
b
E
g
a
D


Phrygian
Ionian
1
5
3
2
7
6
4
Mixolydian
Aeolian
Dorian
Locrian
Lydian