Braids and q-rationals

Alexander Veselov
Loughborough, UK

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Plan

- Braid group B_3 and $PSL(2,\mathbb{Z})$
- Burau representation of B_3: specialization problem
- q-rationals and $PSL(2,\mathbb{Z})_q$
- Singular set of q-rationals and faithful Burau specialisations

References

[MGO-2020] S. Morier-Genoud and V. Ovsienko *q-deformed rationals and q-continued fractions.* Forum Math. Sigma **8** (2020) e13,55 pp.

[LMGOV-2024] L. Leclere, S. Morier-Genoud, V. Ovsienko and A.V. *On radius of convergence of q-deformed real numbers.* Mosc. Math. J. **24:1** (2024), 1-19.

[MGOV-2024] S. Morier-Genoud, V. Ovsienko and A.V. *Burau representation of braid groups and q-rationals.* IMRN, **rnad318** (2024), 1-10.

Artin's braid groups

Emil Artin, 1925: n-strand *braid group* $\mathcal{B}_n$ is generated by $n-1$ elements $\sigma_1, \ldots, \sigma_{n-1}$ with braid relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad i = 1, \ldots, n-1,$$

and $\sigma_i \sigma_j = \sigma_j \sigma_i$ when $|i - j| > 1$.

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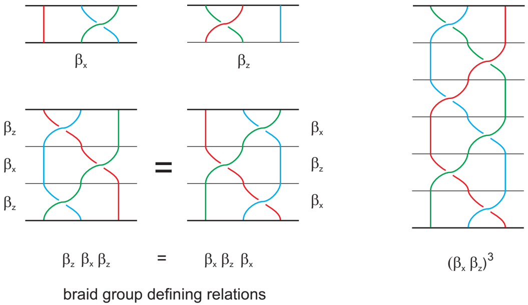

Figure: Braid relations and centre of B_3

The *modular group* $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\pm I$ is generated by matrices $U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, or, by S and $P = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$, satisfying relations $S^2 = P^3 = -I$:

$$PSL(2,\mathbb{Z}) =< P, S | P^3 = S^2 = e > .$$

Braid group B_3 and modular group

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Important fact: $PSL(2,\mathbb{Z}) = \mathcal{B}_3/\mathbb{Z}$, where $\mathbb{Z}$ is the centre of $\mathcal{B}_3$.

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Indeed, let $x = \sigma_1\sigma_2$, $y = \sigma_2\sigma_1\sigma_2$, then modulo braid relation

$$y^2 = \sigma_2\sigma_1\sigma_2\sigma_2\sigma_1\sigma_2 = \sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2 = x^3,$$

so $\mathcal{B}_3 =< x, y | x^3 = y^2 >$. The homomorphism

$$\chi : \mathcal{B}_3 \to PSL(2,\mathbb{Z}), \quad \chi(x) = P,\ \chi(y) = S$$

is surjective with the $Ker\chi = Z(\mathcal{B}_3) \cong \mathbb{Z}$ generated by $x^3 = y^2$.

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is surjective with the $Ker\chi = Z(\mathcal{B}_3) \cong \mathbb{Z}$ generated by $x^3 = y^2$.

Thus χ is a nice representation of $\mathcal{B}_3$ but it is not faithful...
Is there faithful one?

Werner Burau, 1936: *Burau representation* $\rho_n : \mathcal{B}_n \to GL(n-1, \mathbb{Z}[t, t^{-1}])$.

For $n = 3$ the Burau representation $\rho_3 : \mathcal{B}_3 \to GL(2, \mathbb{Z}[t, t^{-1}])$ is defined by

$$\rho_3 \; : \quad \sigma_1 \;\mapsto\; \begin{pmatrix} -t & 1 \\ 0 & 1 \end{pmatrix}, \quad \sigma_2 \;\mapsto\; \begin{pmatrix} 1 & 0 \\ t & -t \end{pmatrix},$$

where t is a formal parameter. When $t = 1$ we have canonical homomorphism $\mathcal{B}_3 \to S_3$, while $t = -1$ corresponds to the representation χ.

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Burau used this representation to introduce the invariant of the link $L = L(\beta)$, $\beta \in \mathcal{B}_n$ by the formula

$$\Delta_L(t) = \frac{1-t}{1-t^n} \det(I - \rho_n(\beta)),$$

which (up to a unit in $\mathbb{Z}[t, t^{-1}]$) turned out to be related to the *Alexander polynomial* (**Alexander, 1928**).

Faithfulness of Burau representation

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Morier-Genoud, Ovsienko and AV 2023: The specialized Burau representation ρ_3^t is faithful for all $t \in \mathbb{C}^*$ outside the annulus

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Conjecture (MGOV): The specialized Burau representation ρ_3^t is faithful for all $t \in \mathbb{C}^*$ outside the annulus

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There is a natural linear-fractional action of $PSL(2,\mathbb{Z})_q$ on the space $\mathbb{Z}(q)$ of rational functions in q with integer coefficients.

The *q-rationals* are the functions from $\mathbb{Z}(q)$ in the orbit of any point from the set $\{0, 1, \infty\}$ for this action.

q-deformed modular group $PSL(2,\mathbb{Z})_q$ and q-rationals

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This agrees with the notion of q-integers due to Euler and Gauss:

$$\begin{aligned} [n]_q &:= 1 + q + q^2 + \ldots + q^{n-1} \\ [-n]_q &:= -q^{-1} - q^{-2} \ldots - q^{-n}. \end{aligned}$$

Let $\frac{r}{s} = [a_1, \ldots, a_{2m}]$ be the continued fraction expansion and consider the corresponding matrix decompositions

$$M(a_1, \ldots, a_{2m}) := \begin{pmatrix} r & v \\ s & u \end{pmatrix} = R^{a_1} L^{a_2} \cdots R^{a_{2m-1}} L^{a_{2m}},$$

$$M_q(a_1, \ldots, a_{2m}) := \begin{pmatrix} \mathcal{R}(q) & \mathcal{V}(q) \\ \mathcal{S}(q) & \mathcal{U}(q) \end{pmatrix} = R_q^{a_1} L_q^{a_2} \cdots R_q^{a_{2m-1}} L_q^{a_{2m}}.$$

q-deformed continued fractions

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For example, for $2 = 1 + \frac{1}{1} = [1, 1]$ we have

$$M(1,1) = RL = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad M_q(1,1) = R_q L_q = \begin{pmatrix} 1+q & q^{-1} \\ 1 & q^{-1} \end{pmatrix},$$

so $[2]_q = 1 + q$ in agreement with Euler and Gauss.

Properties of polynomials $\mathcal{R}(q)$ and $\mathcal{S}(q)$

Suppose that $\frac{r}{s} \geq 1$.

Morier-Genoud, Ovsienko 2020: *The polynomials $\mathcal{R}$ and $\mathcal{S}$ have positive integer coefficients and satisfy the following "reflection and mirror" properties:*

$$\left[\frac{s}{r}\right]_q = \frac{\mathcal{S}(q^{-1})}{\mathcal{R}(q^{-1})}, \qquad \left[-\frac{r}{s}\right]_q = -\frac{\mathcal{R}(q^{-1})}{q\mathcal{S}(q^{-1})}.$$

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Conjecture (LMGOV) *For every rational $\frac{r}{s}$ the roots of the polynomials $\mathcal{R}$ and $\mathcal{S}$ belong to the open annulus*

$$\frac{3 - \sqrt{5}}{2} < |q| < \frac{3 + \sqrt{5}}{2}.$$

Example (**MGO 2020**): Fibonacci polynomials

Let F_n be the n^{th} Fibonacci number, $\frac{F_{n+1}}{F_n}$ are the convergents for $\varphi = \frac{1+\sqrt{5}}{2}$,

$$\left[\frac{F_{n+1}}{F_n}\right]_q =: \frac{\tilde{\mathcal{F}}_{n+1}(q)}{\mathcal{F}_n(q)}$$

be their q-deformed versions. The polynomials $\mathcal{F}_n(q)$ and $\tilde{\mathcal{F}}_n(q)$ are of degree $n-2$ (for $n \geq 2$) and are mirror of each other:

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The *Fibonacci polynomials* $\mathcal{F}_n(q)$ are determined by the recurrence

$$\mathcal{F}_{n+2}(q) = [3]_q\,\mathcal{F}_n(q) - q^2\mathcal{F}_{n-2}(q), \quad [3]_q = 1 + q + q^2$$

with $\mathcal{F}_0(q) = 1,\ \mathcal{F}_2(q) = 1 + q; \quad \mathcal{F}_1(q) = 1,\ \mathcal{F}_3(q) = 1 + q + q^2$:

$$\left[\tfrac{8}{5}\right]_q = \frac{1 + 2q + 2q^2 + 2q^3 + q^4}{1 + 2q + q^2 + q^3},$$

$$\left[\tfrac{13}{8}\right]_q = \frac{1 + 2q + 3q^2 + 3q^3 + 3q^4 + q^5}{1 + 2q + 2q^2 + 2q^3 + q^4},$$

$$\left[\tfrac{21}{13}\right]_q = \frac{1 + 3q + 4q^2 + 5q^3 + 4q^4 + 3q^5 + q^6}{1 + 3q + 3q^2 + 3q^3 + 2q^4 + q^5}.$$

Suppose that a sequence of rationals $\frac{r_m}{s_m}$ converges to an irrational number x.

Morier-Genoud, Ovsienko 2020: The coefficients of the Taylor series of rational functions $\frac{\mathcal{R}_m(q)}{\mathcal{S}_m(q)}$ stabilize as m grows.

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This allows to define a q-deformation $[x]_q$ as Taylor series with integer coefficients. In particular, for the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$ we have

$$[\varphi]_q = \frac{q^2 + q - 1 + \sqrt{(q^2+3q+1)(q^2-q+1)}}{2q}$$

with the radius of convergence $R = \frac{3-\sqrt{5}}{2}$.

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Conjecture [LMGOV 2021] For every real $x > 0$ the radius of convergence $R(x)$ of the series $[x]_q$ satisfies the inequality

$$R(x) \geq R(\varphi) = \frac{3-\sqrt{5}}{2}$$

and the equality holding only for x which are $PSL(2,\mathbb{Z})$-equivalent to φ.

Burau representation and q-rationals

Morier-Genoud, Ovsienko, AV 2023 The q-deformed action of the modular group coincides with projective version of Burau representation with $q = -t$.

Indeed, for $t = -q$ the matrices R_q and L_q coincide with

$$\rho_3(\sigma_1) = \begin{pmatrix} -t & 1 \\ 0 & 1 \end{pmatrix}, \quad \rho_3(\sigma_2)^{-1} = \begin{pmatrix} 1 & 0 \\ t & -t \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & -t^{-1} \end{pmatrix}.$$

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For instance, taking $\beta = \sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-1}$, we have the matrix

$$t^{-2} \begin{pmatrix} -t + t^2 - 2t^3 + t^4 & 1 - t + t^2 \\ -t + t^2 - t^3 & 1 - t \end{pmatrix} = q^{-2} \begin{pmatrix} q + q^2 + 2q^3 + q^4 & 1 + q + q^2 \\ q + q^2 + q^3 & 1 + q \end{pmatrix},$$

so that $\frac{1+q+2q^2+q^3}{1+q+q^2}$ and $\frac{1+q+q^2}{1+q}$ are q-deformed $\frac{5}{3}$ and $\frac{3}{2}$, respectively.

Burau specialisation problem and singular set of q-rationals

Define the **singular set** of q-rationals $\Sigma \subset \mathbb{C}^*$ as the union of complex poles of all q-rationals and the **extended singular set** as $\Sigma_* := \Sigma \cup \{1\}$.

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As a corollary we have

The specialized Burau representation ρ_3^t is faithful for all $t \in \mathbb{C}^*$ outside the annulus

$$3 - 2\sqrt{2} \leq |t| \leq 3 + 2\sqrt{2}$$

and, modulo LMGOV conjecture, outside the annulus

$$\frac{3-\sqrt{5}}{2} \leq |t| \leq \frac{3+\sqrt{5}}{2}.$$

Rational knots

Conway 1969: *Rational (2-bridge, Viergeflechte) link* $K(\frac{r}{s})$ corresponding to $\frac{r}{s} = [a_1, \dots, a_n]$

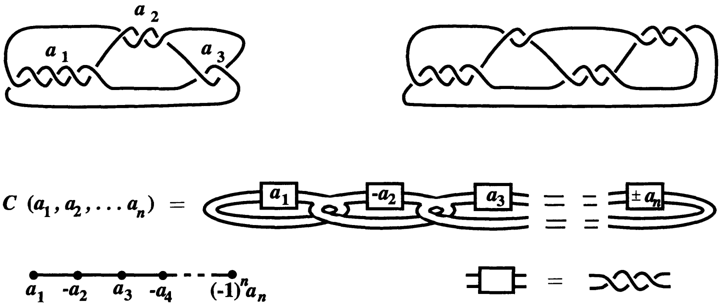

Figure: Rational link description from **Lickorish 1997**

There are different ways to associate a knot/link to a braid $\beta \in \mathcal{B}_3$:

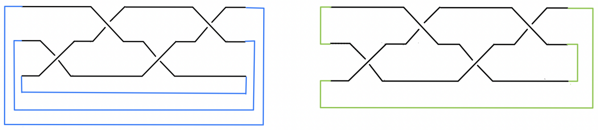

Figure: Standard and Conway closures of $\beta = \sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-1}$

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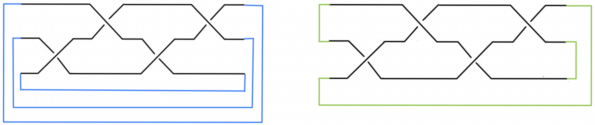

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Jones polynomial of rational knots

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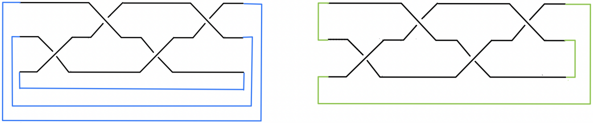

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Lee, Schiffler 2019; MGO 2020 *For the rational knot $K(\frac{r}{s})$ the (normalised) Jones polynomial can be expressed as*

$$J_{\frac{r}{s}}(q) = q\mathcal{R}(q) + (1-q)\mathcal{S}(q).$$

In particular, in our case of figure-eight knot $K(\frac{5}{3})$ the Jones polynomial is

$$J_{\frac{5}{3}}(q) = q^{-2}[q(1+q+2q^2+q^3)+(1-q)(1+q+q^2)] = q^{-2}+q^{-1}+1+q+q^2.$$

Concluding remarks

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Finally, there is an intruguing "left" q-deformation (**MGO; Bapat et al 2023**)

$$\left[\frac{r}{s}\right]_q^{\flat} = \frac{\mathcal{R}^{\flat}(q)}{\mathcal{S}^{\flat}(q)},$$

which, in particular, gives

$$[n]_q^{\flat} = 1 + q + \cdots + q^{n-2} + q^{n}.$$