Lifting SL_2-tilings

Matty van Son, The Open University
(jointly with Ian Short and Andrei Zabolotskii)

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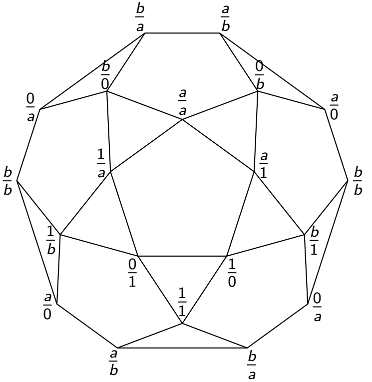
Contents



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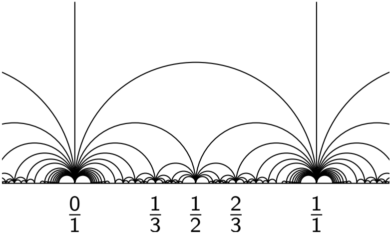

0/1
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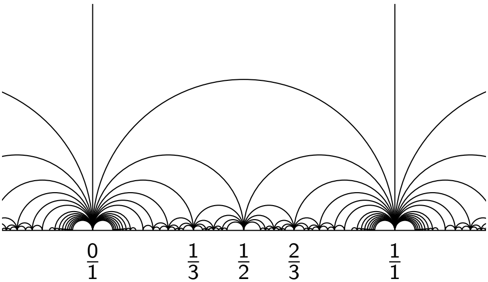
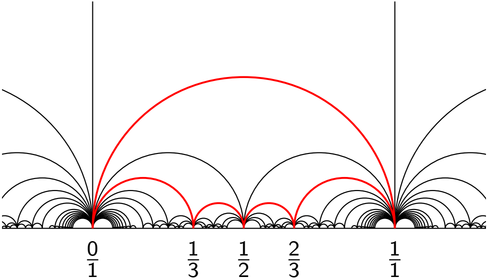
Part I

Tilings and paths in the Farey complex

Friezes and the Farey complex



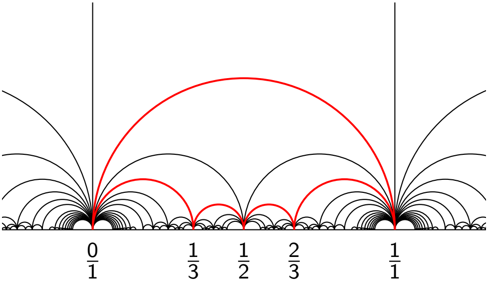
Friezes and the Farey complex



Theorem (S. Morier-Genoud, V. Ovsienko, S. Tabachnikov, (2016))

Positive Conway-Coxeter friezes are in one-to-one correspondence with normalised loops in the Farey complex, up to cyclic equivalence.

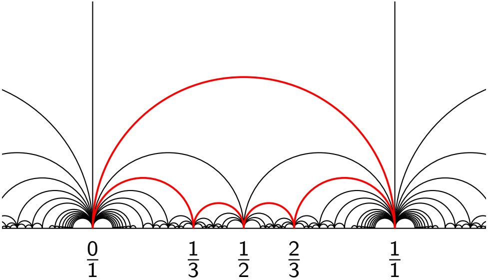
Friezes and the Farey complex



	0		0		0		0		0		0		0	
		1		1		1		1		1		1		
...	2		1		3		1		2		2		1	...
		1		2		2		1		3		1		
	1		1		1		1		1		1		1	
		0		0		0		0		0		0		

	b	
a		d
	c	

$$ad - bc = 1$$

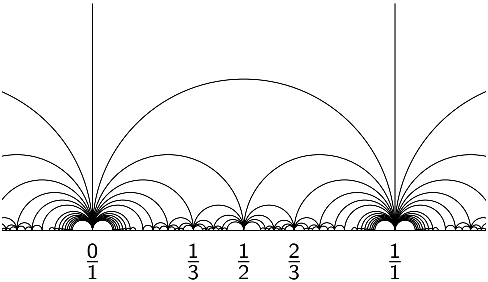


	0		0		0		0		0		0		0	
		1		1		1		1		1		1		
. . .	2		1		3		1		2		2		1	. . .
		1		2		2		1		3		1		
	1		1		1		1		1		1		1	
		0		0		0		0		0		0		

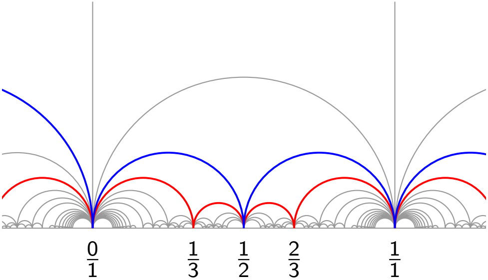
$$\begin{matrix} & b & \\ a & & d \\ & c & \end{matrix}$$

$$ad - bc = 1$$

Tilings and the Farey complex

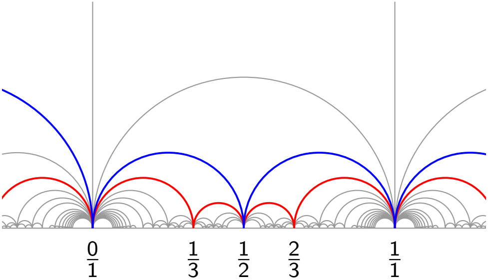


Tilings and the Farey complex



$$\left(\frac{a_i}{b_i}\right) = \left(\ldots, \frac{0}{1}, \frac{1}{2}, \frac{1}{1}, \ldots\right), \quad \left(\frac{c_j}{d_j}\right) = \left(\ldots, \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{2}, \frac{1}{1}, \ldots\right)$$

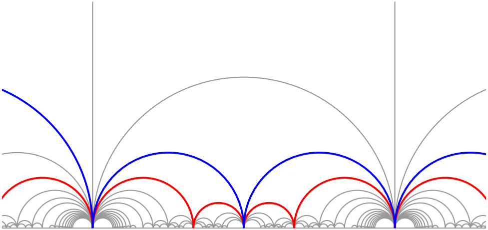
Tilings and the Farey complex



$$\left(\frac{a_i}{b_i}\right) = \left(\dots, \frac{0}{1}, \frac{1}{2}, \frac{1}{1}, \dots\right), \quad \left(\frac{c_j}{d_j}\right) = \left(\dots, \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}, \dots\right)$$

$$\begin{array}{ccccccc} & & & \vdots & & & \\ & 0 & -1 & -1 & -2 & -1 & \\ \cdots & 1 & 1 & 0 & -1 & -1 & \cdots \\ & 1 & 2 & 1 & 1 & 0 & \\ & & & \vdots & & & \end{array} \qquad m_{i,j} = a_i d_j - b_i c_j.$$

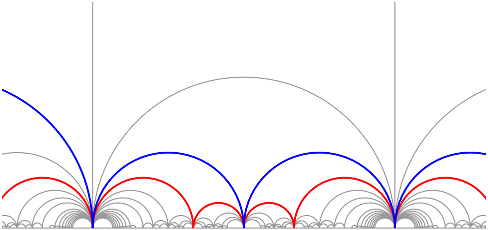
Tilings and the Farey complex



$$\left(\frac{a_i}{b_i}\right) = \left(\dots, \frac{0}{1}, \frac{-1}{-2}, \frac{1}{1}, \dots\right), \quad \left(\frac{c_j}{d_j}\right) = \left(\dots, \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}, \dots\right)$$

$$\begin{array}{ccccccc} & & & \vdots & & & \\ & 0 & -1 & -1 & -2 & -1 & \\ \cdots & -1 & -1 & 0 & 1 & 1 & \cdots \\ & 1 & 2 & 1 & 1 & 0 & \\ & & & \vdots & & & \end{array} \qquad m_{i,j} = a_i d_j - b_i c_j.$$

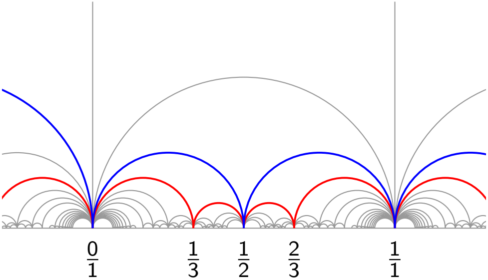
Tilings and the Farey complex



$$\begin{pmatrix} \frac{a_i}{b_i} \end{pmatrix} = \left(\ldots, \frac{0}{-1}, \frac{-1}{-2}, \frac{-1}{-1}, \ldots\right), \quad \begin{pmatrix} \frac{c_j}{d_j} \end{pmatrix} = \left(\ldots, \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}, \ldots\right)$$

$$\begin{array}{ccccccc} & & & \vdots & & & \\ & 0 & 1 & 1 & 2 & 1 & \\ \cdots & -1 & -1 & 0 & 1 & 1 & \cdots \\ & -1 & -2 & -1 & -1 & 0 & \\ & & & \vdots & & & \end{array} \qquad m_{i,j} = a_i d_j - b_i c_j.$$

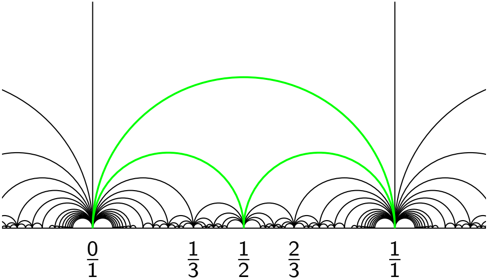
Tilings and the Farey complex



Theorem (I. Short)

There is a one-to-one correspondence between tame SL_2*-tilings over* $\mathbb{Z}$ *and pairs of paths modulo* $\mathrm{SL}_2(\mathbb{Z})$ *in the Farey complex.*

Tilings over $\mathbb{Z}$

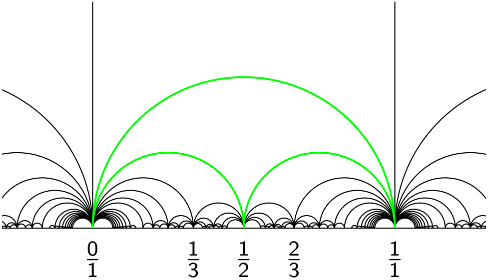


$$\gamma = \left(\dots, \frac{1}{1}, \frac{1}{2}, \frac{0}{-1}, \frac{-1}{-1}, \frac{-1}{-2}, \frac{0}{1}, \dots \right)$$

$$\delta = \left(\dots, \frac{1}{1}, \frac{1}{2}, \frac{0}{-1}, \frac{-1}{-1}, \frac{-1}{-2}, \frac{0}{1}, \dots \right)$$

$$\begin{array}{rrrrrrrr}
0 & -1 & -1 & 0 & 1 & 1 & 0 & -1 \\
1 & 0 & -1 & -1 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & -1 & -1 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & -1 & -1 & 0 & 1 \\
-1 & 0 & 1 & 1 & 0 & -1 & -1 & 0 \\
-1 & -1 & 0 & 1 & 1 & 0 & -1 & -1 \\
0 & -1 & -1 & 0 & 1 & 1 & 0 & -1 \\
1 & 0 & -1 & -1 & 0 & 1 & 1 & 0
\end{array}$$

Tilings over $\mathbb{Z}$



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$$\delta = \left(\ldots, \frac{1}{1}, \frac{1}{2}, \frac{0}{-1}, \frac{-1}{-1}, \frac{-1}{-2}, \frac{0}{1}, \ldots\right)$$

$$\begin{array}{cccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
\cdots & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots \\
 & & 0 & & 0 & & 0 & & 0 & & 0 &
\end{array}$$

Question

We can define tilings over general rings:

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$$\begin{matrix} a & b \\ c & d \end{matrix}, \qquad ad - bc = 1.$$

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$$\begin{matrix} a & b \\ c & d \end{matrix}, \qquad ad - bc = 1.$$

Question: Can we define a Farey complex for any ring R?

Farey complex $\mathscr{F}_{R,U}$

Definition

Let R be a commutative ring and let U be a group of units of R. Let $\sim$ be the relation $\frac{a}{b} \sim \frac{ua}{ub}$ for all $u \in U$. We call $\mathscr{F}_{R,U}$ a *Farey complex over* R where:

Farey complex $\mathscr{F}_{R,U}$

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- the vertices of $\mathscr{F}_{R,U}$ are equivalence classes $\left\{\frac{a}{b} : a, b \in R,\ aR + bR = R\right\} / \sim$;

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- the vertices of $\mathscr{F}_{R,U}$ are equivalence classes $\left\{\frac{a}{b} : a, b \in R,\ aR + bR = R\right\} / \sim$;
- an edge goes from $\frac{a}{b}$ to $\frac{c}{d}$ whenever $ad - bc \in U$.

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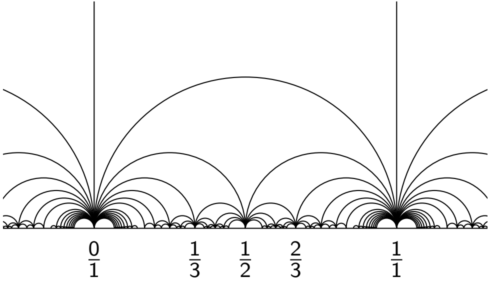

$\mathscr{F}_{\mathbb{Z},\{\pm 1\}}$

Farey complex $\mathscr{F}_{R,U}$

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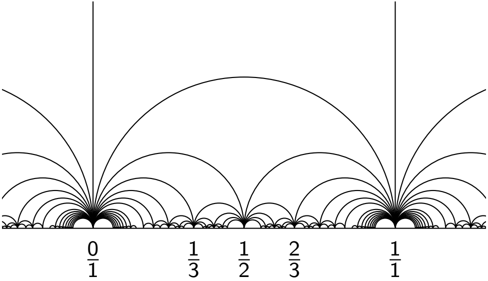

$\mathscr{F}_{\mathbb{Z},\{\pm 1\}}$

$\left\{\frac{a}{b} : a, b \in \mathbb{Z},\ a, b \text{ coprime}\right\}.$

Farey complex $\mathscr{F}_{R,U}$

Definition

Let R be a commutative ring and let U be a group of units of R. Let $\sim$ be the relation $\frac{a}{b} \sim \frac{ua}{ub}$ for all $u \in U$. We call $\mathscr{F}_{R,U}$ a *Farey complex over R* where:

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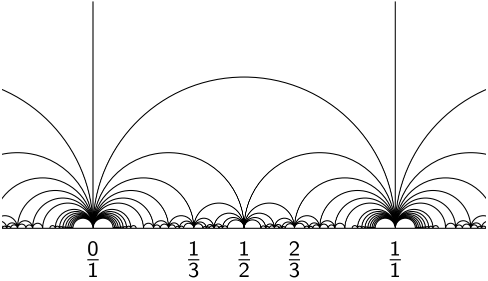
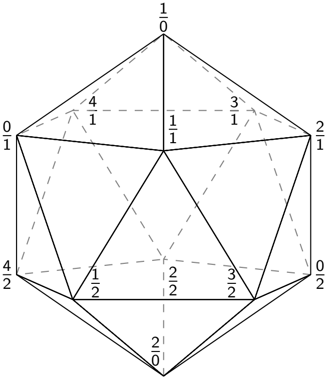


$\mathscr{F}_{\mathbb{Z},\{\pm 1\}}$

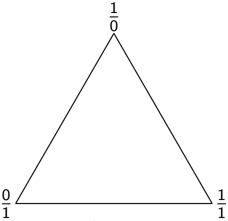
$\left\{\frac{a}{b} : a, b \in \mathbb{Z},\ a, b \text{ coprime}\right\}$.

$\frac{a}{b}$ — $\frac{c}{d}$ when $ad - bc = \pm 1$.

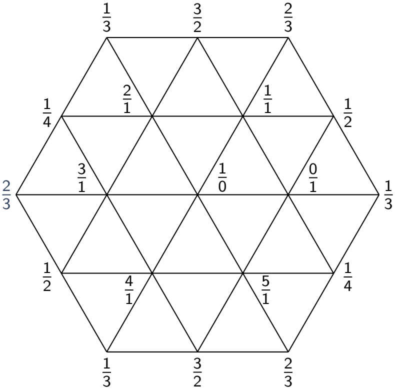
Farey complex $\mathscr{F}_{R,U}$



$\mathscr{F}_{\mathbb{Z}/5\mathbb{Z}}$



$\mathscr{F}_{\mathbb{Z}/2\mathbb{Z}}$

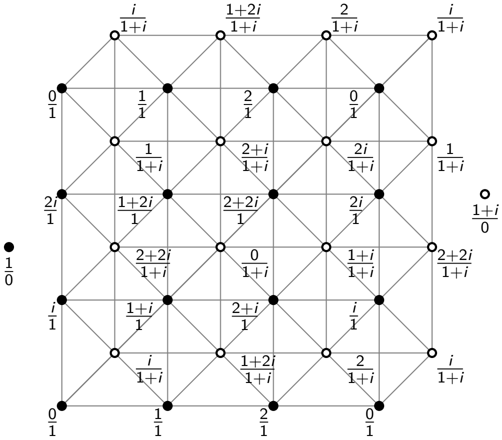


$\mathscr{F}_{\mathbb{Z}/6\mathbb{Z}}$

Farey complex $\mathscr{F}_{R,U}$

Example

Consider $R = \mathbb{Z}[i]/3\mathbb{Z}[i]$ and $U = \{\pm 1, \pm i\}$. Then $\mathscr{F}_{R,U}$ looks like



Tilings and the Farey complex

Consider the connected edges in $\mathscr{F}_{R,\{\pm 1\}}$

$$\cdots \leftrightarrows \frac{a}{b} \leftrightarrows \frac{c}{d} \leftrightarrows \frac{e}{f} \leftrightarrows \cdots$$

Assume $ad - bc = 1$ and $cf - de = -1$.

Tilings and the Farey complex

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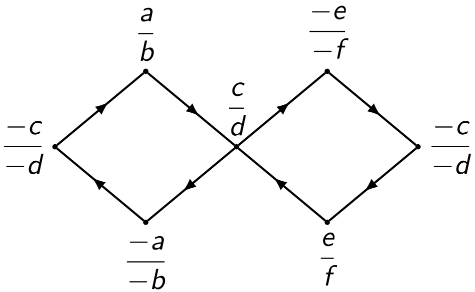

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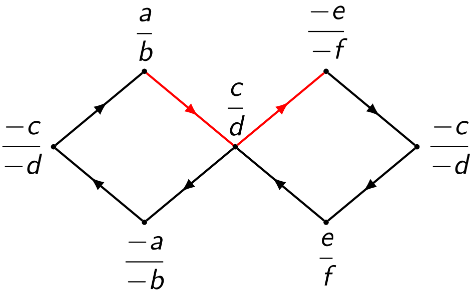

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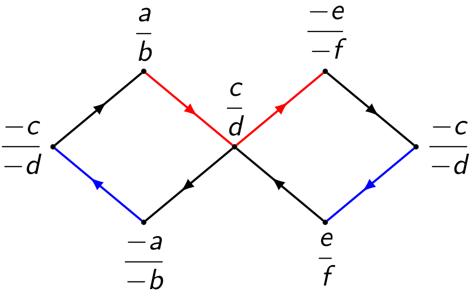
Assume $ad - bc = 1$ and $cf - de = -1$.



Tilings and the Farey complex

Recall, over $\mathbb{Z}$,

$$M \qquad\qquad\qquad\qquad -M$$

$$\begin{array}{ccccccc} & & & \vdots & & & \\ & 0 & -1 & -1 & -2 & -1 & \\ \cdots & 1 & 1 & 0 & -1 & -1 & \cdots \\ & 1 & 2 & 1 & 1 & 0 & \\ & & & \vdots & & & \end{array} \qquad \begin{array}{ccccccc} & & & \vdots & & & \\ & 0 & 1 & 1 & 2 & 1 & \\ \cdots & -1 & -1 & 0 & 1 & 1 & \cdots \\ & -1 & -2 & -1 & -1 & 0 & \\ & & & \vdots & & & \end{array}$$

Tilings and the Farey complex

Consider the tiling over $\mathbb{Q}$ with $U = \{\pm 1, \pm 2, \pm 2^{-1}\}$,

$$\begin{array}{ccccccc} & & & \vdots & & & \\ & 0 & -1 & -1 & -2 & -1 & \\ \cdots & 1 & 1 & 0 & -1 & -1 & \cdots \\ & 1 & 2 & 1 & 1 & 0 & \\ & & & \vdots & & & \end{array} \qquad \left(\frac{a_i}{b_i}\right)_{i\in\mathbb{Z}}, \quad \left(\frac{c_j}{d_j}\right)_{j\in\mathbb{Z}}.$$

Tilings and the Farey complex

Consider the tiling over $\mathbb{Q}$ with $U = \{\pm 1, \pm 2, \pm 2^{-1}\}$,

$$\begin{array}{ccccccc} & & & \vdots & & & \\ \cdots & 0 & -1 & -1 & -2 & -1 & \cdots \\ & 1 & 1 & 0 & -1 & -1 & \\ & 1 & 2 & 1 & 1 & 0 & \\ & & & \vdots & & & \end{array} \qquad \left(\frac{a_i}{b_i}\right)_{i\in\mathbb{Z}}, \quad \left(\frac{c_j}{d_j}\right)_{j\in\mathbb{Z}}.$$

Then multiplying alternate rows by 2 and $\frac{1}{2}$ we get another tiling

$$\begin{array}{cccccccc} & & & & \vdots & & & \\ 2 & & 0 & -2 & -2 & -4 & -2 & \\ \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & \cdots \\ 2 & & 2 & 4 & 2 & 2 & 0 & \\ & & & & \vdots & & & \end{array} \qquad \left(\frac{2a_i}{2b_i}, \frac{2^{-1}a_{i+1}}{2^{-1}b_{i+1}}\right)_{i\in\mathbb{Z}}, \quad \left(\frac{c_j}{d_j}\right)_{j\in\mathbb{Z}}.$$

Tilings and the Farey complex

Theorem (IS, MvS, AZ.)

There is a one-to-one correspondence

$$\mathrm{SL}_2(R)\Big\backslash\left\{\begin{matrix}\text{pairs of bi-infinite}\\ \text{paths in } \mathscr{F}_{R,U}\end{matrix}\right\} \longleftrightarrow (U\times U)\Big\backslash\left\{\begin{matrix}\text{tame } \mathrm{SL}_2\text{-tilings}\\ \text{over } R\end{matrix}\right\}.$$

Tilings and the Farey complex

Example

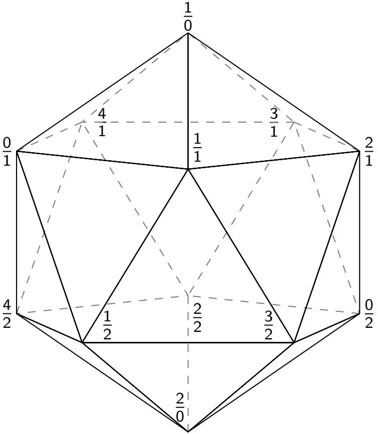


Figure: The Farey graph $\mathscr{F}_{\mathbb{Z}/5\mathbb{Z},\{\pm 1\}}$

Tilings and the Farey complex

Example

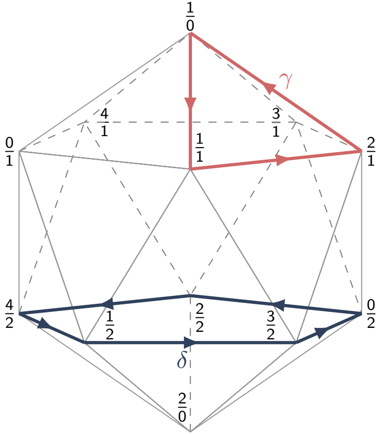


Figure: Two periodic bi-infinite paths on $\mathscr{F}_{\mathbb{Z}/5\mathbb{Z},\{\pm 1\}}$

Tilings and the Farey complex

Example

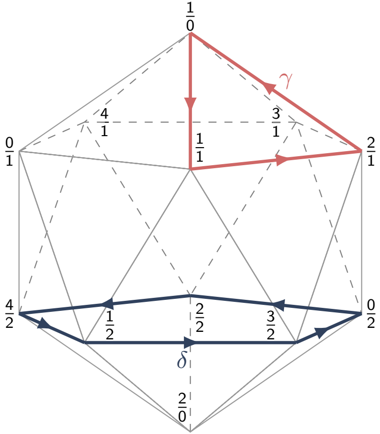


Figure: Two periodic bi-infinite paths on $\mathscr{F}_{\mathbb{Z}/5\mathbb{Z},\{\pm1\}}$

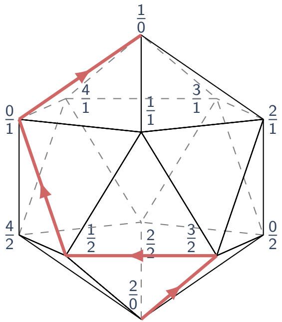
$$m_{i,j} = a_i d_j - b_i c_j$$

⋮

	2	2	2	2	2	2	2	2	2	
	2	0	3	1	4	2	0	3	1	
	1	3	0	2	4	1	3	0	2	
	2	2	2	2	2	2	2	2	2	
···	2	0	3	1	4	2	0	3	1	···
	1	3	0	2	4	1	3	0	2	
	2	2	2	2	2	2	2	2	2	
	2	0	3	1	4	2	0	3	1	
	1	3	0	2	4	1	3	0	2	

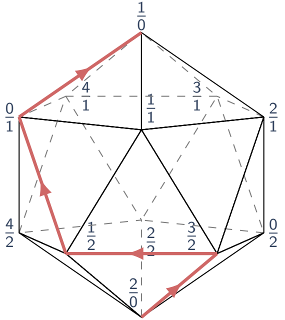
⋮

Friezes over R



$$
\begin{array}{ccccccccccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \\
\cdots & 1 & & 3 & & 3 & & 1 & & 2 & & 4 & & 4 & & 2 & & 1 & \cdots \\
 & & 2 & & 3 & & 2 & & 3 & & 2 & & 3 & & 2 & & 3 & & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 &
\end{array}
$$

Friezes over R



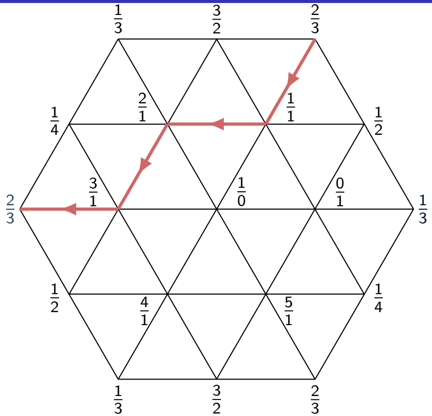
$$
\begin{array}{ccccccccccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \\
\cdots & 1 & & 3 & & 3 & & 1 & & 2 & & 4 & & 4 & & 2 & & 1 & \cdots \\
 & & 2 & & 3 & & 2 & & 3 & & 2 & & 3 & & 2 & & 3 & & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 &
\end{array}
$$

Theorem (IS, MvS, AZ.)

There is a one-to-one correspondence

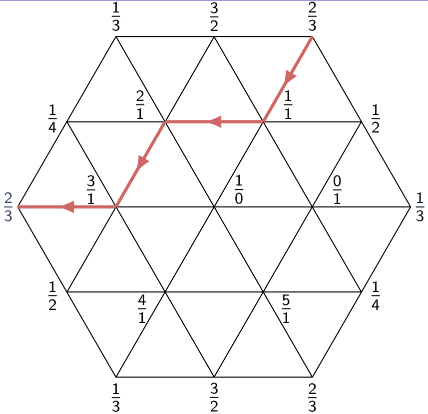
$$
\mathrm{SL}_2(R) \Big\backslash \left\{ \begin{array}{c} \text{paths of length } n \\ \text{between equivalent} \\ \text{vertices in } \mathscr{F}_{R,U} \end{array} \right\} \longleftrightarrow U \Big\backslash \left\{ \begin{array}{c} \text{tame semiregular} \\ \text{friezes of width } n \\ \text{over } R \end{array} \right\}.
$$

Friezes over R



$$
\begin{array}{ccccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & & & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & & \\
\cdots & 2 & & 4 & & 2 & & 4 & & 2 & & \cdots & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & &
\end{array}
$$

Friezes over R



$$
\begin{array}{ccccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & & & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & & \\
\cdots & 2 & & 4 & & 2 & & 4 & & 2 & & \cdots & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & &
\end{array}
$$

Theorem (IS, MvS, AZ.)

There is a one-to-one correspondence

$$\mathrm{SL}_2(R)\Big\backslash\left\{\begin{array}{c}\text{semiclosed paths of}\\ \text{length } n \text{ in } \mathscr{F}_{R,\{1\}}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{tame regular friezes}\\ \text{over } R \text{ of width } n\end{array}\right\}.$$

Part II

Lifting Tilings and Friezes

Lifting Tilings

$\mathscr{F}_{\mathbb{Z},\{\pm 1\}}$

1	−1	−2	−9	−7	−5	−3
1	0	−1	−5	−4	−3	−2
1	1	0	−1	−1	−1	−1
4	5	1	0	−1	−2	−3
7	9	2	1	−1	−3	−5
10	13	3	2	−1	−4	−7
3	4	1	1	0	−1	−2

$\mathscr{F}_{\mathbb{Z}/5\mathbb{Z},\{\pm 1\}}$

1	4	3	1	3	0	2
1	0	4	0	1	2	3
1	1	0	4	4	4	4
4	0	1	0	4	3	2
2	4	2	1	4	2	0
0	3	3	2	4	1	3
3	4	1	1	0	4	3

Lifting Tilings

Theorem (IS, MvS, AZ.)

For any ideal I in R, the map Φ from tame SL_2-tilings over R to tame SL_2-tilings over R/I

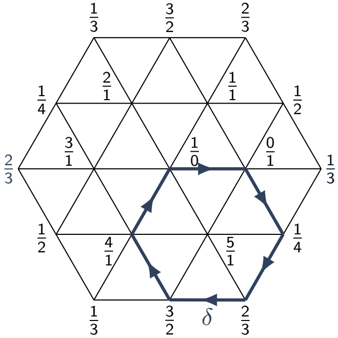
$$\Phi : m_{i,j} \mapsto m_{i,j} + I$$

is surjective if and only if the map $\varphi : \mathrm{SL}_2(R) \to \mathrm{SL}_2(R/I)$

$$\varphi : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a+I & b+I \\ c+I & d+I \end{pmatrix}$$

is surjective.

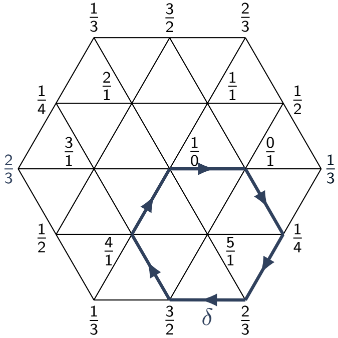
Lifting Friezes



$\mathscr{F}_{\mathbb{Z}/6\mathbb{Z},\{\pm 1\}}$

	0		0		0		0		0			
		1		1		1		1		1		
	2		2		2		2		2			
⋯		3		3		3		3		3		⋯
	1		1		1		1		1			
		0		0		0		0		0		

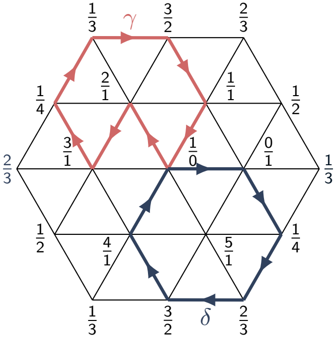
Lifting Friezes



$\mathscr{F}_{\mathbb{Z}/6\mathbb{Z},\{\pm 1\}}$

$$
\begin{array}{ccccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
\cdots & 2+6k_1 & & 2+6k_2 & & 2+6k_3 & & 2+6k_4 & & 2+6k_5 & & \cdots \\
 & & & & & \vdots & & & & & &
\end{array}
$$

Lifting Friezes

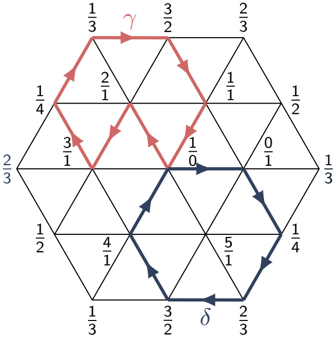


Definition

We call a closed path γ in $\mathscr{F}_{\mathbb{Z}/n\mathbb{Z},\{\pm 1\}}$ *strongly contractible* if it can be transformed to a point by application of the following homotopies:

- replacing a subpath $\langle v, u, v \rangle$ with $\langle v \rangle$;
- replacing a subpath $\langle v, u, w \rangle$ with $\langle v, w \rangle$, where $v \leftrightarrow w$ is an edge in $\mathscr{F}_{\mathbb{Z}/n\mathbb{Z},\{\pm 1\}}$;

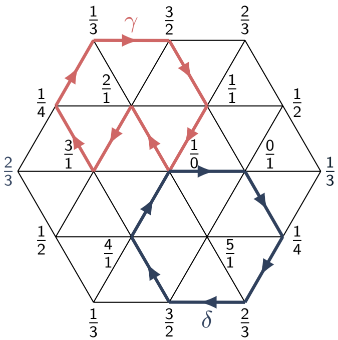
Lifting Friezes



$\mathscr{F}_{\mathbb{Z}/6\mathbb{Z},\{\pm 1\}}$

	0		0		0		0		0		0		0		
		1		1		1		1		1		1		1	
	2		2		2		2		1		5		1		
		3		3		3		1		4		4		1	
...	1		4		4		1		3		3		3		...
		1		5		1		2		2		2		2	
	1		1		1		1		1		1		1		
		0		0		0		0		0		0		0	

Lifting Friezes



Theorem (IS, MvS, AZ.)

A tame semiregular frieze **F** *over* $\mathbb{Z}/n\mathbb{Z}$ *lifts to a tame frieze over* $\mathbb{Z}$ *of the same width if and only if any path* γ *in* $\mathscr{F}_{\mathbb{Z}/n\mathbb{Z},\{\pm 1\}}$ *corresponding to* **F** *is a strongly contractible closed path.*

References

- S. Morier-Genoud, V. Ovsienko, S. Tabachnikov, $\mathrm{SL}_2(\mathbb{Z})$*-tilings of the torus, Coxeter-Conway friezes and Farey triangulations*, Enseign. Math. **61** (2015), no. 1-2, 71–92.
- I. Short, *Classifying* SL_2*-tilings*, Trans. Amer. Math. Soc., **376** (2023), no. 1, 1–38.
- I. Short, M. van Son, A. Zabolotskii, *Frieze patterns and Farey complexes*, arXiv: 2312.12953.

Thank you!