Hyperbolic Coxeter polytopes

A. Felikson

Independent University of Moscow, IHES

(joint with P. Tumarkin)

March 6, Lille 1.

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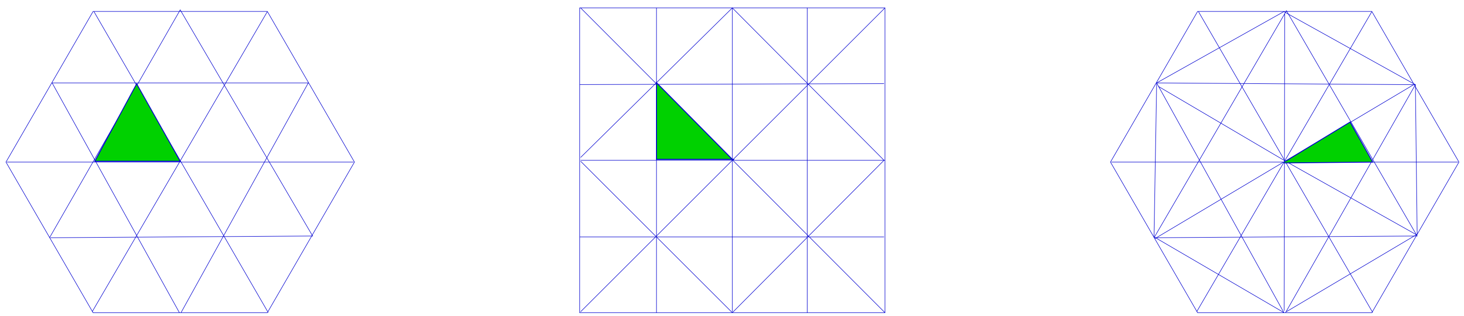
Example: Euclidean Coxeter triangles.



$(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3})$ $(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4})$ $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6})$

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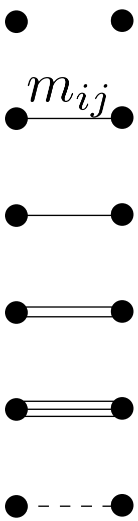
- $P \subset \mathbb{H}^n$. Infinitely many, No classification.

Coxeter diagrams

- Nodes $\longleftrightarrow$ facets f_i of P
- Edges:



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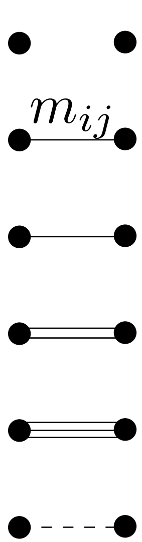

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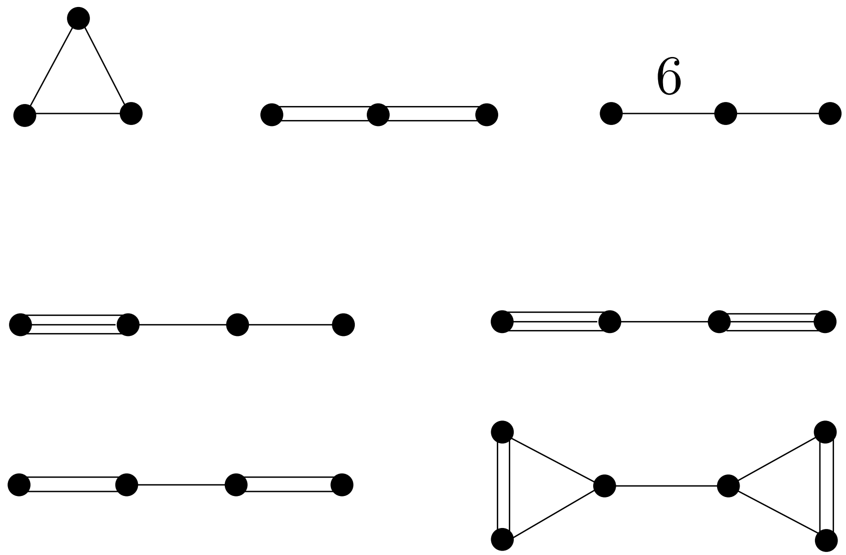
Examples:



Gram matrix

$$P \subset \mathbb{S}^n, \mathbb{E}^n \text{ or } \mathbb{H}^n \quad \longrightarrow \quad \text{Symmetric matrix } G = \{g_{ij}\}$$

- $g_{ii} = 1, \qquad g_{ij} = \begin{cases} -cos(\frac{\pi}{m_{ij}}), & \text{if } \angle(f_i f_j) = \pi/m_{ij}, \\ -1, & \text{if } f_i \text{ is parallel to } f_j, \\ -ch(\rho(f_i, f_j)), & \text{if } f_i \text{ and } f_j \text{ deverge.} \end{cases}$

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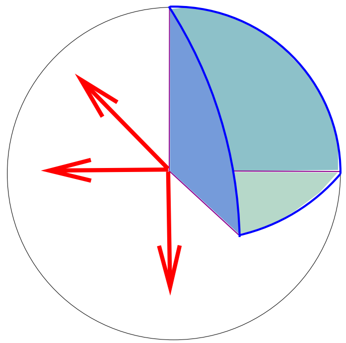

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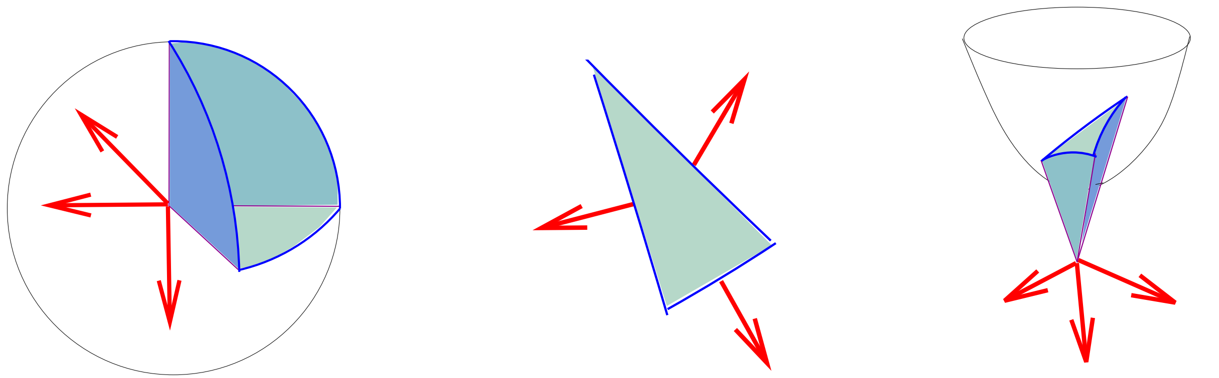

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	$\mathbb{S}^d$	$\mathbb{E}^d$	$\mathbb{H}^d$
$sgn(G)$	$(d+1, 0)$	$(d, 0)$	$(d, 1)$

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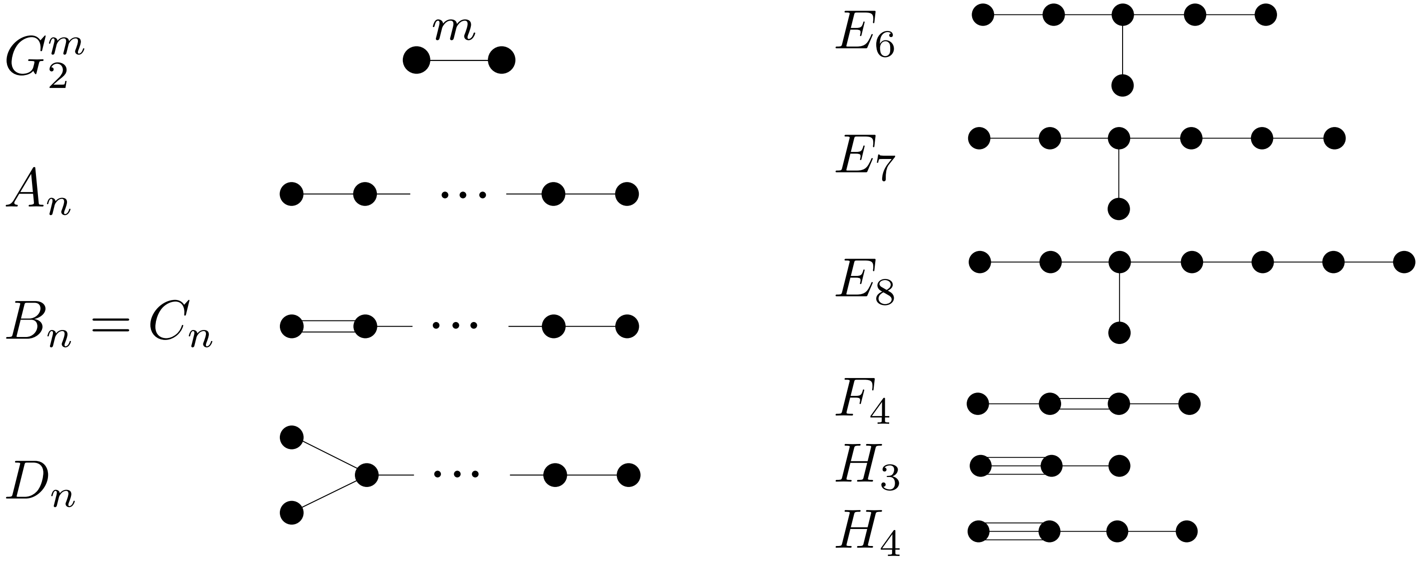

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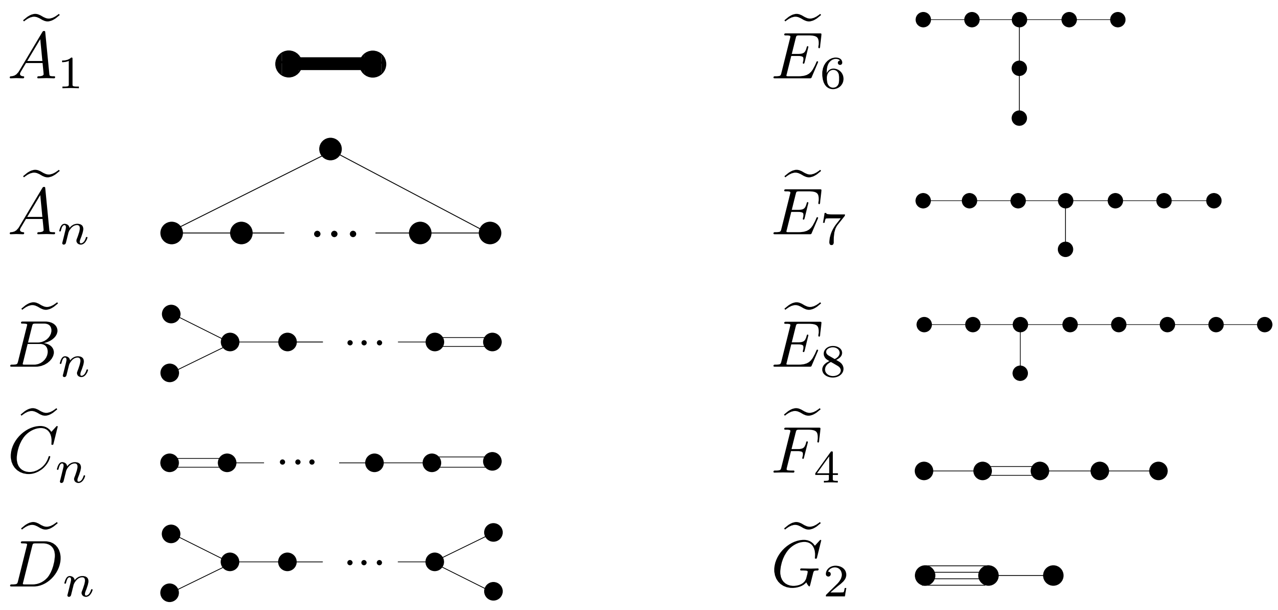

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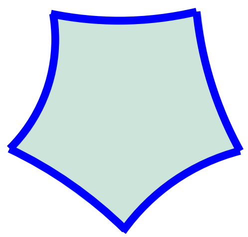

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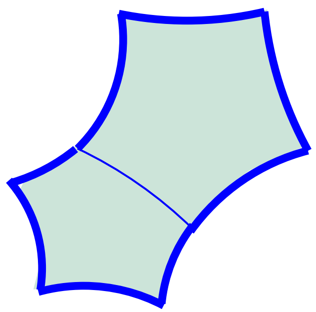

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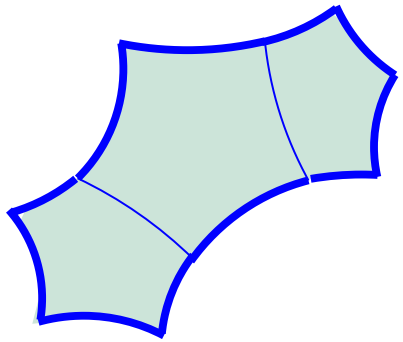

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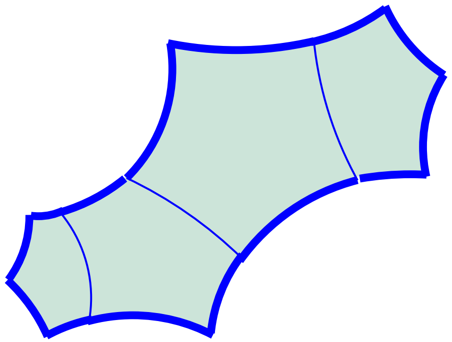

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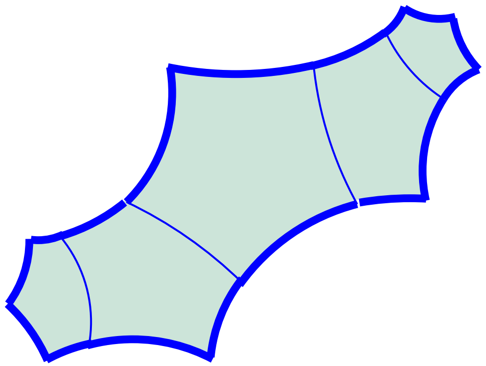

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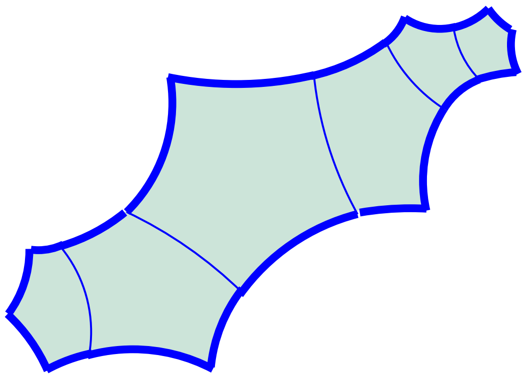

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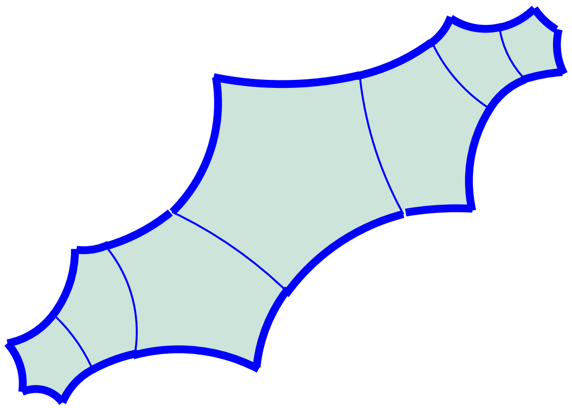

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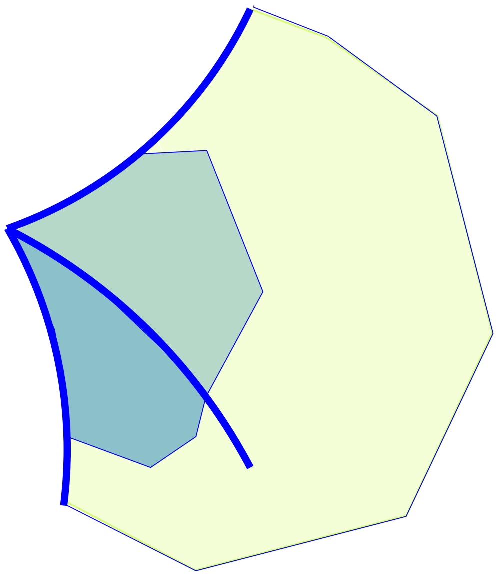
- **Thm.** (Allcock' 05) *There are infinitely many finite-volume Coxeter polytopes in* $\mathbb{H}^d$*, for every* $d \leq 19$.

  There are infinitely many compact Coxeter polytopes in $\mathbb{H}^d$*, for every* $d \leq 6$.

- If P is compact then P is simple.

  (i.e. d facets through each vertex)



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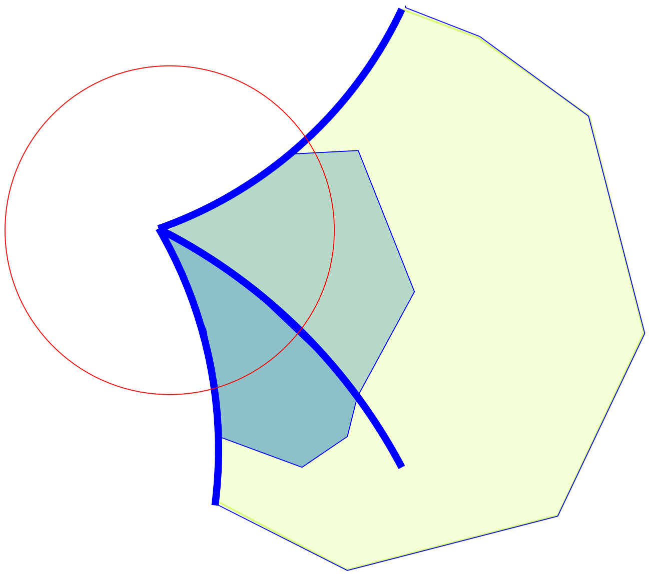

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- (Vinberg'85) Indecomposible, symm. matrix G, $sgn(G) = (d, 1)$
  $\Rightarrow$ $\exists!$ $P \in \mathbb{H}^d$, $G = G(P)$.

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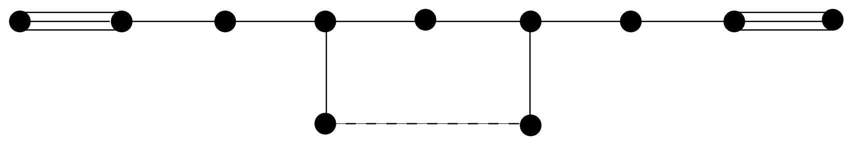

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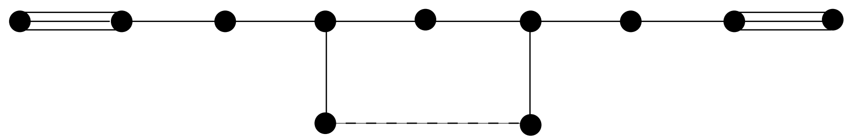

- If $P \subset \mathbb{H}^d$ is of finite volume then $d \leq 996$.
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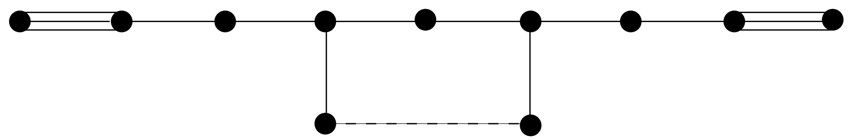

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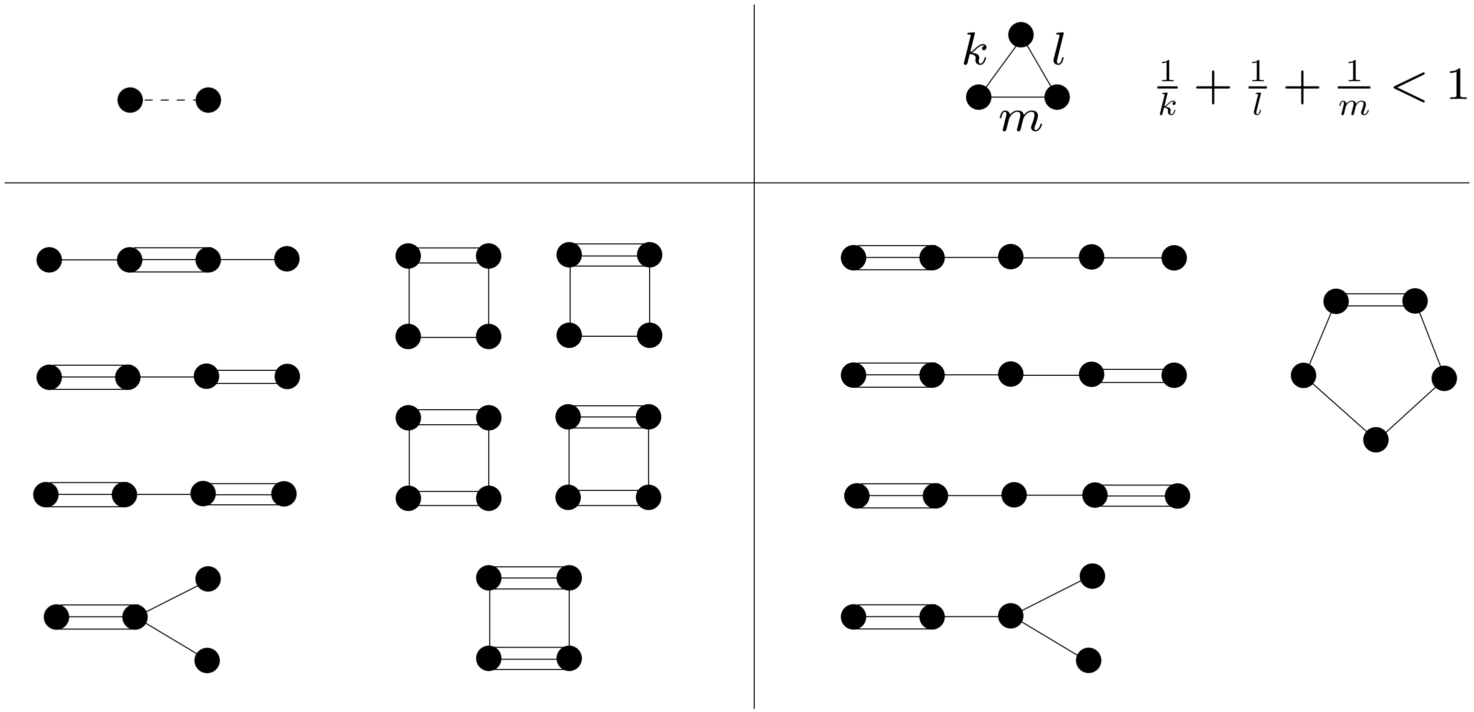

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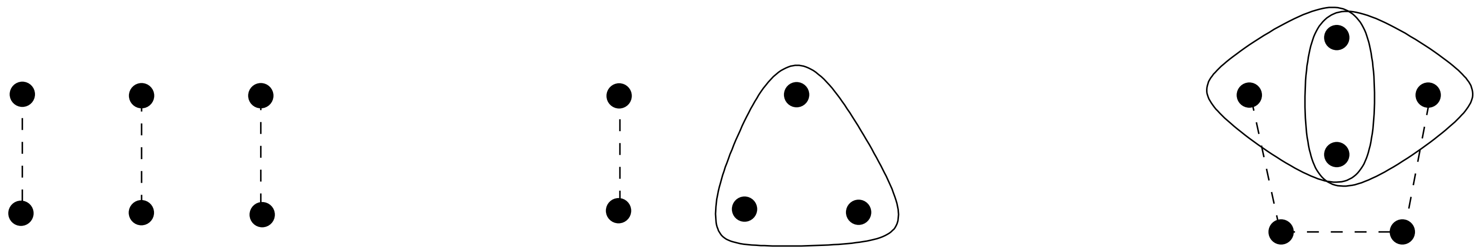
Tools

- Given a combinatorial type, may try to "reconstruct" the polytope (i.e. to find its dihedral angles).

Combinatorics:	Dihedral angles:
Diagram of missing faces	Coxeter diagram

Diagram of missing faces

- Nodes $\longleftrightarrow$ facets of P
- Missing face is a minimal set of facets $f_1, \ldots, f_k$, such that $\bigcap_{i=1}^{k} f_i = \emptyset$.
- Missing faces are encircled.
- Ex:



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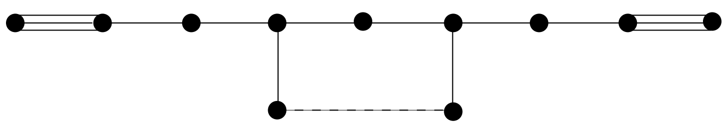

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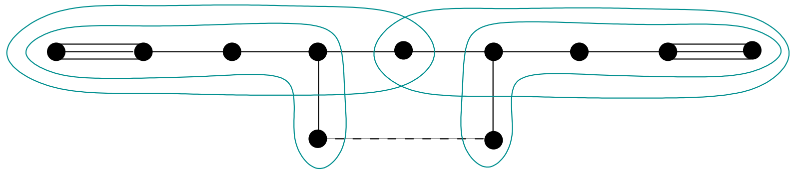

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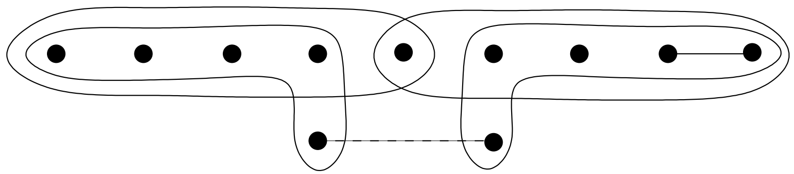

Lanner subdiagrams $\longleftrightarrow$ Missing faces

- If L is a Lanner diagram then $|L| \leq 5$.
- $\#$ of Lanner diagrams of order $4, 5$ is finite.
- For any two Lanner subdiagrams s.t. $L_1 \cap L_2 = \emptyset$, $\exists$ an edge joining these subdiagrams.

Given a combinatorial type may try to check

if there is a Coxeter polytope of this type.

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- $n = d + 1$, simplices (Lanner'82): $d \leq 4$, fin. many for $d > 2$.
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  (T,F'05): $d \leq 9$.

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- Coxeter faces.
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  - Allcock'05: Angles of this face are easy to find.

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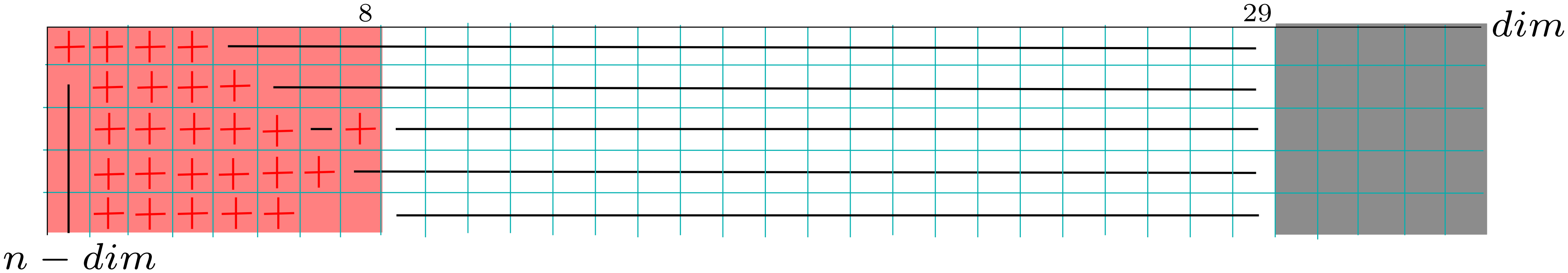
  (T,F'06): $d \leq 7$, unique example in $d = 7$.
- $n = d+5$, (T,F'06): $d \leq 8$.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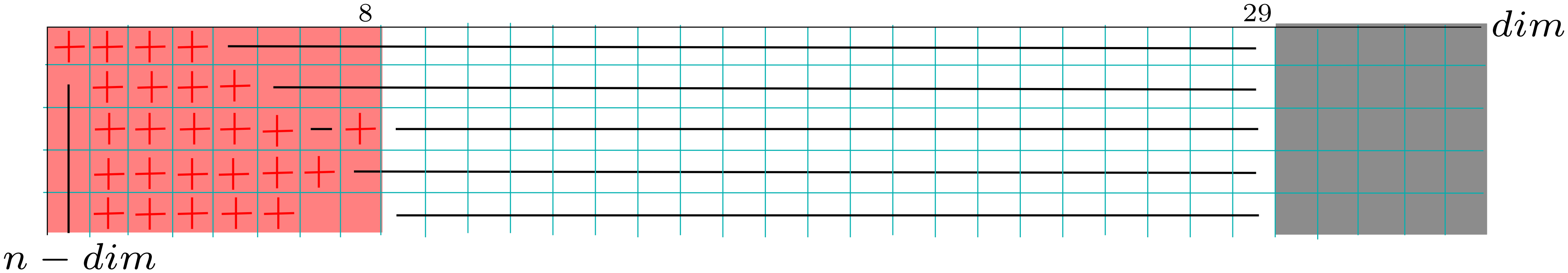
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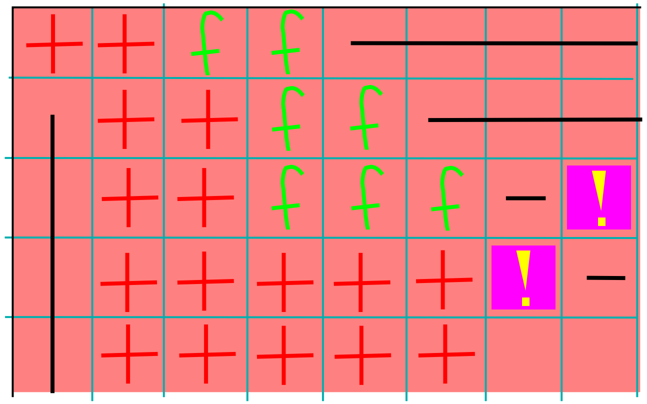

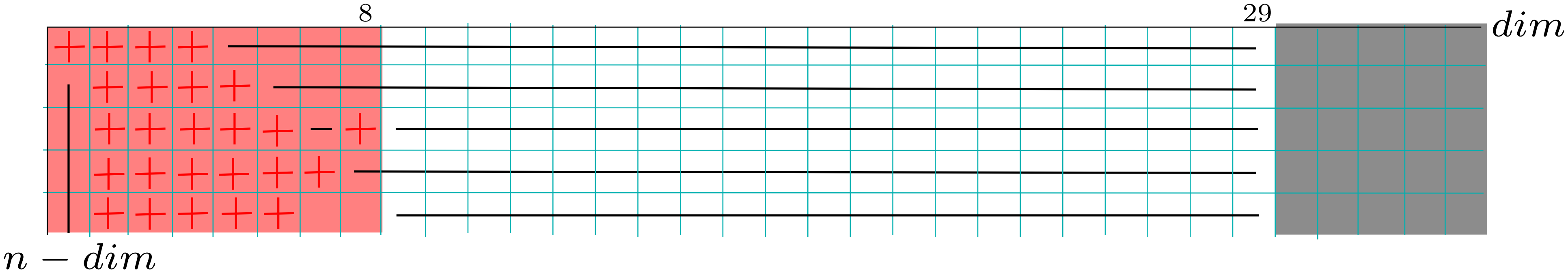

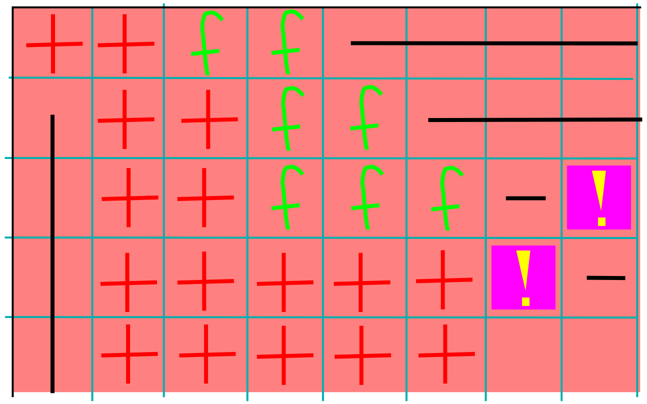

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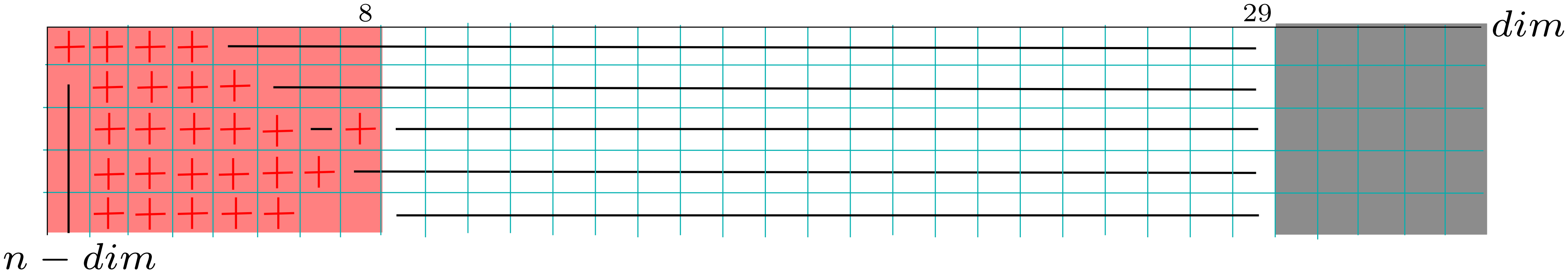

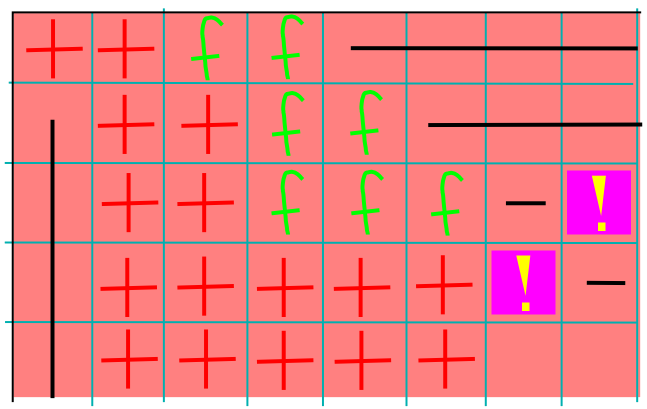

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Inductive algorithm?

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- (T,F'06): If all Lanner subdiagrams are of order 2, then $d \le 13$.
  (for compact or simple finite volume polytops).

Essential polytopes

A Coxeter polytope P is essential iff

- P generates a maximal reflection group;
- P is not glued of two smaller Coxeter polytopes.

Question: Is the number of essential polytopes finite?

Is there any in $dim > 8$?

Problem: How to determine if two combinatorial polytopes are of the same combinatorial types?

- Polytopes where presented by lists of vertices

	triangular 3-prism		1,3,4 2,3,4
Example:	with bases 1 and 2	$\longleftrightarrow$	1,3,5 2,3,5
	and sides 3,4,5		1,4,5 2,4,5

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Idea: compare the numbers of missing faces of each size.

$(t_2, t_3, t_4, \dots, t_n), \quad t_i = \#$(missing faces of order i).

Simple polytope	$\longleftrightarrow$	Dual simplicial complex Σ	$\longleftrightarrow$	Complex Σ_1 of missing faces of Σ

Missing face is a minimal set of vertices defining no simplex in the complex.

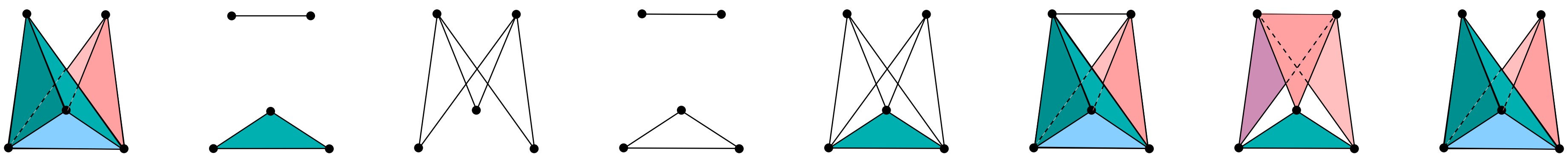
- Complex of missing faces for any simplicial complex:

$$\Sigma \longrightarrow \Sigma_1 \longrightarrow \Sigma_2 \longrightarrow \Sigma_3 \longrightarrow \dots$$

$(t_2^j, t_3^j, t_4^j, \dots, t_n^j), \quad t_i^j = \#$(missing faces of order i in Σ_j).

Example: triangular prism.

Σ	Σ_1	Σ_2	Σ_3	Σ_4	Σ_5	Σ_6	Σ_7
1,3,4	1,2	1,3	1,2	1,3	1,2	3,4,5	1,3,4
1,3,5	3,4,5	1,4	3,4	1,4	1,3,4	1,2,3	1,3,5
1,4,5		1,5	3,5	1,5	1,3,5	1,2,4	1,4,5
2,3,4		2,3	4,5	2,3	1,4,5	1,2,5	2,3,4
2,3,5		2,4		2,4	2,3,4		2,3,5
2,4,5		2,5		2,5	2,3,5		2,4,5
				3,4,5	2,4,5		

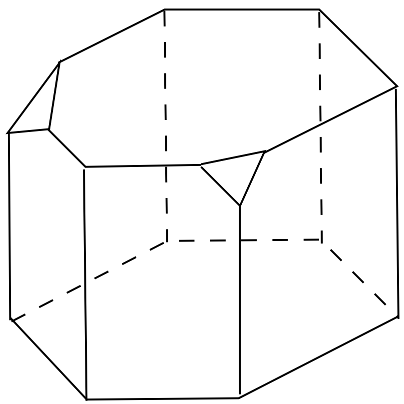
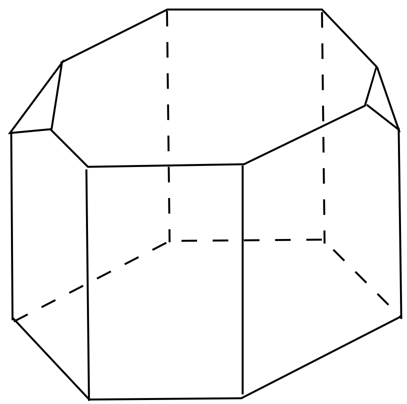


$$\begin{tikzcd} \Sigma \arrow[r] & \Sigma_1 \arrow[r] & \Sigma_2 \arrow[r] & \dots \arrow[r] & \Sigma_k \arrow[llll, bend left] \end{tikzcd}$$

$$\begin{array}{c}
(t_2^0, t_3^0, t_4^0, \dots, t_n^0) \\
(t_2^1, t_3^1, t_4^1, \dots, t_n^1) \\
(t_2^2, t_3^2, t_4^2, \dots, t_n^2) \\
\dots\dots\dots\dots\dots\dots \\
(t_2^k, t_3^k, t_4^k, \dots, t_n^k)
\end{array}$$

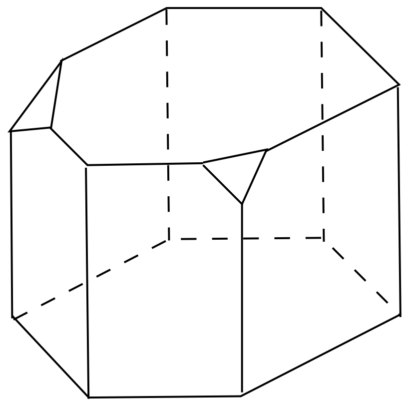
Conjecture: Do these numbers determine a polytope?

Example:

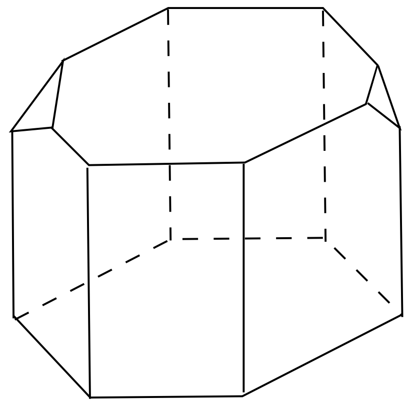
$$
\begin{array}{cc}
(0,16,0,0,0,0,0,0,0) & (0,16,0,0,0,0,0,0,0) \\
(21,2,0,0,0,0,0,0,0) & (21,2,0,0,0,0,0,0,0) \\
(18,20,0,0,0,0,0,0,0) & (18,21,0,0,0,0,0,0,0) \\
(1,92,2,0,0,0,0,0,0) & (1,91,2,0,0,0,0,0,0) \\
\cdots\cdots\cdots & \cdots\cdots\cdots
\end{array}
$$

Example:



$$(0,16,0,0,0,0,0,0,0)$$
$$(21,2,0,0,0,0,0,0,0)$$
$$(18,20,0,0,0,0,0,0,0)$$
$$(1,92,2,0,0,0,0,0,0)$$
$$\dots\dots\dots\dots\dots\dots$$

k=158381



$$(0,16,0,0,0,0,0,0,0)$$
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$$(18,21,0,0,0,0,0,0,0)$$
$$(1,91,2,0,0,0,0,0,0)$$
$$\dots\dots\dots\dots\dots\dots$$

k=666517

Question: Do these numbers determine a simplicial complex?

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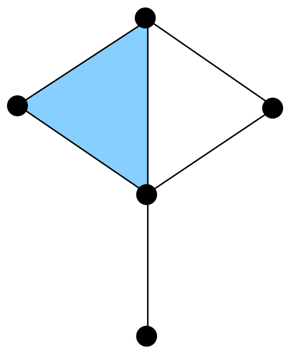
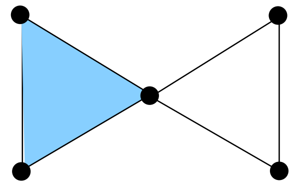
No: They do not differ



from



Def: Σ^* is Alexander dual to Σ if

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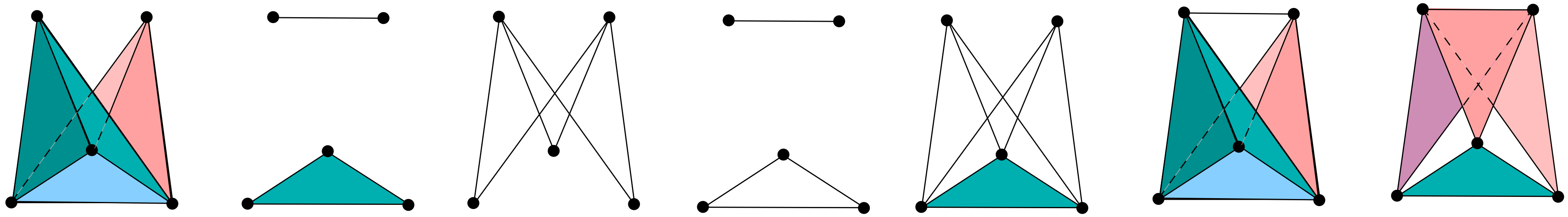

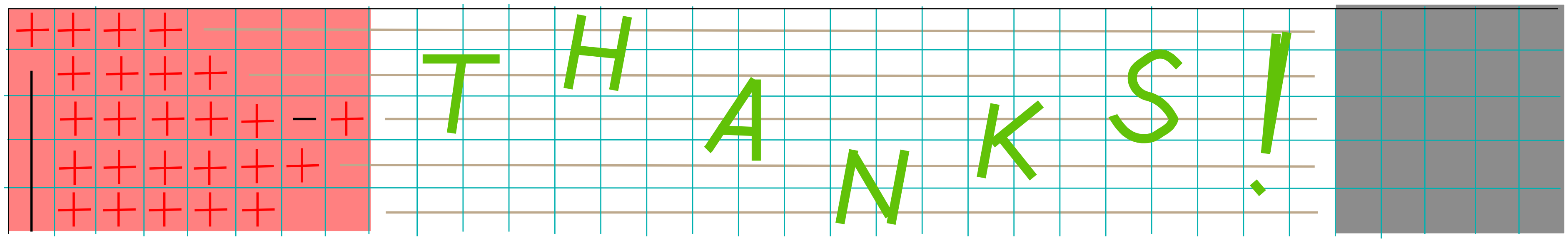
THANKS!