Hyperbolic Coxeter polytopes

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(joint with Pavel Tumarkin)

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Hyperbolic Coxeter polytopes

Giving talks

usually: because of reacent progress;

today: less progress than expected.

- Collect what we know;
- connect to another classification problem.

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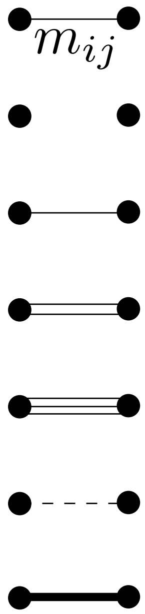
Coxeter polytopes ...

- are polytopes in $\mathbb{S}^d$, $\mathbb{E}^d$ or $\mathbb{H}^d$ whose all dihedral angles are submultiples of π;
- are fundamental domains of discrete reflections groups;
- are represented by Coxeter diagrams or by Gram matrices.

Coxeter diagrams

- Nodes $\longleftrightarrow$ facets f_i of P

- Edges:



if $\angle(f_i f_j) = \pi/m_{ij}$

if $\angle(f_i f_j) = \pi/2$

if $\angle(f_i f_j) = \pi/3$

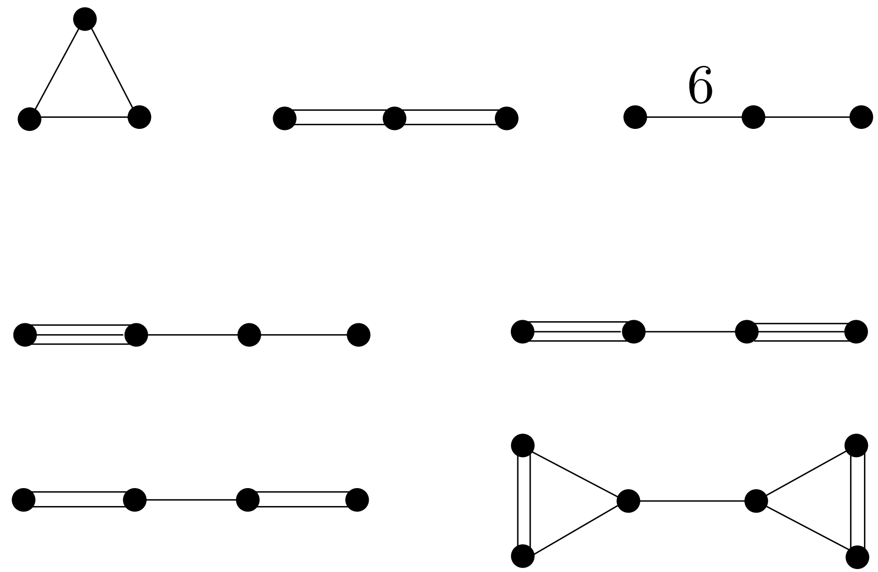
if $\angle(f_i f_j) = \pi/4$

if $\angle(f_i f_j) = \pi/5$

if $f_i \cap f_j = \emptyset$

if $f_i \cap f_j \in \partial\mathbb{H}^n$

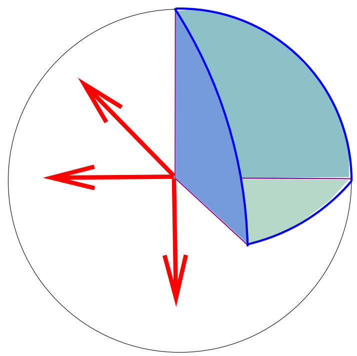
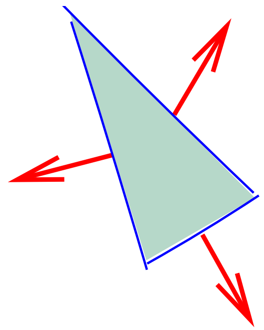
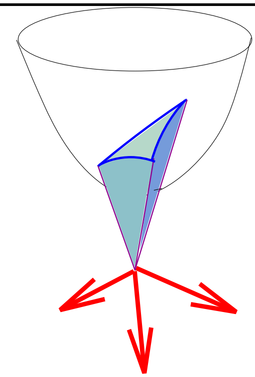
Examples:



Gram matrix

$P \subset \mathbb{S}^n, \mathbb{E}^n$ or $\mathbb{H}^n \quad \longrightarrow \quad$ Symmetric matrix $G = \{g_{ij}\}$

- $g_{ii} = 1, \qquad g_{ij} = \begin{cases} -\cos(\frac{\pi}{m_{ij}}), & \text{if } \angle(f_i f_j) = \pi/m_{ij}, \\ 1, & \text{if } f_i \text{ is parallel to } f_j, \\ -\cosh(\rho(f_i, f_j)), & \text{if } f_i \text{ and } f_j \text{ deverge.} \end{cases}$

	$\mathbb{S}^d$	$\mathbb{E}^d$	$\mathbb{H}^d$
			
$sgn(G)$	$(d+1, 0)$	$(d, 0)$	$(d, 1)$

Coxeter polytopes ...

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  whose all dihedral angles are submultiples of π;

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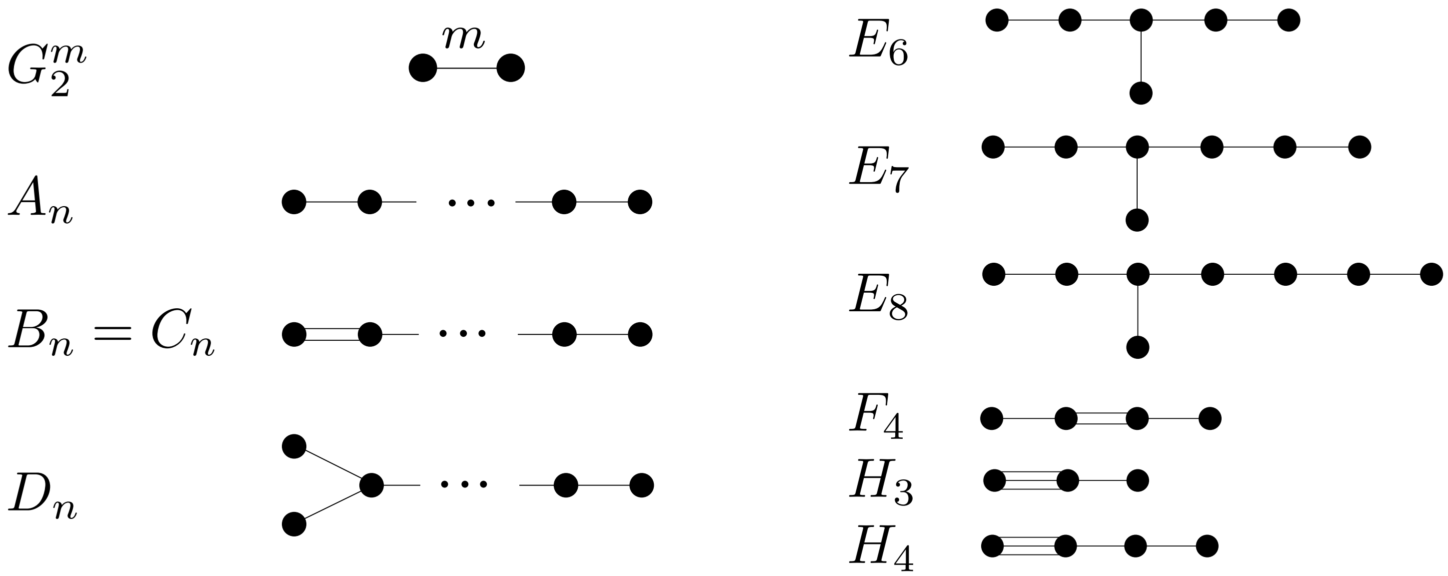
  or by Gram matrices.

- spherical and Euclidean Coxeter polytopes:
  - finitely many types in each dimension;
  - classified by Coxeter in 1934.

Spherical Coxeter polytopes

- $P \subset \mathbb{S}^n \Rightarrow\ P$ is a simplex.

- Coxeter diagram of P is called elliptic, it is a union of



Euclidean Coxeter polytopes

- $P \subset \mathbb{E}^n \Rightarrow$ P is a product of simplices.
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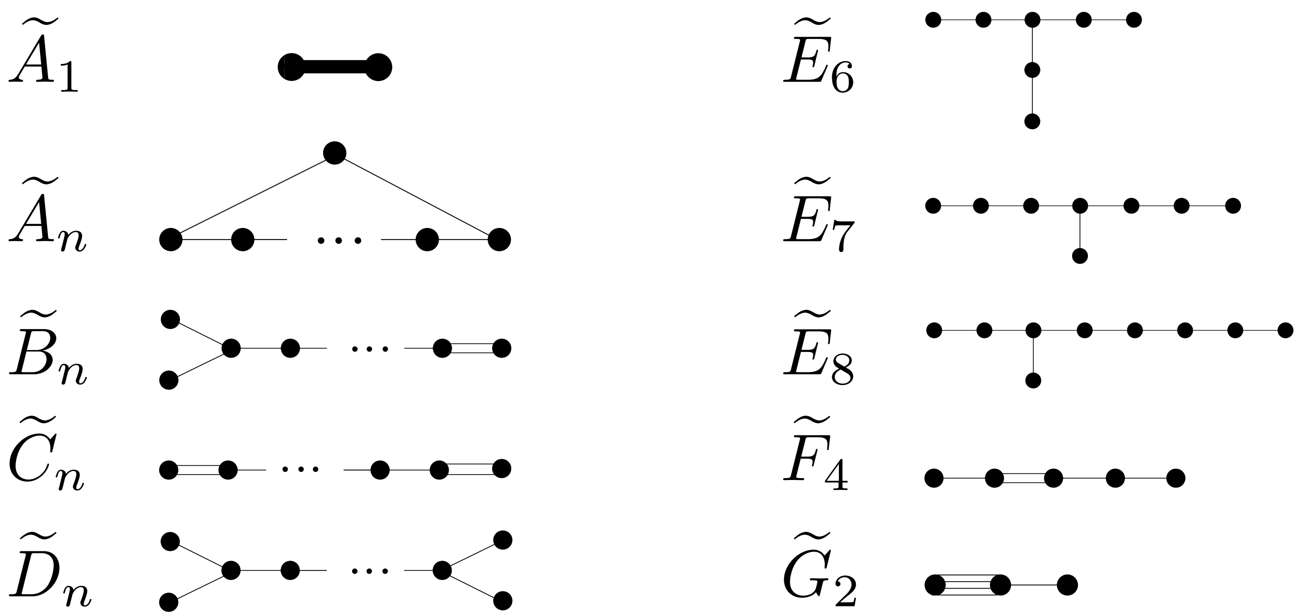

Hyperbolic Coxeter polytopes

- Wide veriety of compact and finite-volume polytopes.

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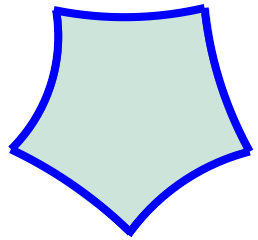

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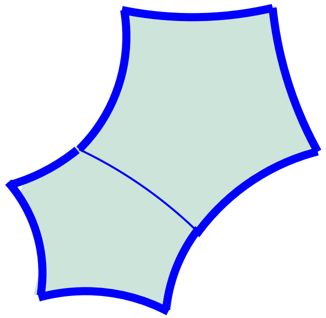

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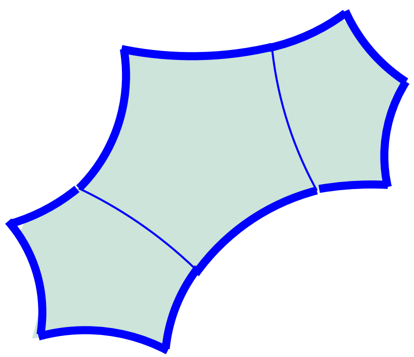

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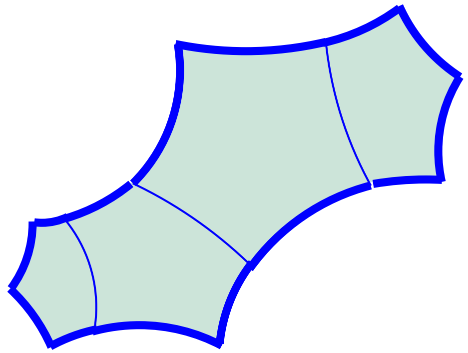

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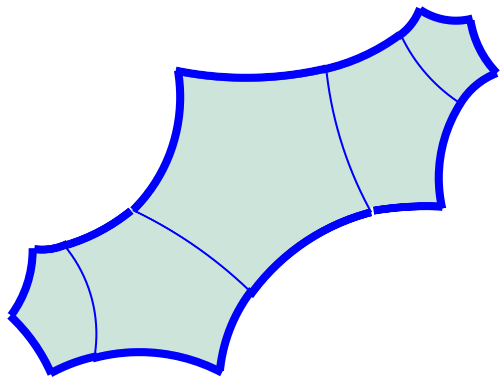

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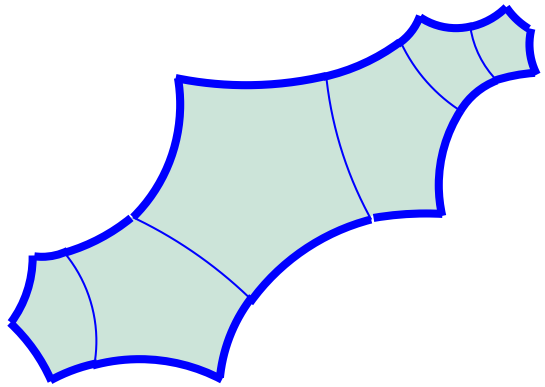

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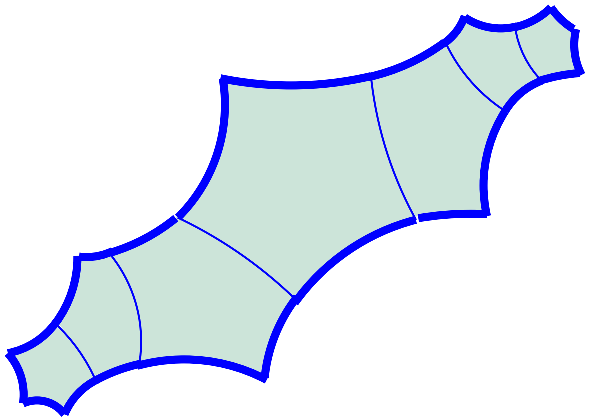

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Compact polytopes: infinitely many in $\mathbb{H}^d$ for all $d \leq 6$.
Finite volume polytopes: infinitely many in $\mathbb{H}^d$ for all $d \leq 19$.

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- Plan: 1. How badly we don't know
  2. Small bits we know
  3. How to add a bit of structure
  4. How to use

1. Hyperbolic Coxeter polytopes: how we don't know

Absence in large dimensions:

- If $P \subset \mathbb{H}^d$ is compact then $d \leq 29$. [Vinberg'84].

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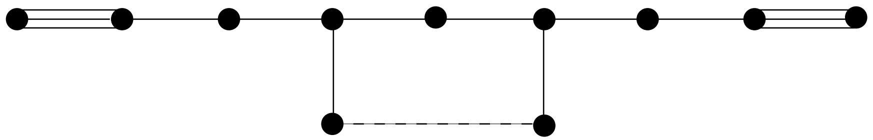
Unique Ex. for $d = 8$ [Bugaenko'92]:



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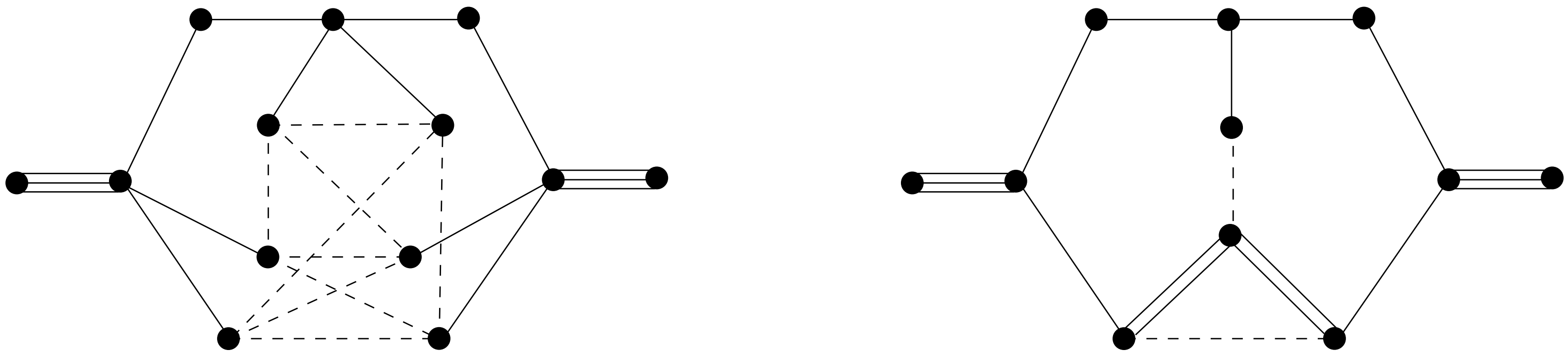

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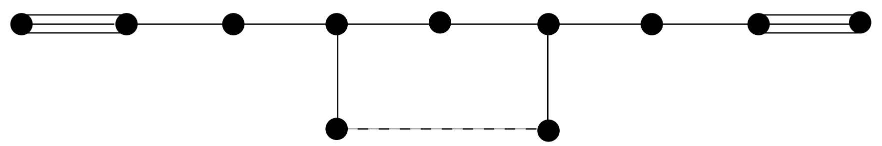

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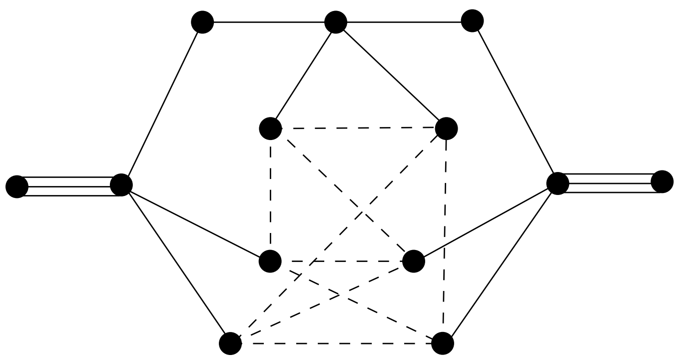

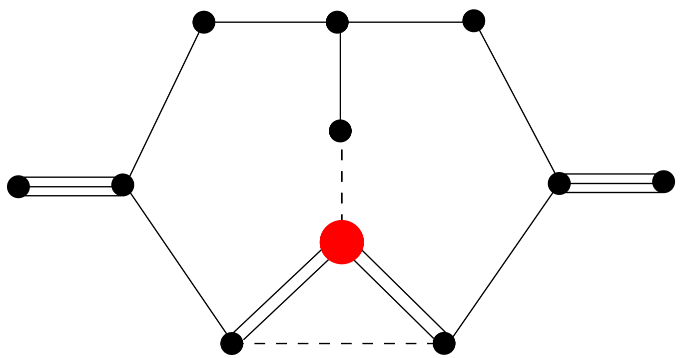

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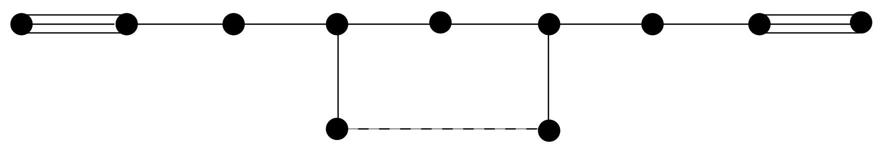

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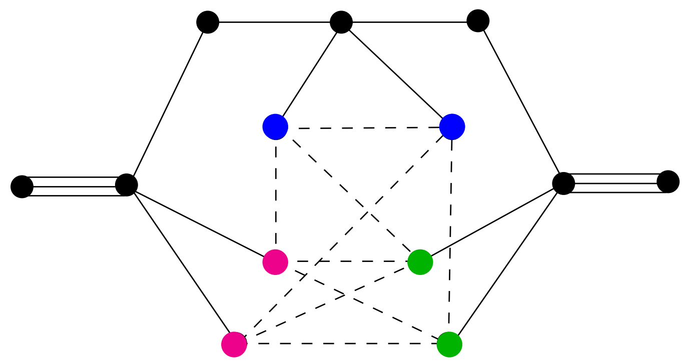

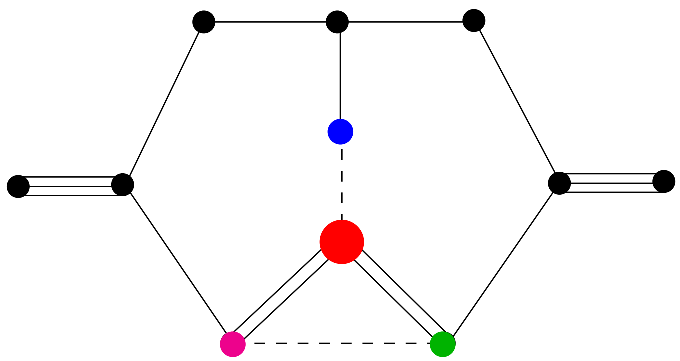

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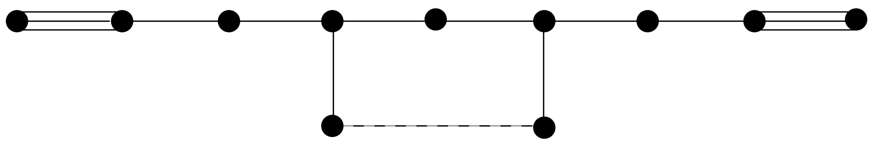

- If $P \subset \mathbb{H}^d$ is of finite volume than $d \leq 996$.
  [Prochorov'85, Khovanskiy'86].

  Examples known for $d \leq 19$ [Vinberg, Kaplinskaya'78],
  $d = 21$ [Borcherds'87].

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    (elliptic $=$ Coxeter diagrams of spherical simplices).
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- [Vinberg'85] Indecomposible, symm. matrix G, $sgn(G)=(d,1)$
  $\Rightarrow$ $\exists!\ P \in \mathbb{H}^d$, $G=G(P)$.

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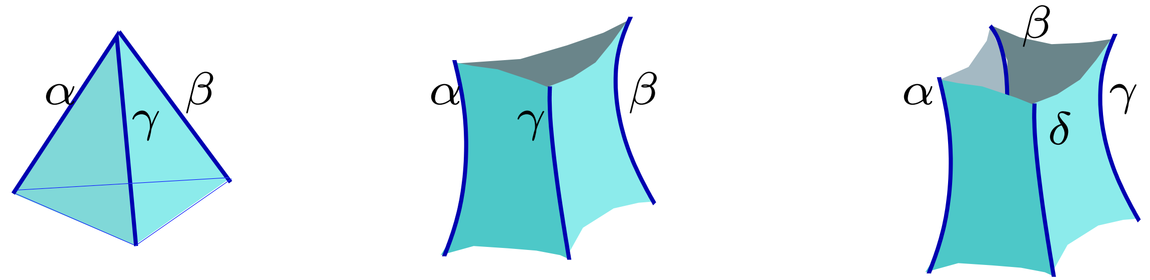

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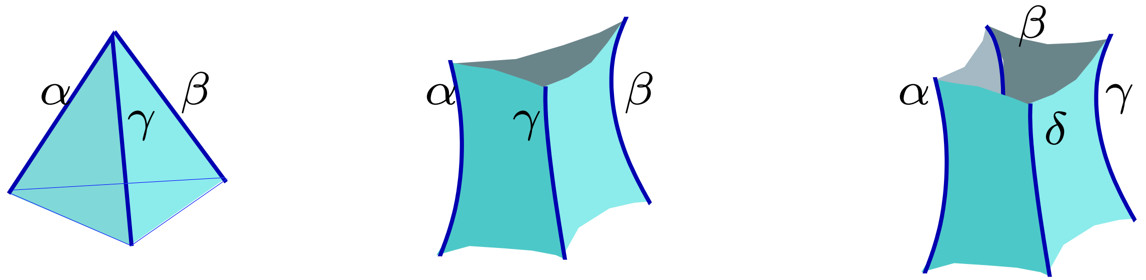

$\alpha+\beta+\gamma>\pi$ $\qquad$ $\alpha+\beta+\gamma<\pi$ $\qquad$ $\alpha+\beta+\gamma+\delta<2\pi$

- $dim \ge 4$. ?????

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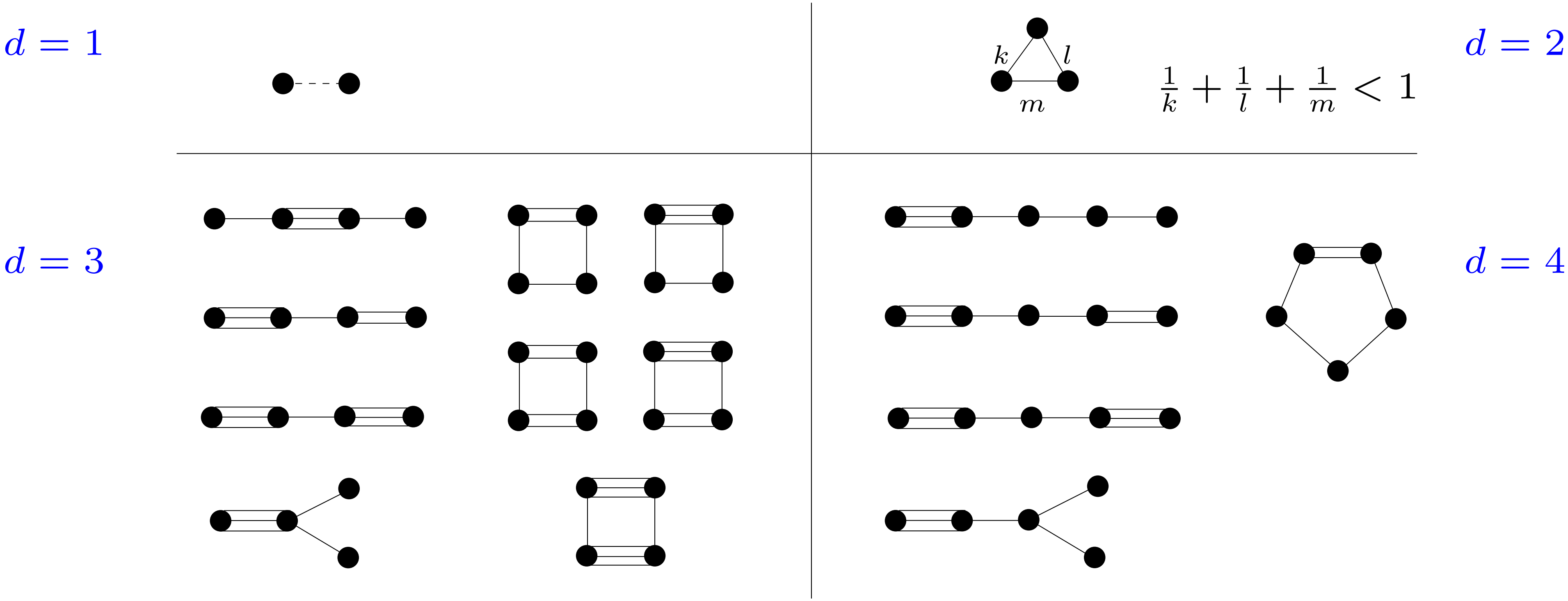
- $n = d + 1$, simplices [Lanner'82], Lanner diagrams



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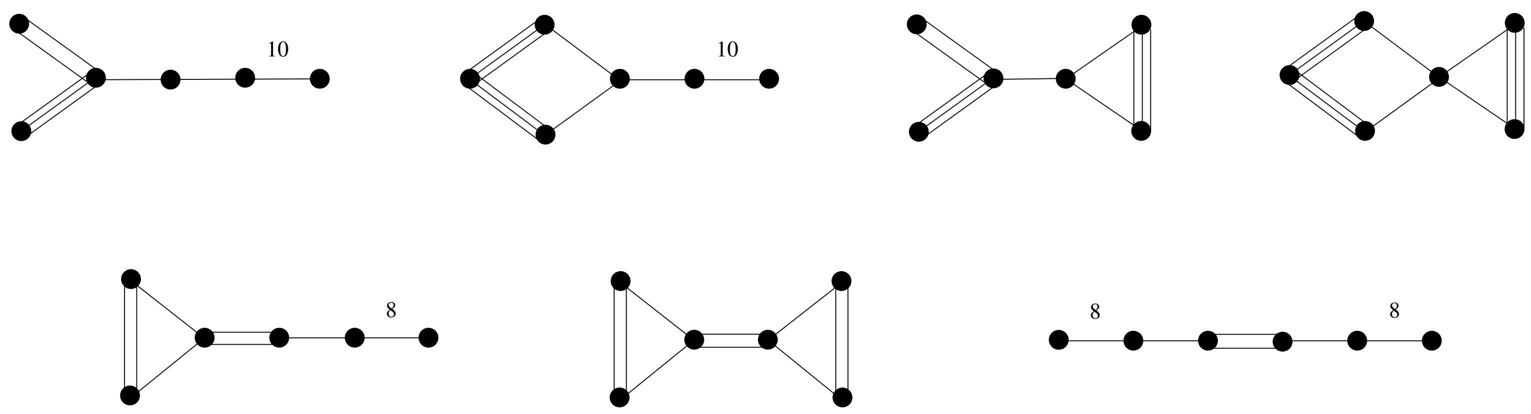

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Tools: diagram of missing faces

- Nodes $\longleftrightarrow$ facets of P

- Missing face is a minimal set of facets $f_1, ..., f_k$, such that $\bigcap_{i=1}^{k} f_i = \emptyset$.

- Missing faces are encircled.

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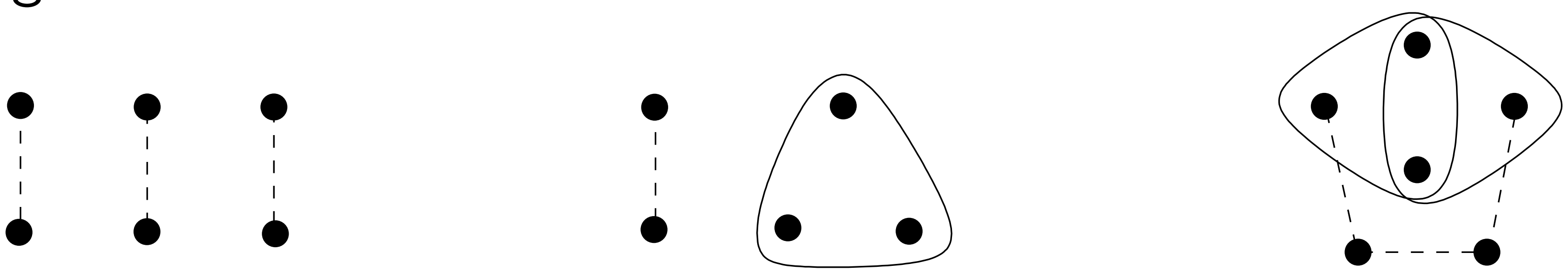

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- Given a combinatorial type, may try to "reconstruct" the polytope (i.e. to find its dihedral angles).

Combinatorics: Diagram of missing faces		Dihedral angles: Coxeter diagram
Missing faces	$\longleftrightarrow$	Lanner subdiagrams (minimal non-eliptic subd.)

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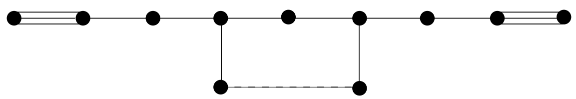

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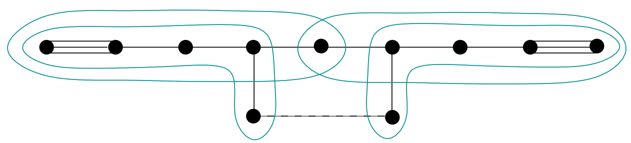

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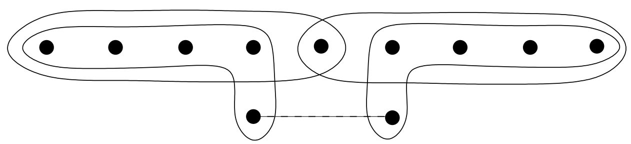

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Tools: diagram of missing faces

Lanner subdiagrams $\longleftrightarrow$ Missing faces

- If L is a Lanner diagram then $|L| \leq 5$.
- $\#$ of Lanner diagrams of order $4, 5$ is finite.
- For any two Lanner subdiagrams s.t. $L_1 \cap L_2 = \emptyset$, $\exists$ an edge joining these subdiagrams.

Combinatorial type $\longrightarrow$ Coxeter polytope

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  [F,T'05]: $d \leq 9$.

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Tools: Coxeter faces

- [Borcherds'98]: Elliptic subdiagram without A_n and D_5 $\rightarrow$ Coxeter face
- [Allcock'05]: Angles of this face are easy to find.

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Coxeter diagram

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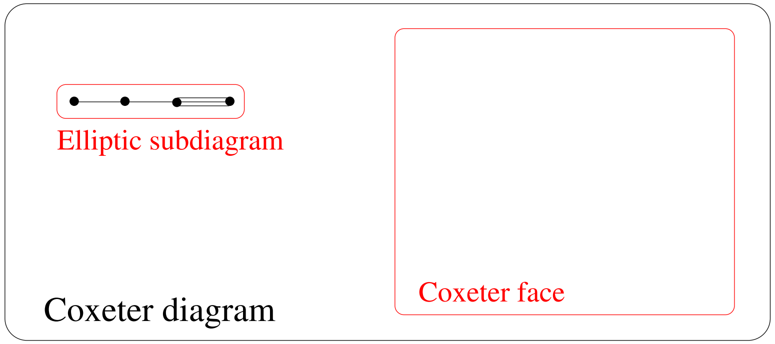

2. Compact hyp. Coxeter polytopes: bits we know

1. By dimension.

2. By number of facets.

Tools: Coxeter faces

- [Borcherds'98]: Elliptic subdiagram without A_n and D_5 $\rightarrow$ Coxeter face

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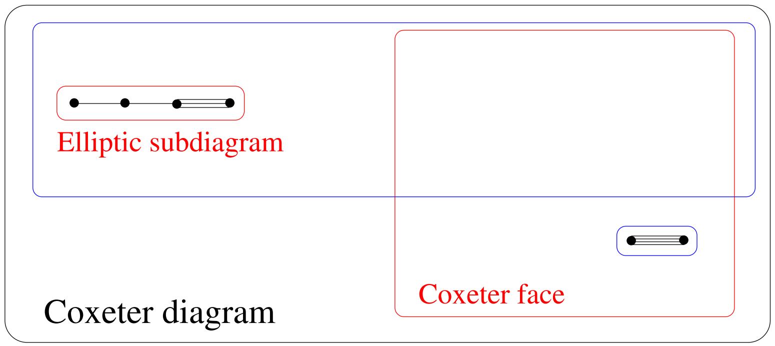

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1. By dimension.

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- $n = d + 1$, simplices [Lanner'82]: $d \leq 4$, fin. many for $d > 2$.

- $n = d + 2$, $\Delta^k \times \Delta^l$
  - prisms [Kaplinskaya'74]: $d \leq 5$, fin. many for $d > 3$.
  - others [Esselmann'96]: $d = 4$, $\Delta^2 \times \Delta^2$, 7 items.

- $n = d + 3$, many combinatorial types
  [Tumarkin'03]: $d \leq 6$ or $d = 8$, fin.many for $d > 3$.

- $n = d + 4$, really many combinatorial types...
  [F,T'06]: $d \leq 7$, unique example in $d = 7$.

- $n = d + 5$, [F,T' "06"]: $d \leq 8$.

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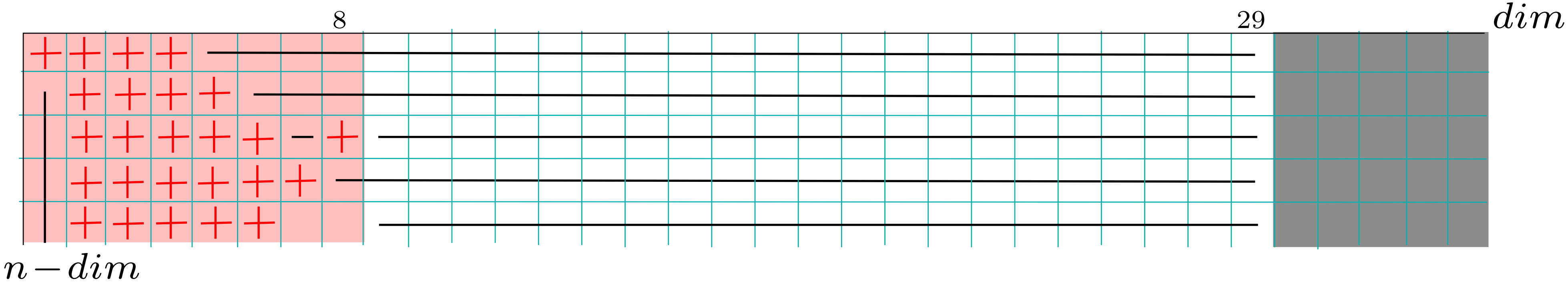

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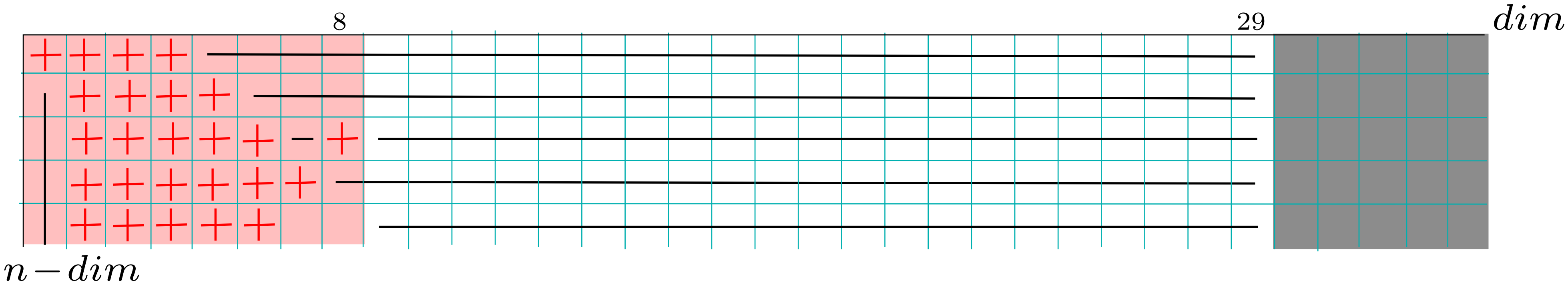

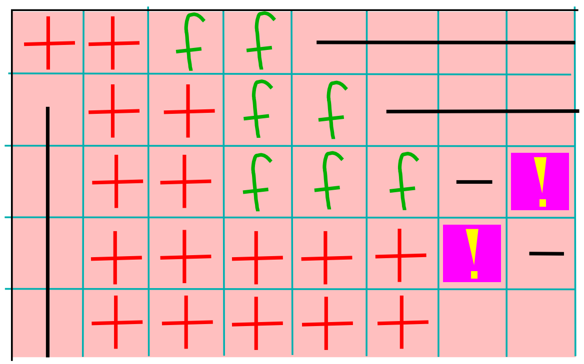

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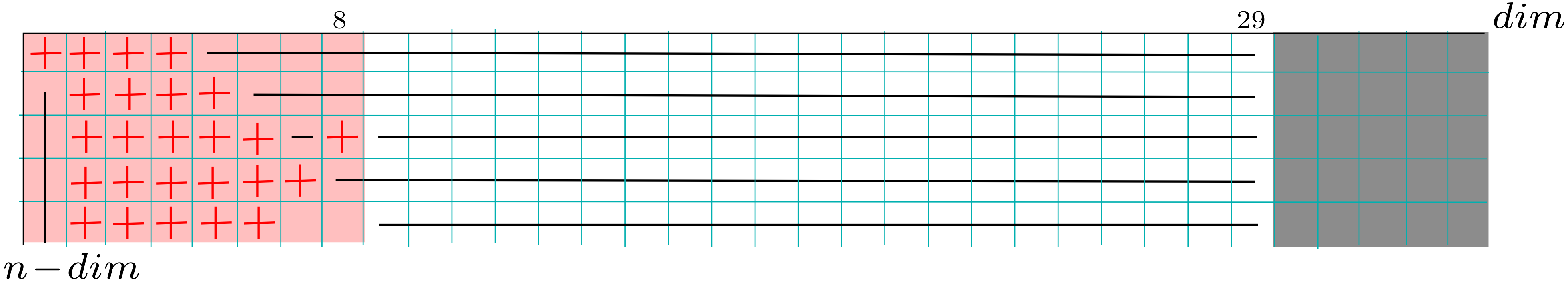

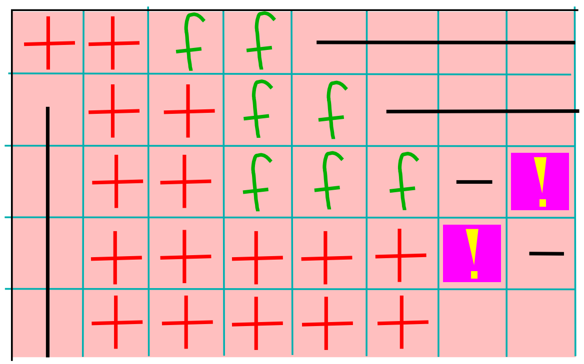

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2) use previous cases

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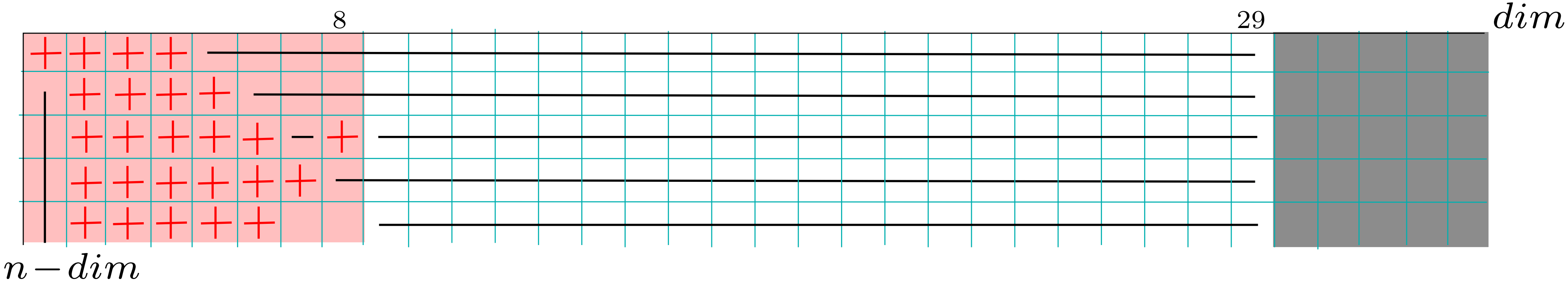

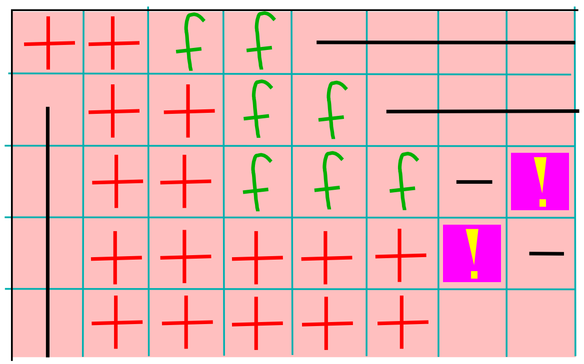

1) proofs are similar
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Inductive algorithm?

2. Compact hyp. Coxeter polytopes: bits we know

1. By dimension.
2. By number of facets.
3. By largest denominator:
   - Right-angled polytopes [Potyagailo, Vinberg' 05]:
     $d \leq 4$, examples for $d = 2, 3, 4$.
   - (Some) polytopes with angles $\pi/2$ and $\pi/3$ [Prokhorov' 88].

2. Compact hyp. Coxeter polytopes: bits we know

1. By dimension.
2. By number of facets.
3. By largest denominator.
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- $p = 1$, [F,T'07]: Only polytopes with $n \leq d + 3$.
  $d \leq 6$ and $d = 8$.
- $p \leq n - d - 2$, [F,T,'07]: finitely many polytopes. Algorithm.
  - Implemented the algorithm for $d = 4$:
    nothing new.

3. Compact hyp. Coxeter polytopes: structure?

Essential polytopes

A Coxeter polytope P is essential iff

- P generates a maximal reflection group;
- P is not glued of two smaller Coxeter polytopes.

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Is there any in $dim > 8$?

Evidence: Finitely many max. groups in the arithmetic case. [Nikulin'07] and [Agol, Belolipetsky, Storm, Whyte'08].

Since then?

- Some other combinatorial types

  - cubes [Jacquemet' 16; Jacquemet-Tschantz' 17 ?].

- Some results for finite volume polytopes

  - pyramids over products of more than two simplices [Mcleod' 13];

  - n=dim+3, with one non-simple vertex [Roberts' 15];

  - non-arithmetic examples in $dim \leq 12$ and $dim = 14, 18$ [Vinberg' 15].

4. Compact hyp. Coxeter polytopes: how to use?

Another question:

- Quiver = oriented graph;
- Mutation of quiver = local operation.
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- Idea: – look at minimal quivers non-decomposable into blocks
  (mimicking "missing faces $=$ minimal non-faces");
  – upside-down technique $\longrightarrow$ minimal;
  – add more vertices one by one $\longrightarrow$ all.

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- Mutation of quiver = local operation.
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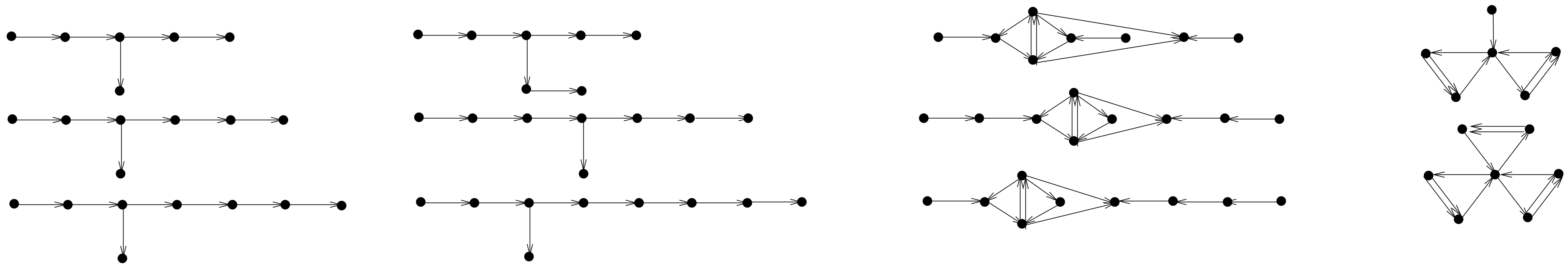
Thm. [F,Shapiro,T'08]: Let Q be a quiver of finite mutation type.
Then either Q has 2 vertices,
or Q comes from triangulated surfaces,
or Q mutation-equivalent to one of:



5. Back to polytopes?

Why worked for quivers and not for polytopes?

- integer number of arrows / any numbers (distances) in the polytopes;
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- Webpage

  http://www.maths.dur.ac.uk/users/anna.felikson/Polytopes/polytopes.html

Mozilla Firefox

Anna Felikson × file:///hom...ytopes.html ×

file:///home/anna/Dropbox/A/Polytopes/polytopes.html

Hyperbolic Coxeter polytopes

- **Disclaimer:**
  - this is an attempt to collect some results concerning classification and properties of hyperbolic Coxeter polytopes.
  - This page is under construction. Any corrections, suggestions or other comments are very welcome.
- **Arithmetic groups:** for detailed discussion of advances in the arithmetic case see the recent survey by M. Belolipetsky [Bel].
- **Why "hyperbolic":** spherical and Euclidean Coxeter polytopes are classified by H.S.M.Coxeter in 1934 [Cox].

Basic definitions **(see [Vin1], [Vin3], [Vin6], [Vin7]):**

- Definitions of Coxeter polytope, Gram matrix, Coxeter diagram.
- Faces of Coxeter polytopes.
- Existence and Uniqueness of a polytope with given Gram matrix.

Absence in large dimensions:

- Compact hyperbolic Coxeter polytopes:
  - do not exist in dimensions dim>29 [Vin2];
  - examples are known only up to dim=8, the unique known example in dim=8 and both known examples in dim=7 are due to Bugaenko [Bug1].
- Finite volume hyperbolic Coxeter polytopes:
  - do not exist in dimensions dim>995 [Pr];
  - examples are known in dimensions dim≤19 [Vin4], [KV] and dim=21 [Bor].

Some known classifications:

By dimension (dim):

- dim=2: there exists a polygon with given angles if and only if the sum of angles is less than π [Po].
- dim=3: see Andreev's theorem [And1], [And2], [RHD].

By number of facets (n):

- n=dim+1: compact simplices (Lannér diagrams [Lan], dim=2,3,4) and non-compact simplices (quasi-Lannér diagrams [Ch], [Vin7], [Bou], dim=2,...,9).
- n=dim+2:
  - Products of two simplices:
    - Simplicial prisms exist in dim=3,4,5 [Kap], see also [Vin3].
    - Other products of two simplices (exist in dim=4 only): Esselmann polytopes [Ess] and the unique non-compact polytope [Tum1].
  - Pyramids over a product of two simplices [Tum1], dim=3,...,13, 17.
- n=dim+3:
  - Compact: exist in dim=2,...,6,8 only; see the list [Tum2]. First high-dimensional results are due to V. Bugaenko [Bug2].
  - Finite volume:
    - do not exist in dim≥17 [Tum3], [Tum3'].
    - the unique polytope in dim=16 [Tum3], [Tum3'].
    - polytopes with exactly one non-simple vertex exist in dim=4,...,10, see the list (see pp. 8-33) [Rob].
- n=dim+4: compact polytopes with n=dim+4 facets do not exist in dim>7 [FT7]. There is a unique compact polytope in dim=7 with 11 facets [FT7].

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