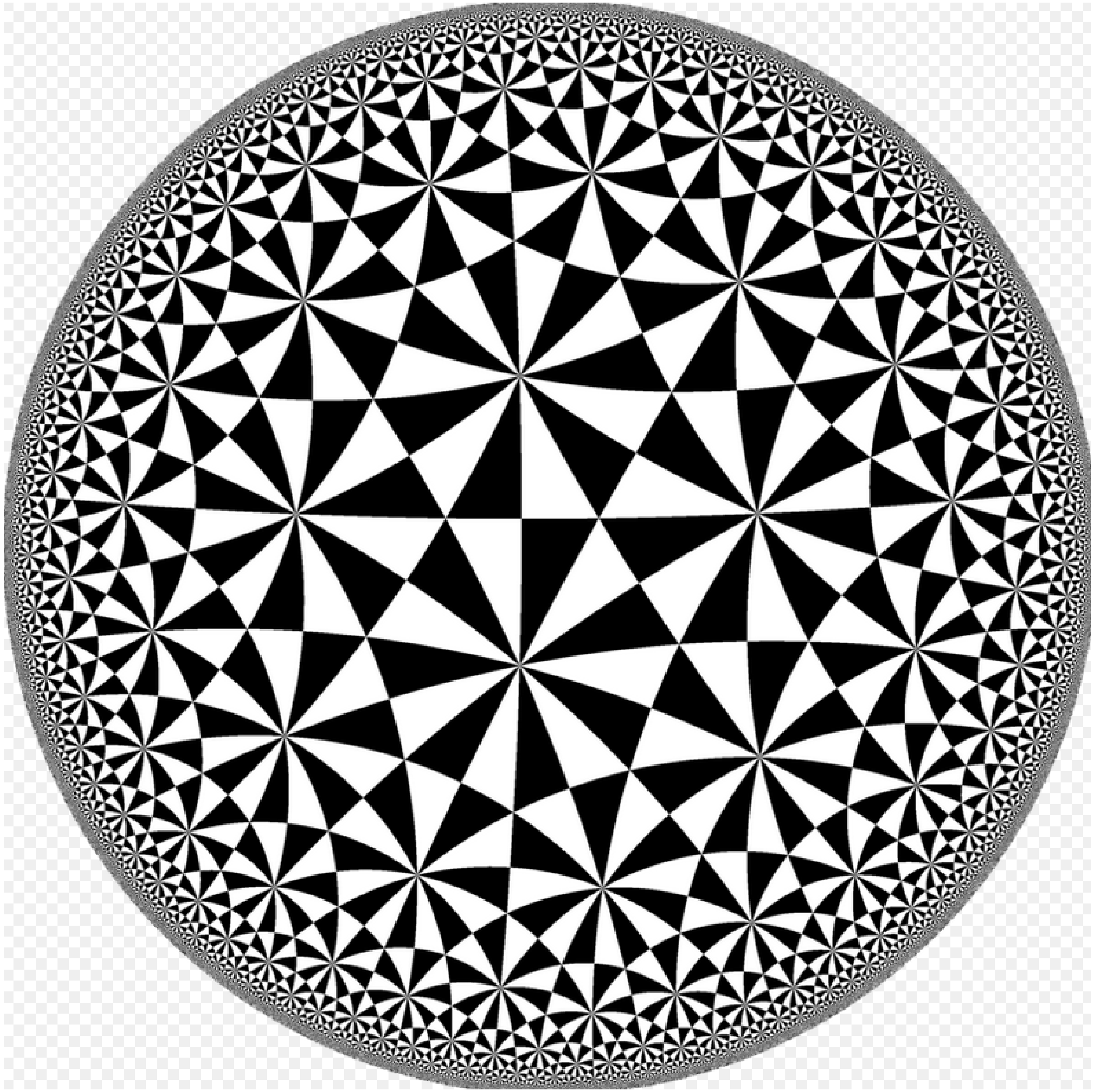
Quiver mutations and triangulated surfaces

Anna Felikson

Durham University

joint work with **Michael Shapiro** and **Pavel Tumarkin**

LMS Undergraduate Summer School
July 16, 2021



Cluster algebras, quiver mutations and triangulated surfaces

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Cluster algebras

(Fomin, Zelevinsky, 2002)

Andrei Zelevinsky

Cluster algebras

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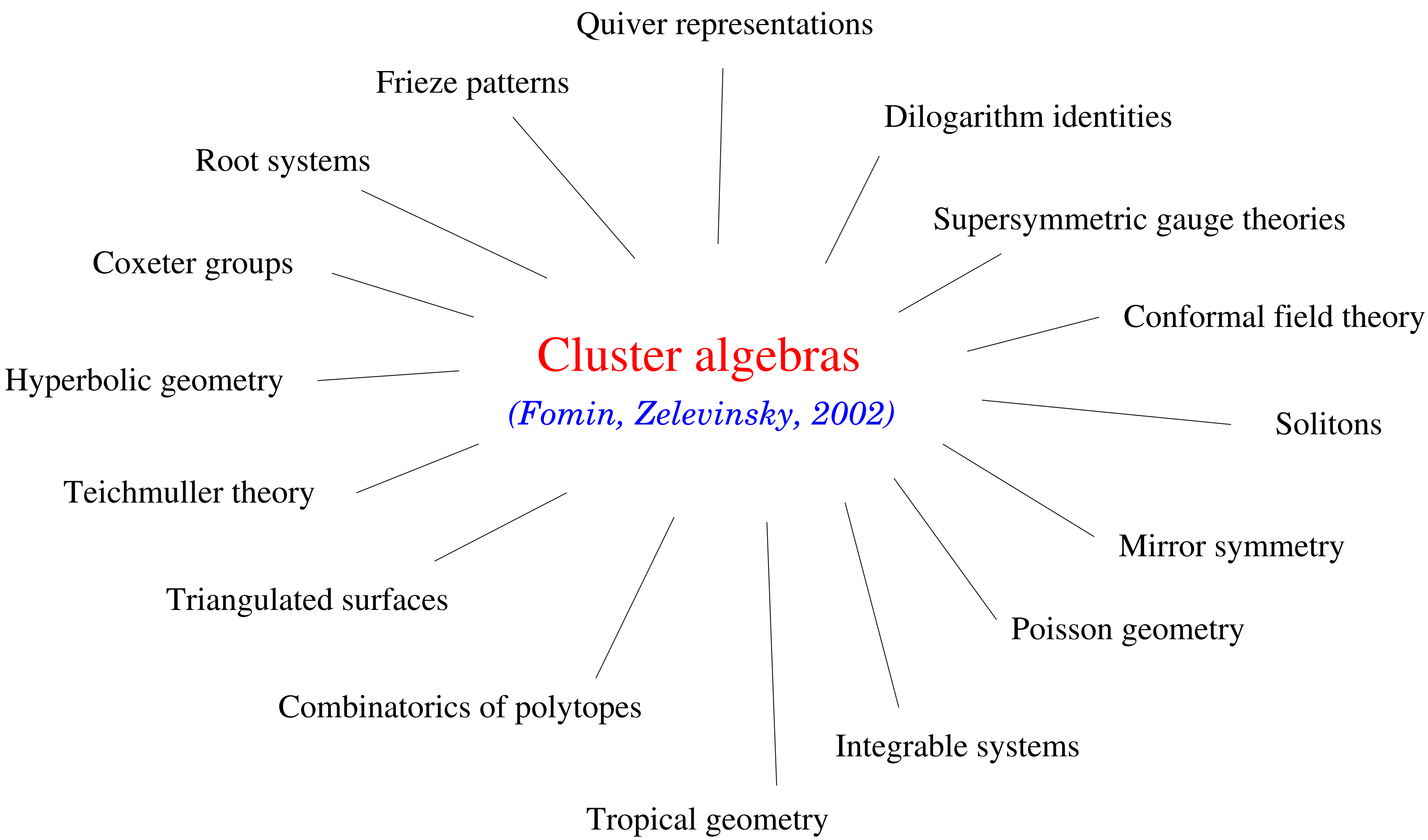
Sergey Fomin


Quiver representations
Frieze patterns
Dilogarithm identities
Root systems
Supersymmetric gauge theories
Coxeter groups
Conformal field theory
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Teichmuller theory
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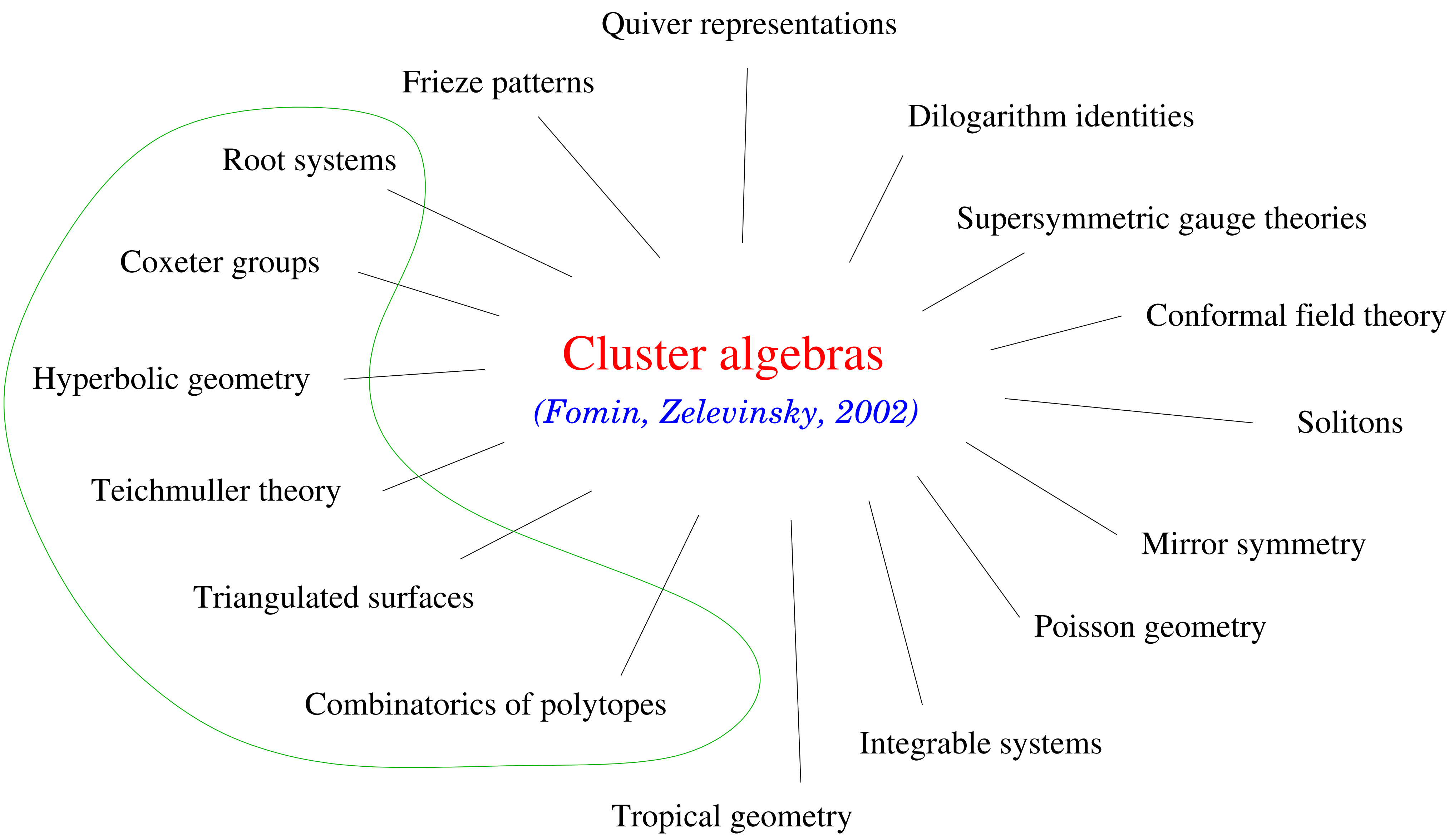
Triangulated surfaces
Poisson geometry
Combinatorics of polytopes
Integrable systems
Tropical geometry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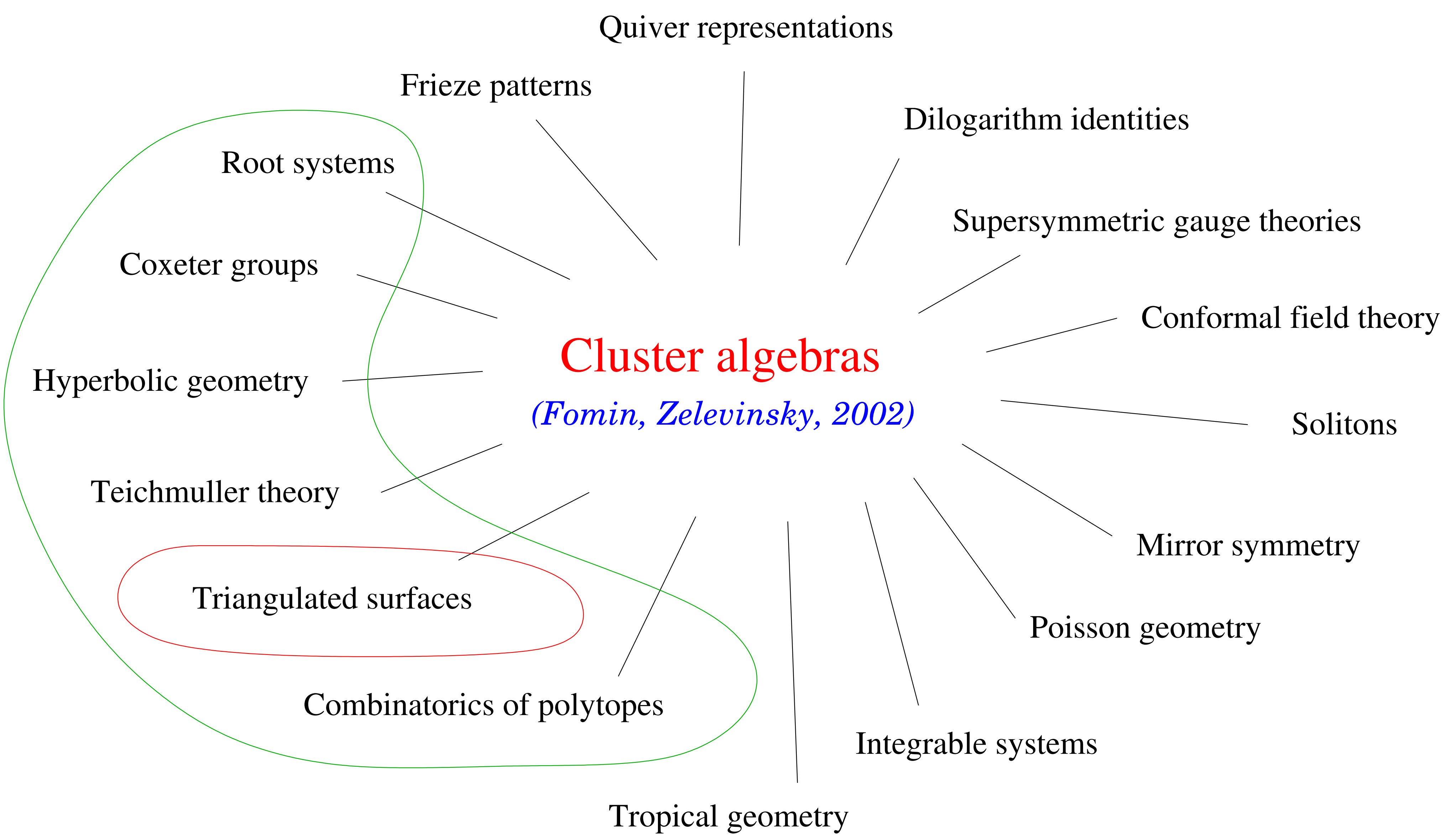
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0. Prologue: Ptolemy Theorem.

Claudius Ptolemy (AD 100 – 170):
Greco-Roman mathematician, astronomer, geographer and astrologer.

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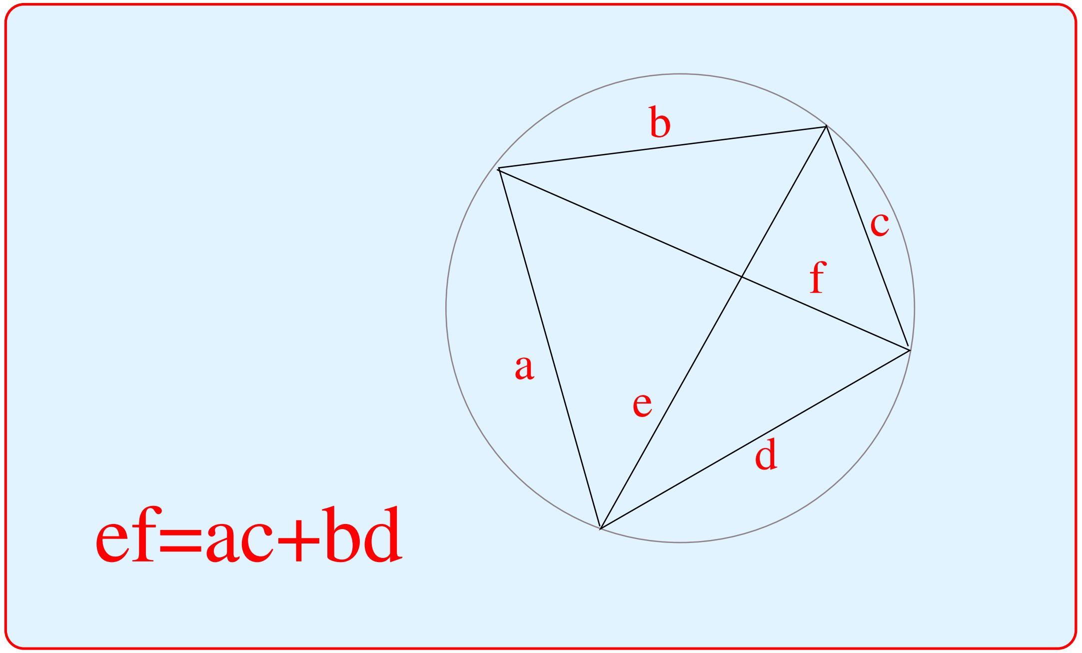

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1. Quiver mutation

- Quiver is a directed graph without loops and 2-cycles.

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- Mutation μ_k of quivers:
  - reverse all arrows incident to k;
  - for every path through k with $\bullet \xrightarrow{p} \overset{k}{\bullet} \xrightarrow{q} \bullet$ and $p, q > 0$ do:

$$\begin{tikzcd} & k \arrow[dr, "q"] & \\ \bullet \arrow[ur, "p"] & & \bullet \arrow[ll, "r"] \end{tikzcd} \quad \overset{\mu_k}{\rightsquigarrow} \quad \begin{tikzcd} & k \arrow[dl, "p"'] & \\ \bullet \arrow[rr, "r' = pq - r"'] & & \bullet \arrow[ul, "q"'] \end{tikzcd}$$

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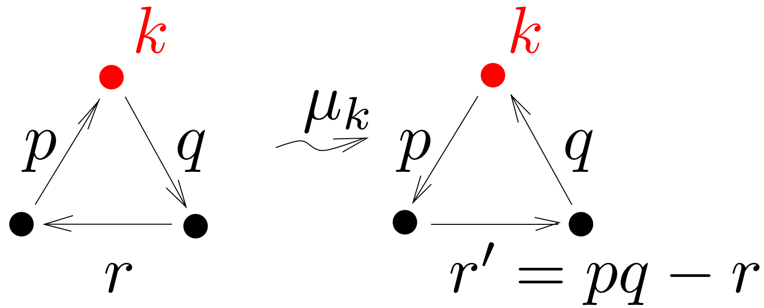
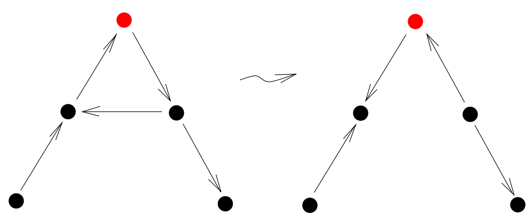


Example:



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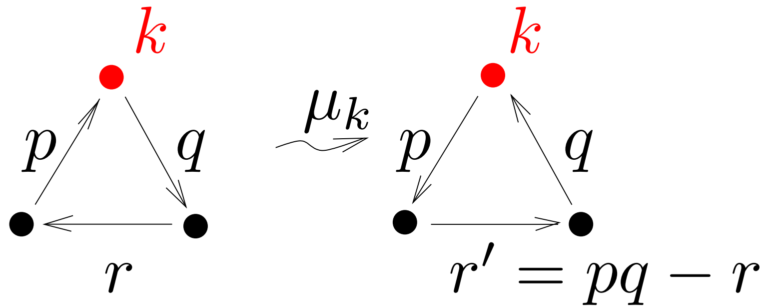

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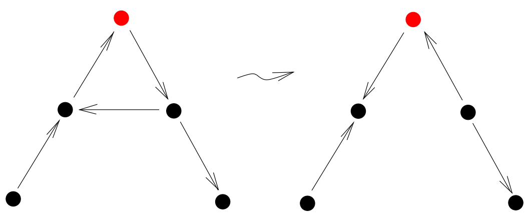
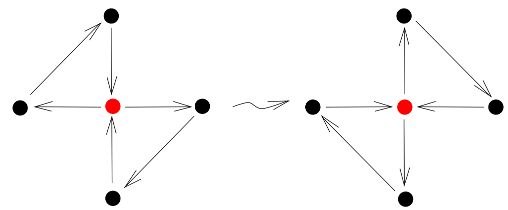
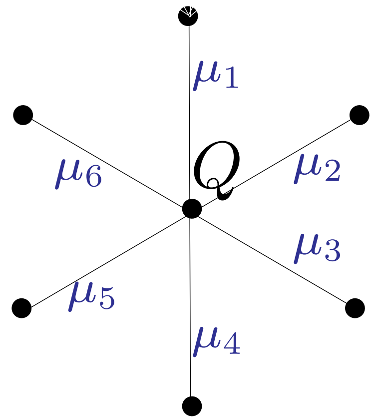




1. Quiver mutation

Iterated mutations $\longrightarrow$ many other quivers
$Q \longrightarrow$ its mutation class



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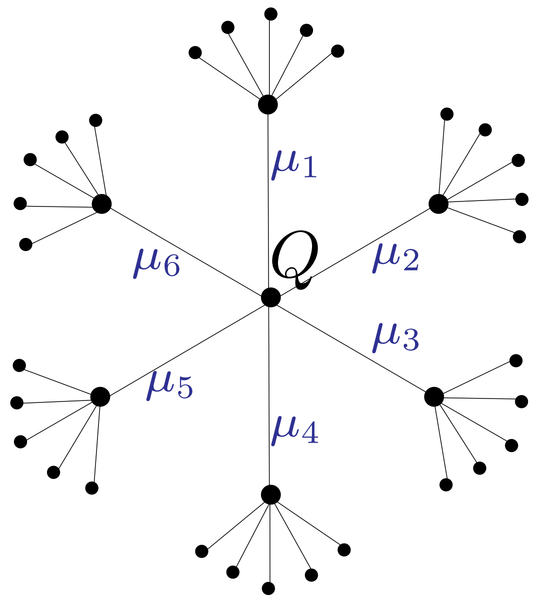

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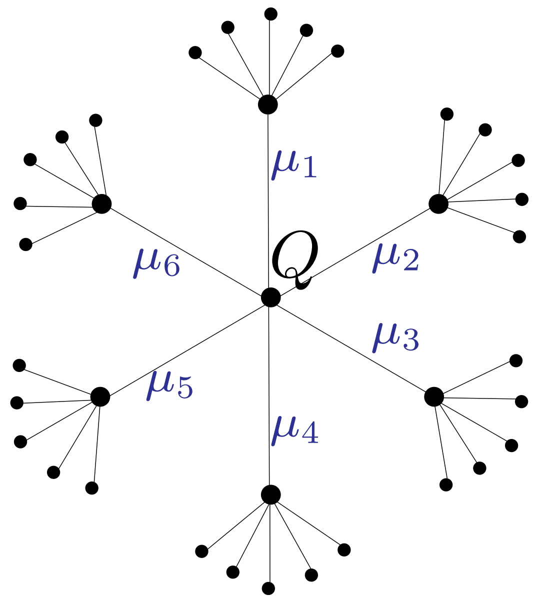

Property: $\mu_k \circ \mu_k(Q) = Q$ for any quiver Q.

Definition. A quiver is of finite mutation type if its mutation class contains finitely many quivers.

Question. Which quivers are of finite mutation type?

1. Quiver mutation

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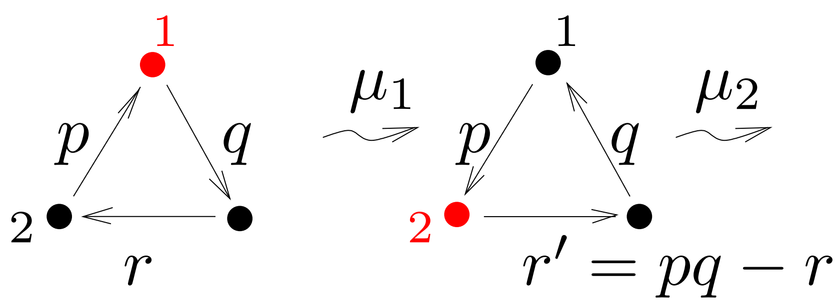
Quick answer. Not many:

If Q is connected, $|Q| \geq 3$ and Q contains arrow $\xrightarrow{p}$ with $p > 2$, then Q is mutation infinite.

Why: if $\quad q > r > 0, \quad p > 2 \qquad$ then $r' = pq - r > q > r$, so the weghts grow under alternating mutations μ_1, μ_2.



2. Cluster algebra: seed mutation

A seed is a pair $(Q, \boldsymbol{u})$ where

Q is a quiver with $n := |Q|$ veritices,

$\boldsymbol{u} = (u_1, \ldots, u_n)$ is a set of rational functions in variables $(x_1, \ldots, x_n)$.

Initial seed: $(Q_0, \boldsymbol{u}_0)$, where $\boldsymbol{u}_0 = (x_1, \ldots, x_n)$.

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Seed mutation: $\mu_k(Q, (u_1, \ldots, u_n)) = (\mu_k(Q), (u'_1, \ldots, u'_n))$

where $u'_k = \frac{1}{u_k}(\prod_{i \to k} u_i + \prod_{k \to j} u_j)$

products over all incoming/outgoing arrows

$u'_i = u_i$ if $i \neq k$.

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products over all incoming/outgoing arrows

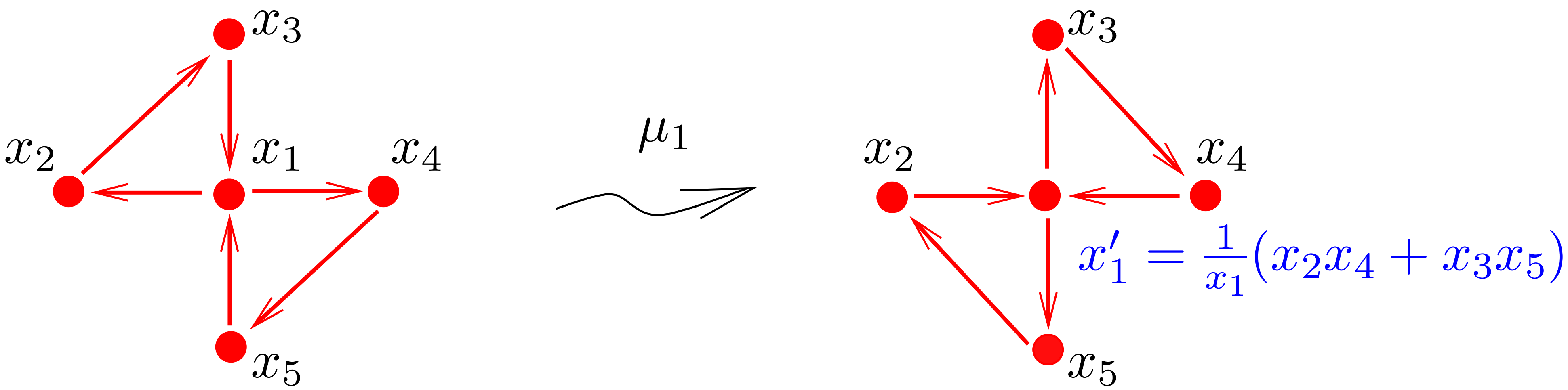
Cluster variable: a function u_i in one of the seeds.

Cluster algebra: all one can form from cluster variables, "$+$", "*" and rational numbers.

2. Cluster algebra: seed mutation

How to think about this?

Example:



Cluster variables: $x_1, x_2, x_3, x_4, x_5, x_1', \ldots$

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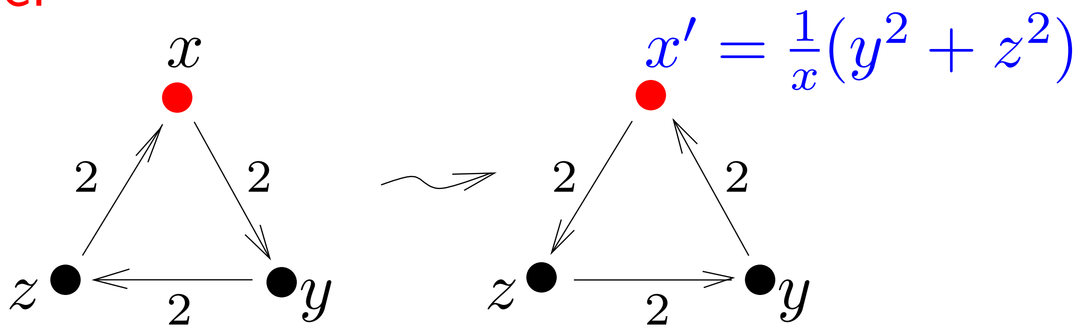
How to think about this?

Example: Markov quiver



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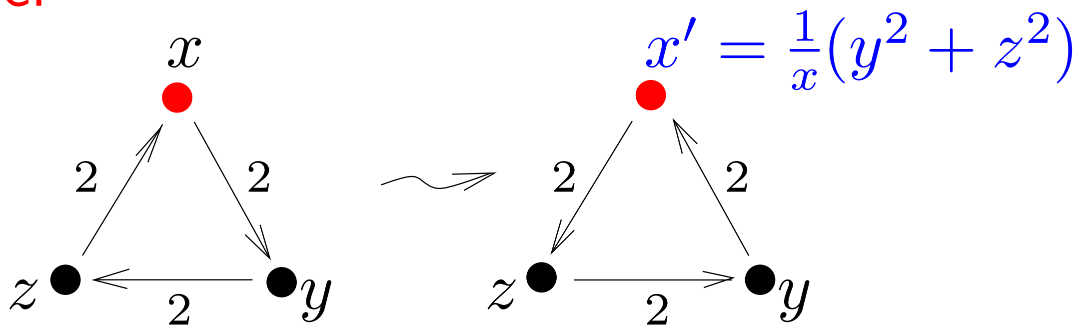

Markov equation: $$x^2 + y^2 + z^2 = 3xyz$$
$$(x, y, z) \to (3yz - x, y, z)$$

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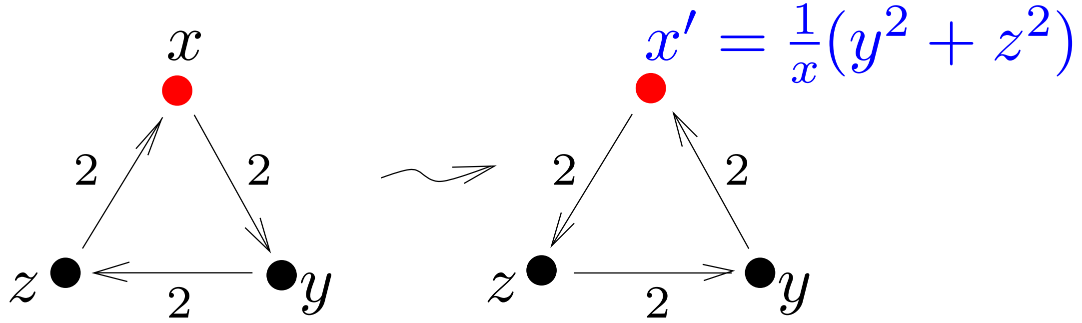

Markov equation:

$$x^2 + y^2 + z^2 = 3xyz$$
$$(x, y, z) \to (3yz - x, y, z)$$

So, if $(x, y, z) = (1, 1, 1)$ then seed mutation produces all Markov triples!

2. Cluster algebra: Two remarkable properties

By definition $u_i = \frac{P(x_1,\ldots,x_n)}{R(x_1,\ldots,x_n)}$, where P and R are polynomials.

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In fact:

- Laurent phenomenon [Fomin, Zelevinsky' 2001]:
  $R(x_1, \ldots, x_n)$ is a monomial, $R = x_1^{d_1} \ldots x_n^{d_n}$.

It is a miracle as computing $u'_k = \frac{1}{u_k}(\prod\limits_{i \to k} u_i + \prod\limits_{k \to j} u_j)$ we divide by u_k!

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- Positivity [Conj.: Fomin, Zelevinsky' 2001; proved: Lee, Schiffler' 2013]:
  $P(x_1,\ldots,x_n)$ has positive coefficients.

  It is a miracle as we divide: $\frac{a^3+b^3}{a+b} = a^2 - ab + b^2$.

2. Cluster algebra: finite type

A cluster algebra is of finite type

if it contains finitely many cluster variables.

Theorem. (Fomin, Zelevinsky' 2002)

A cluster algebra $\mathcal{A}(Q)$ is of finite type iff

Q is mutation-equivalent to an orientation

of a Dynkin diagram A_n, D_n, E_6, E_7, E_8.

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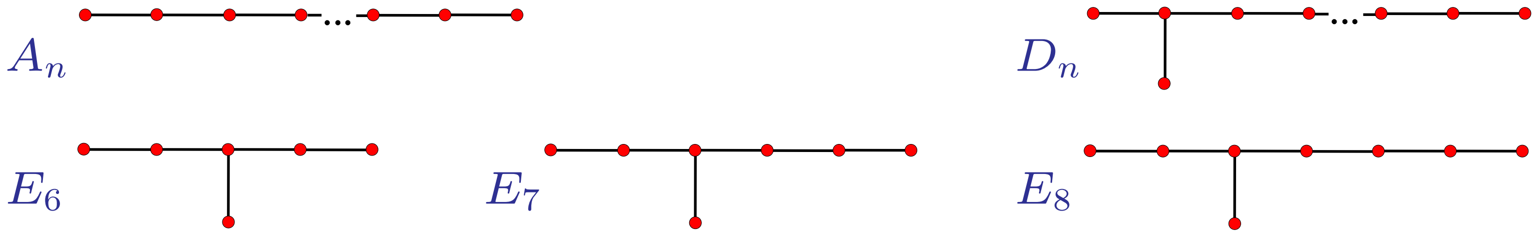

Note: Dynkin diagrams describe:

finite reflection groups, semisimple Lie algebras, surface singularities...

2. Cluster algebra: finite mutation type

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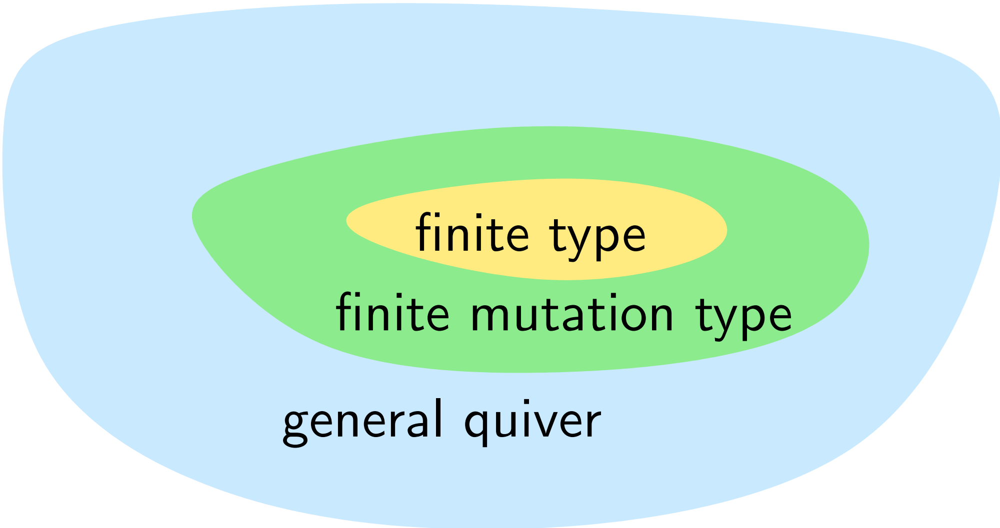


mutation class	cluster variables
$< \infty$	$< \infty$
$< \infty$	∞
∞	∞
$\cancel{\infty}$	$\cancel{< \infty}$

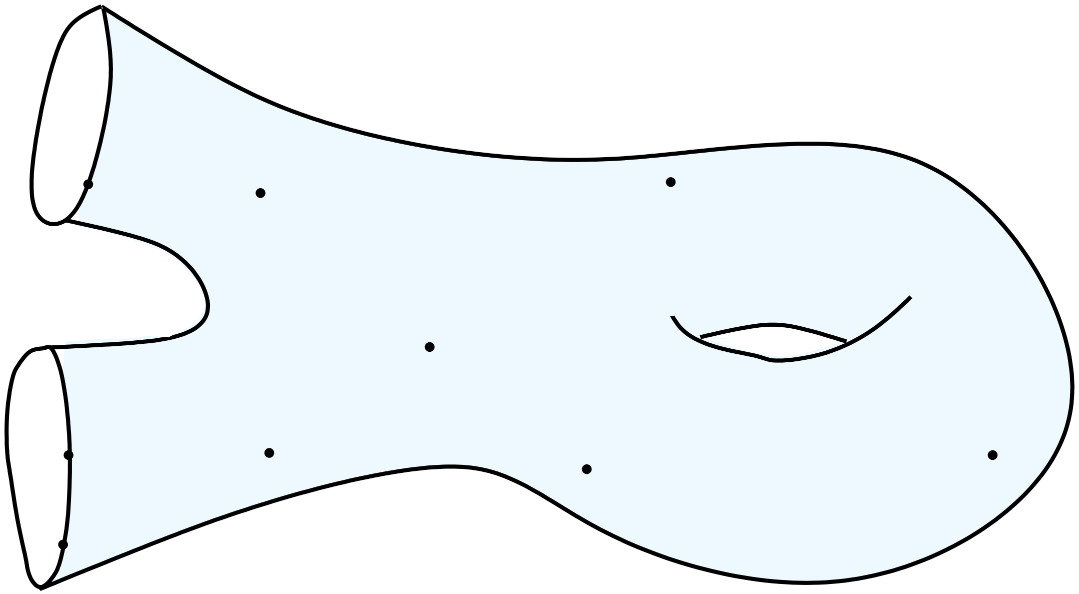
3. Finite mutation type: examples

1. $n = 2$.

2. Quivers arising from triangulated surfaces.

3. Finitely many except that.
   (conjectured by Fomin, Shapiro, Thurston)

4. Quivers from triangulated surfaces



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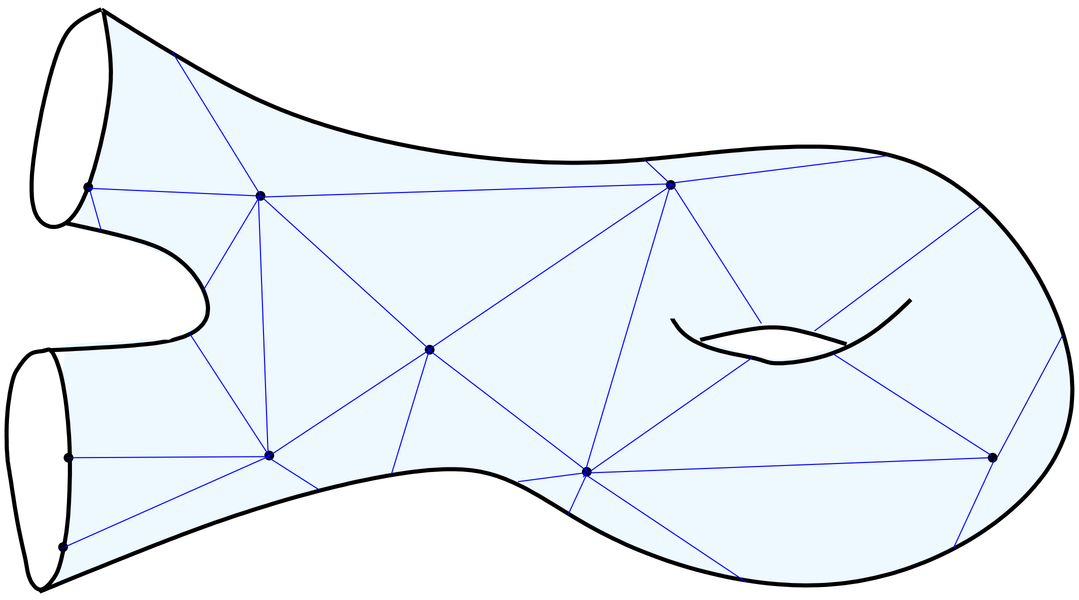
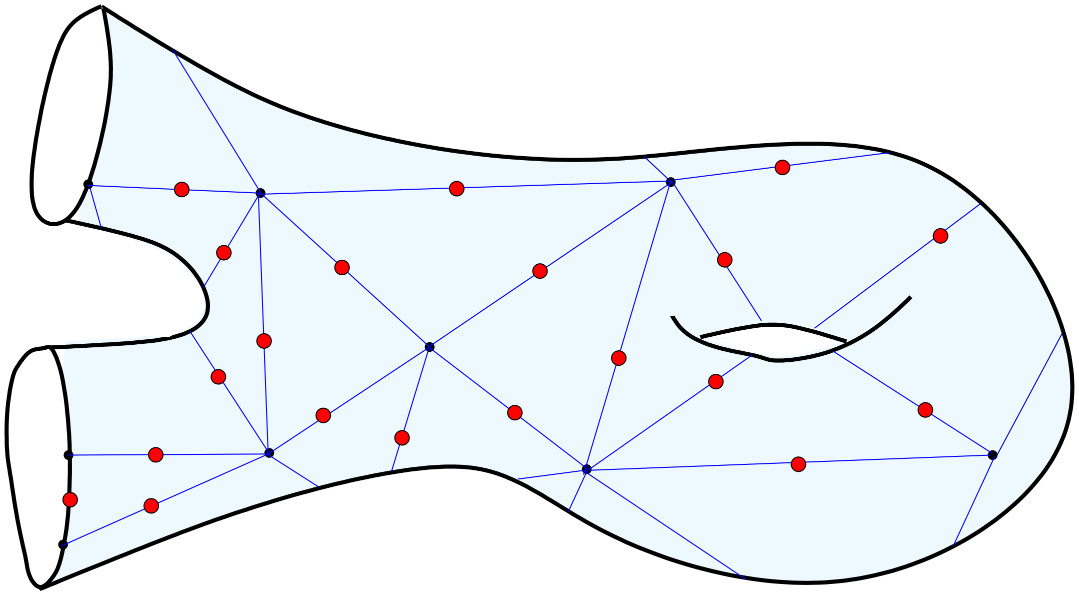


4. Quivers from triangulated surfaces

Triangulated surface $\longrightarrow$ Quiver



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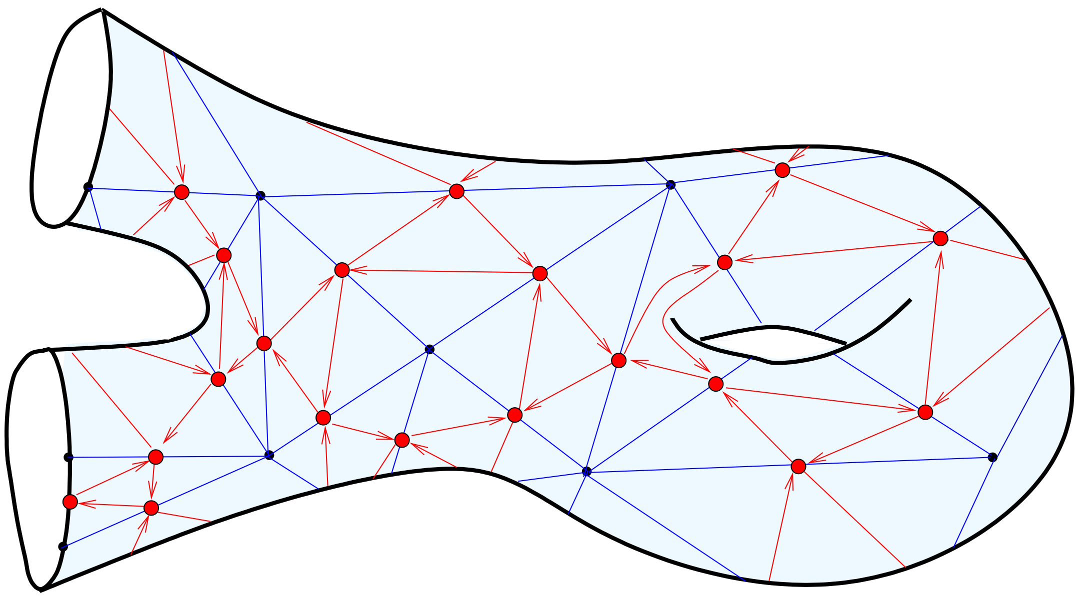
Triangulated surface	$\longrightarrow$	Quiver
edge of triangulation		vertex of quiver
two edges of one triangle		arrow of quiver



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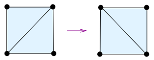
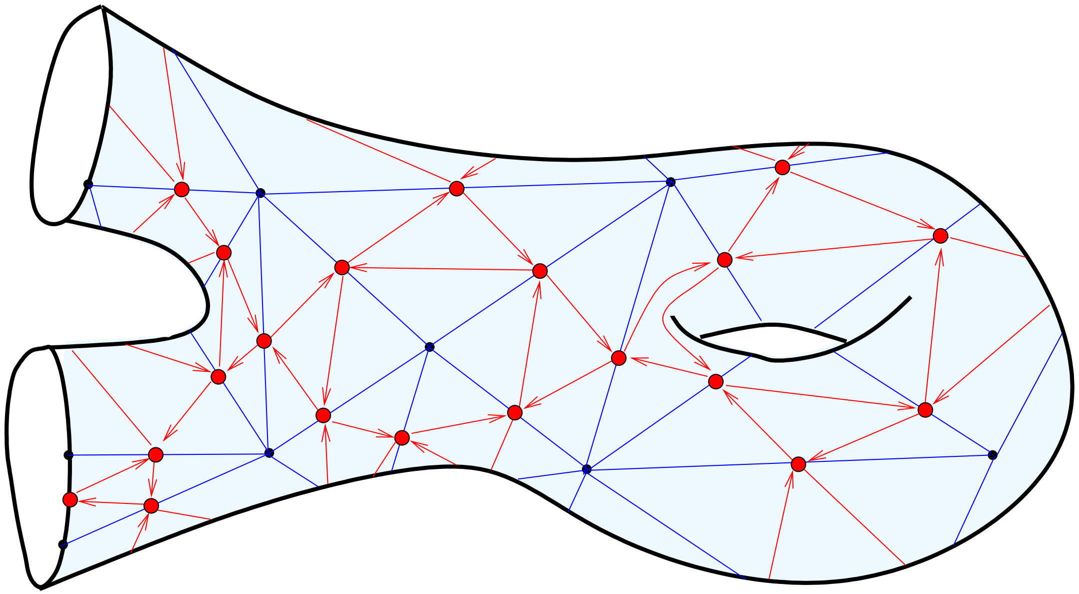
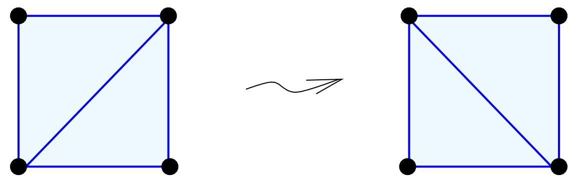
flip of triangulation 		





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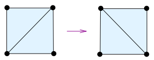
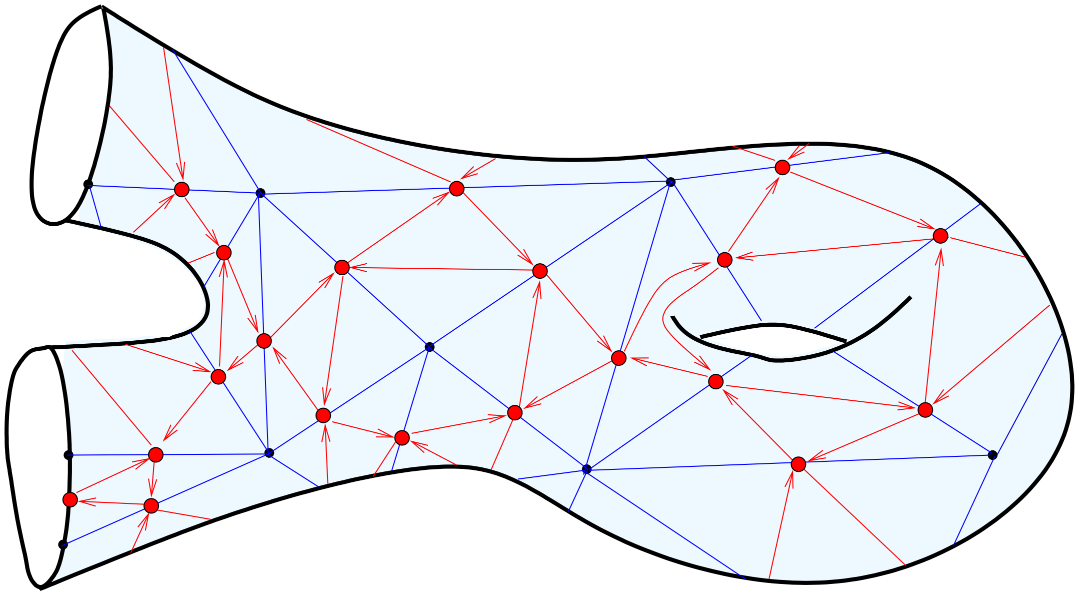
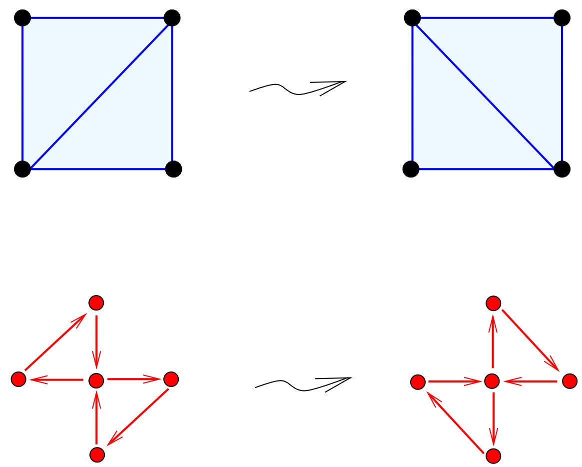
flip of triangulation 		mutation of quiver





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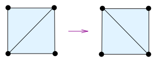
flip of triangulation 		mutation of quiver

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(as every arc lies at most in two triangles)

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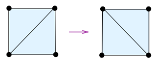
flip of triangulation 		mutation of quiver

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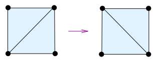
flip of triangulation 		mutation of quiver

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Corollary. (a) Quivers from triangulations of the same surface are mutation-equivalent (and form the whole mutation class).

(b) Quivers from triangulations are mutation-finite.

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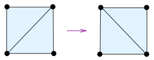
flip of triangulation 		mutation of quiver

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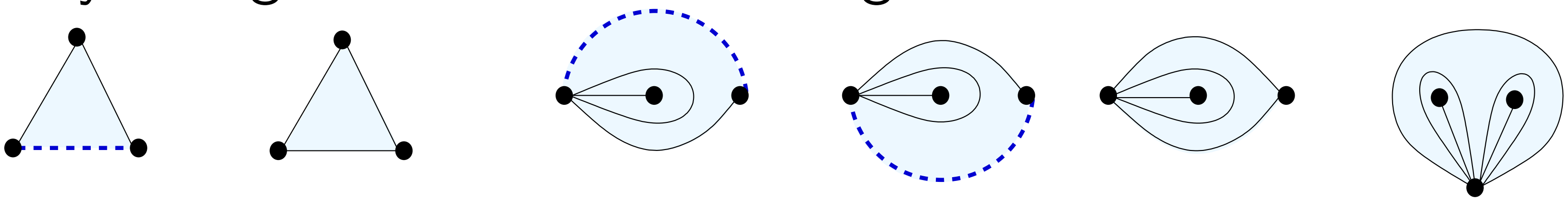
Question. What else is mutation-finite?

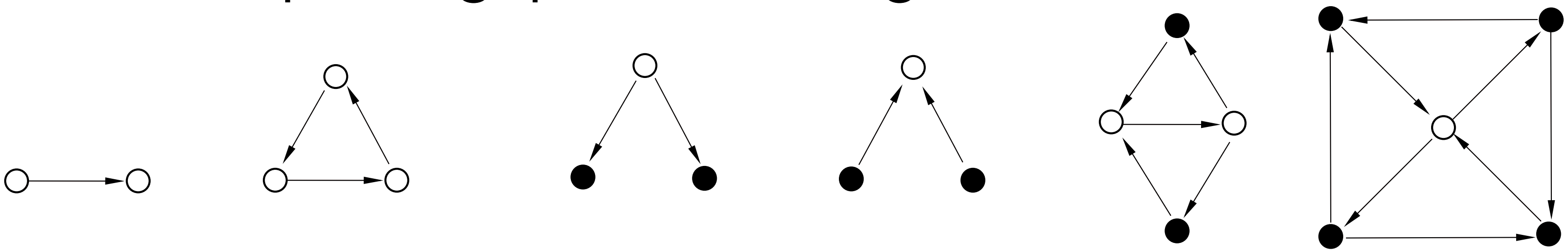
4. Quivers from triangulations: description

(Fomin-Shapiro-Thurston)

Any triangulated surface can be glued of:



The corresponding quiver can be glued of blocks:



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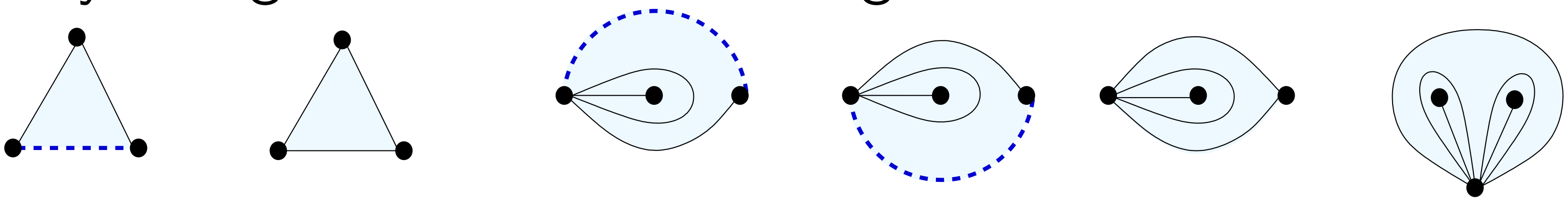

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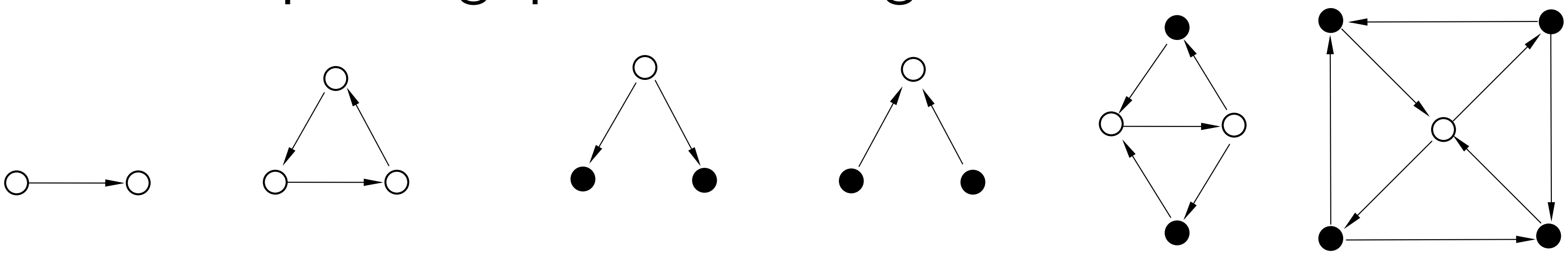

Proposition. (Fomin-Shapiro-Thurston)

$$\{Q \text{ is from triangulation }\} \Leftrightarrow \{Q \text{ is block-decomposable }\}$$

Question: How to find all mutation-finite but not block-decomposable quivers?

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Interlude:

How to classify tesselations in hyperbolic space?

1. They correspond to some polytopes (described by some diagrams);
2. Combinatorics of these polytopes is described by:
   a. subdiagrams corresponding to finite objects (classified) ;
   b. minimal subdiagrams correponding to infinite objects.

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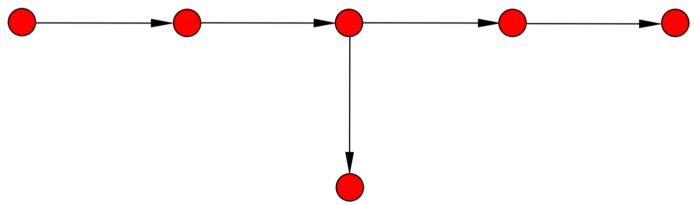
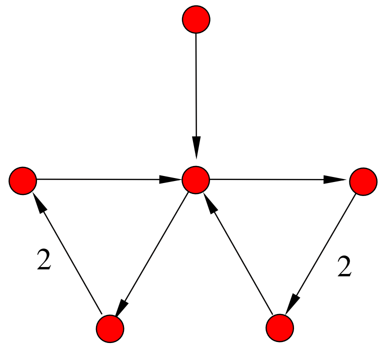
1. They correspond to some polytopes (described by some diagrams);
2. Combinatorics of these polytopes is described by:
   a. subdiagrams corresponding to finite objects (classified) ;
   b. minimal subdiagrams correponding to infinite objects.

Idea: Classify minimal non-decomposable quivers.

Lemma 1. If Q is a minimal non-decomposable quiver then $|Q| \leq 7$.

Lemma 2. If Q is a minimal non-decomposable mutation-finite quiver then is mutation equivalent to one of



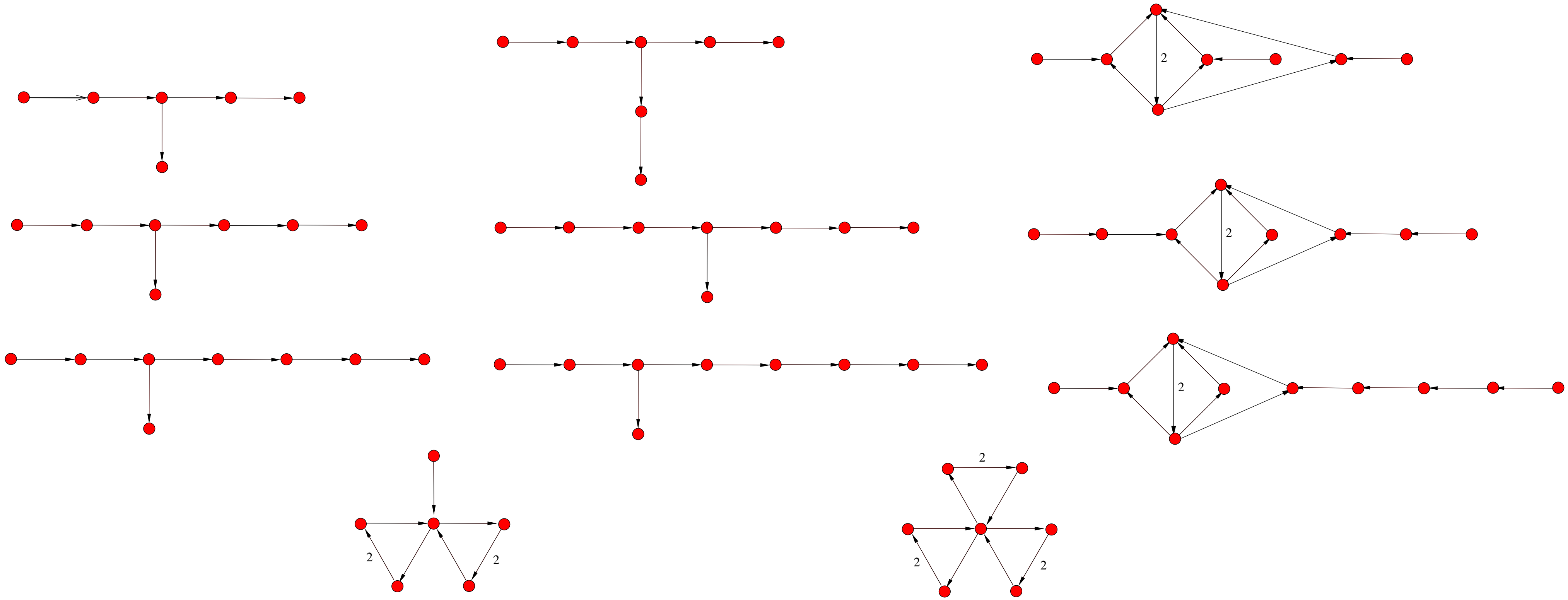


Now: - add vertices to these quivers (and their mutations) one by one
- check the obtained quiver is still mutation-finite.

Theorem 1. (A.F, M.Shapiro, P.Tumarkin' 2008)

Let Q be a connected quiver of finite mutation type. Then

- either $|Q| = 2$;
- or Q is obtained from a triangulated surface;
- or Q is mut.-equivalent to one of the following 11 quivers:



Proof:

Proof: **terrible, technical ....** -but follows the same steps
as some classifications of tessellations

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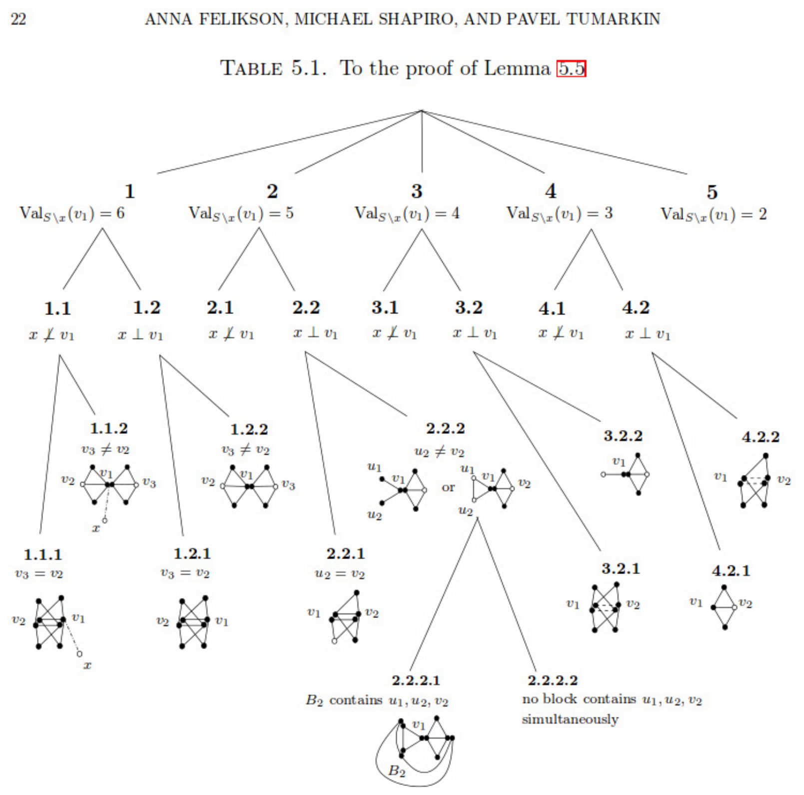
Example. Logic scheme for a proof of some small lemma:



Proof: **terrible, technical ....** -but follows the same steps
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Andrei Zelevinsky: "For this sort of proofs the authors should be sent to Solovki"

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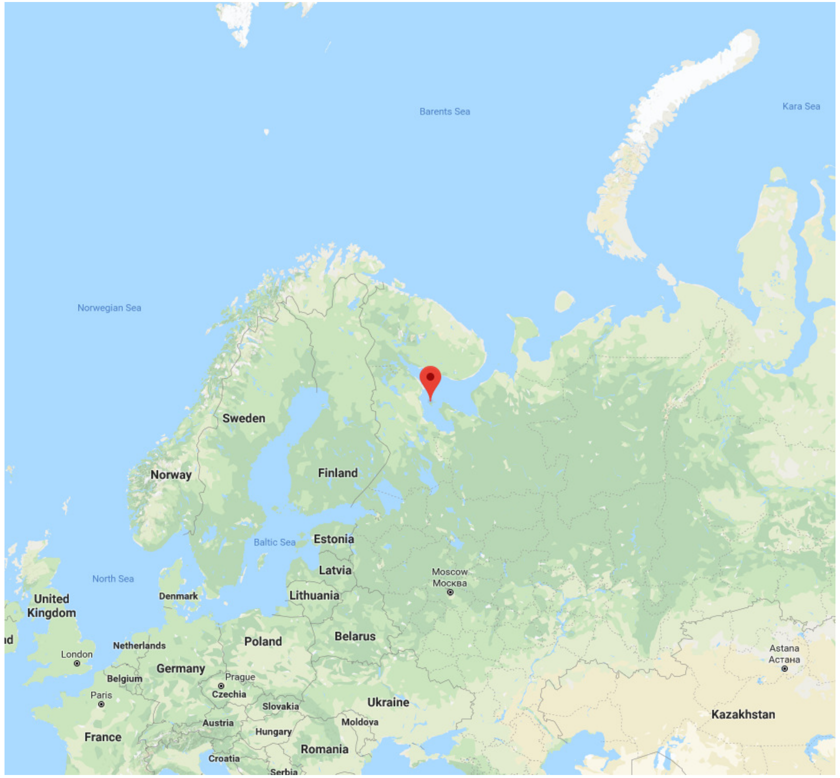
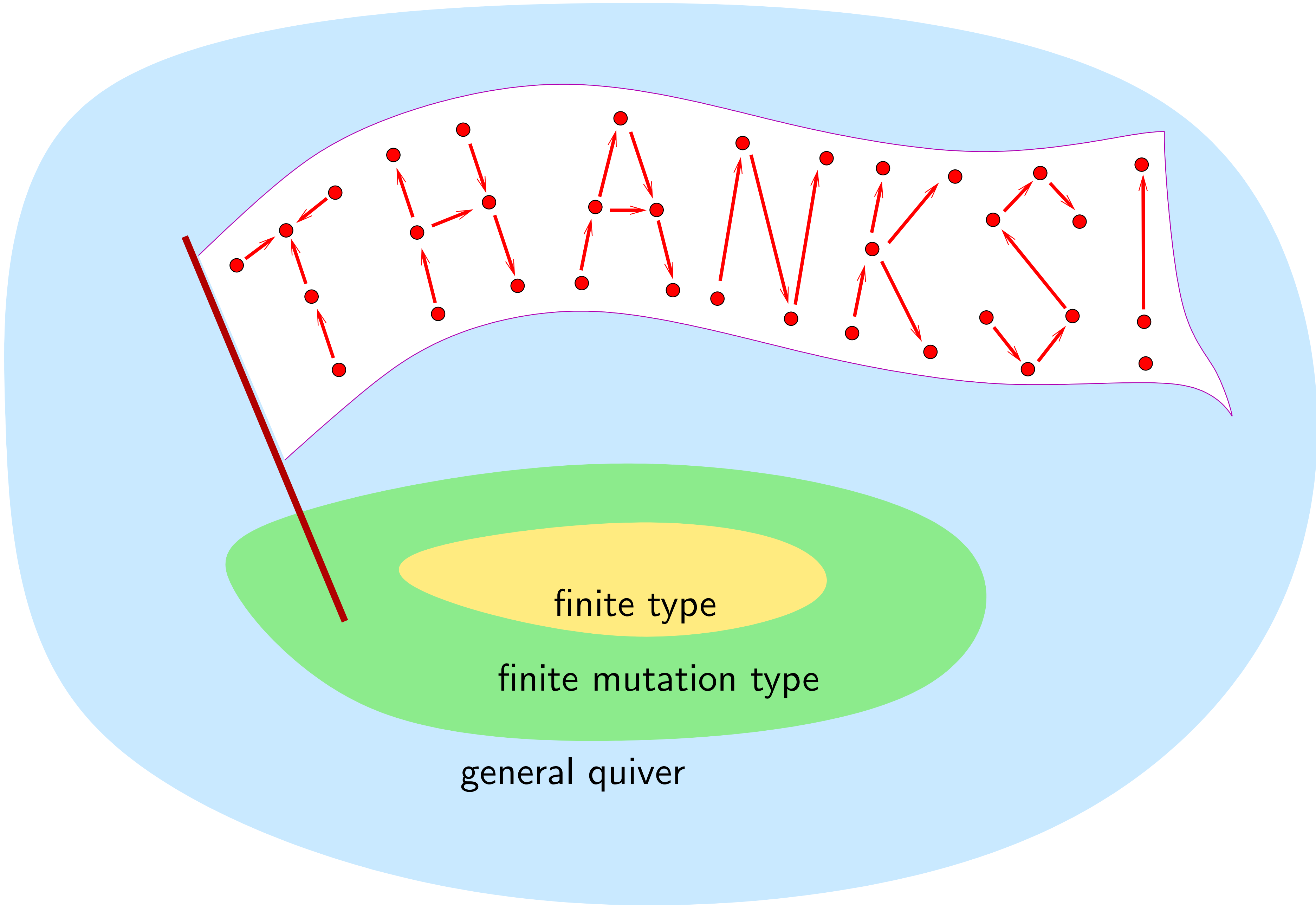
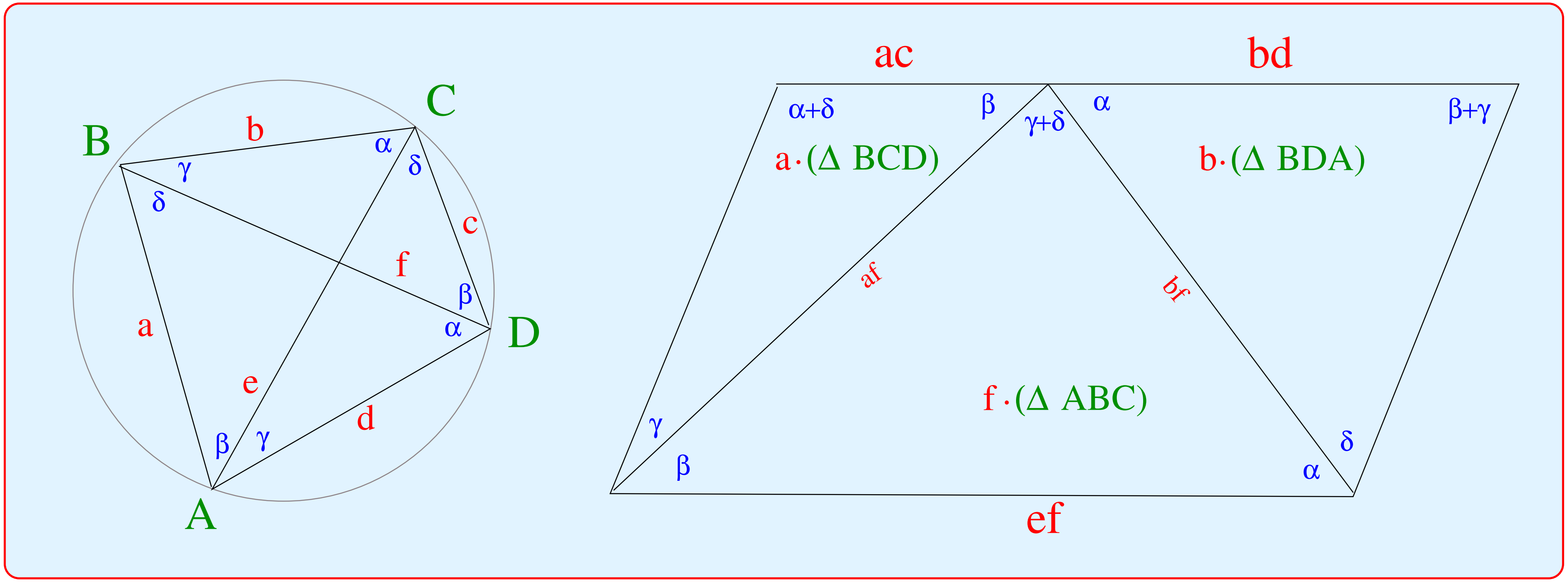



THANKS!
finite type
finite mutation type
general quiver

Bonus: proof of Ptolemy Theorem.



ef=ac+bd

Proof from:
https://www.cut-the-knot.org/proofs/PtolemyTheoremPWW.shtml