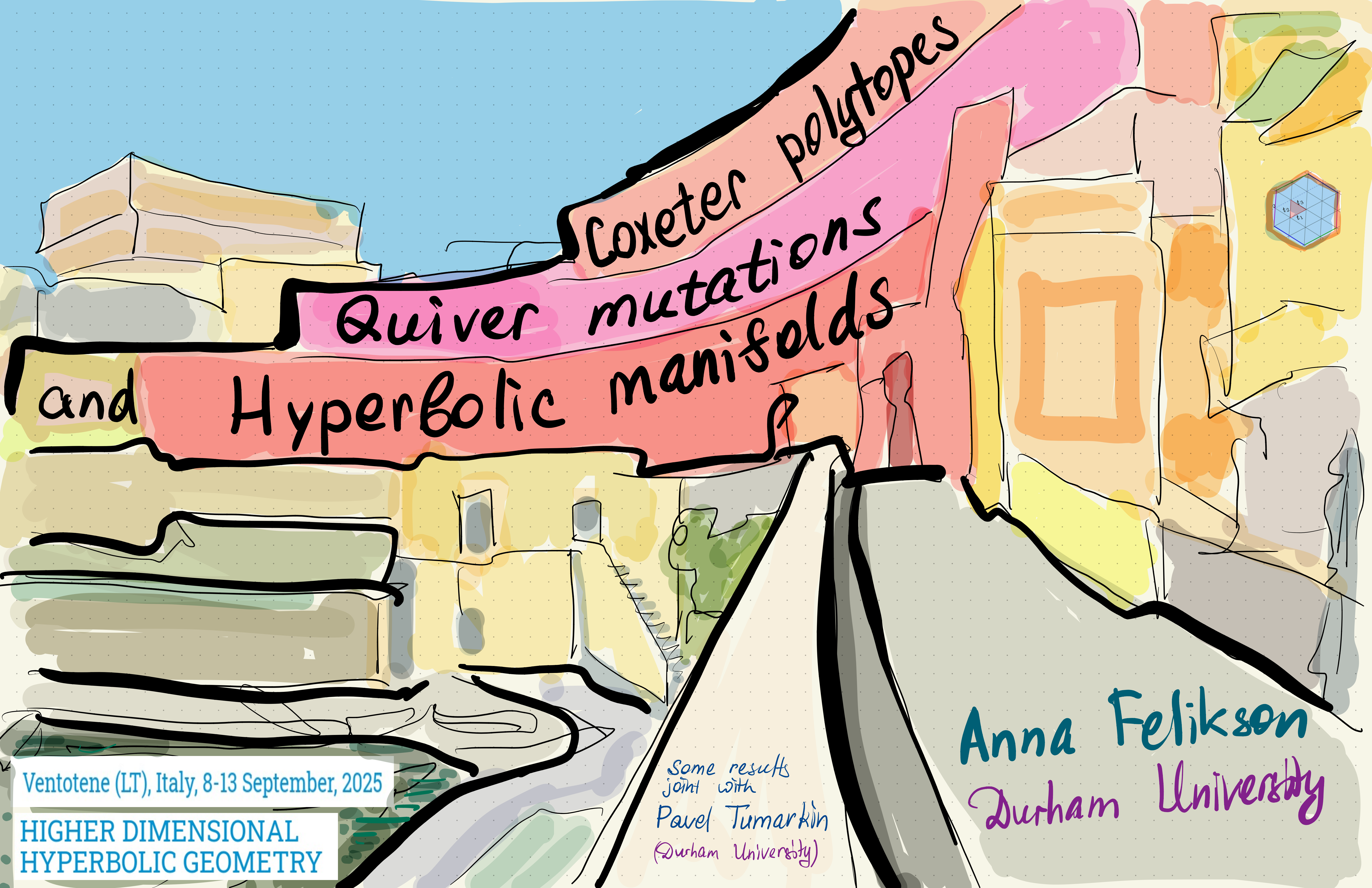

Coxeter polytopes
Quiver mutations
and Hyperbolic manifolds
Anna Felikson
Durham University
Some results joint with
Pavel Tumarkin
(Durham University)
Ventotene (LT), Italy, 8-13 September, 2025
HIGHER DIMENSIONAL HYPERBOLIC GEOMETRY

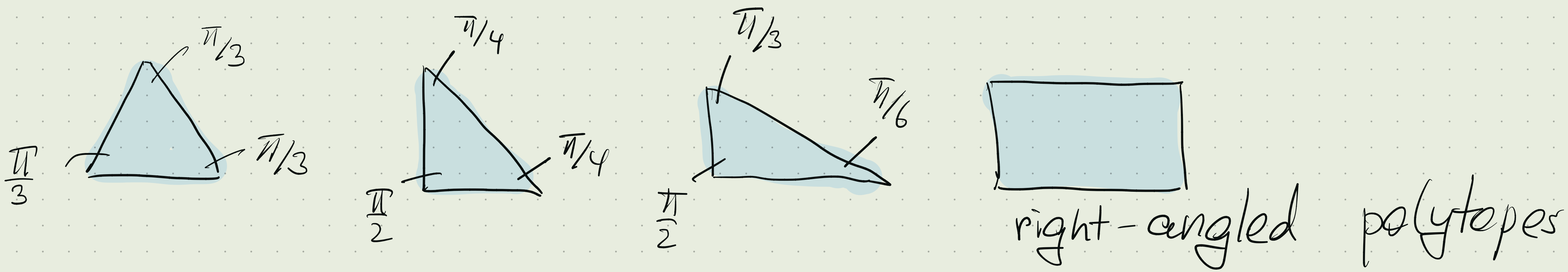
I On classification of hyperbolic Coxeter polytopes

II Application to a classification problem about quiver mutation

III Constructing hyperbolic manifolds via quiver mutations

Coxeter polytopes are polytopes in $\mathbb{S}^d$, $\mathbb{E}^d$ or $\mathbb{H}^d$ whose all dihedral angles are submultiples of π

Ex:



- Coxeter polytopes are
  - fundamental domains of discrete reflection groups
  - a tool to construct hyperbolic manifolds [by gluings]

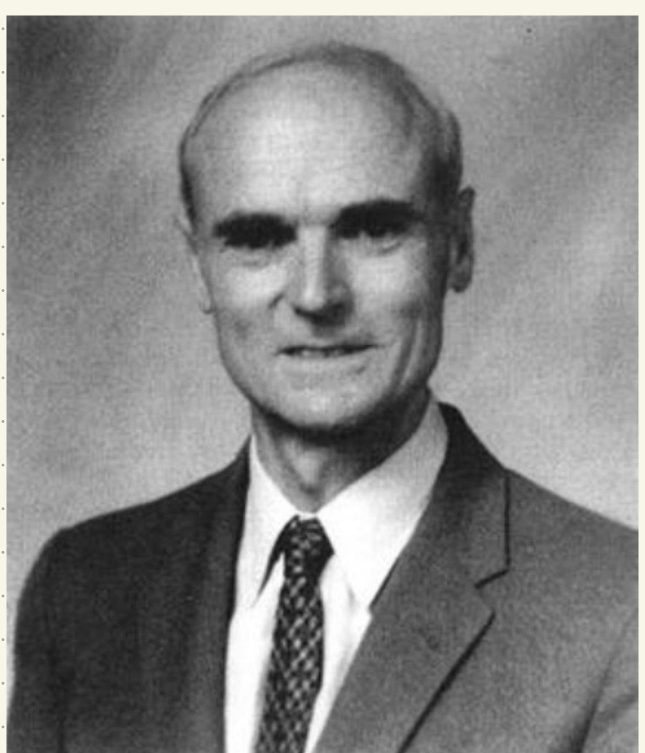
How to classify Coxeter polytopes?

Coxeter polytopes in $\mathbb{S}^n$, $\mathbb{E}^n$ and $\mathbb{H}^n$:

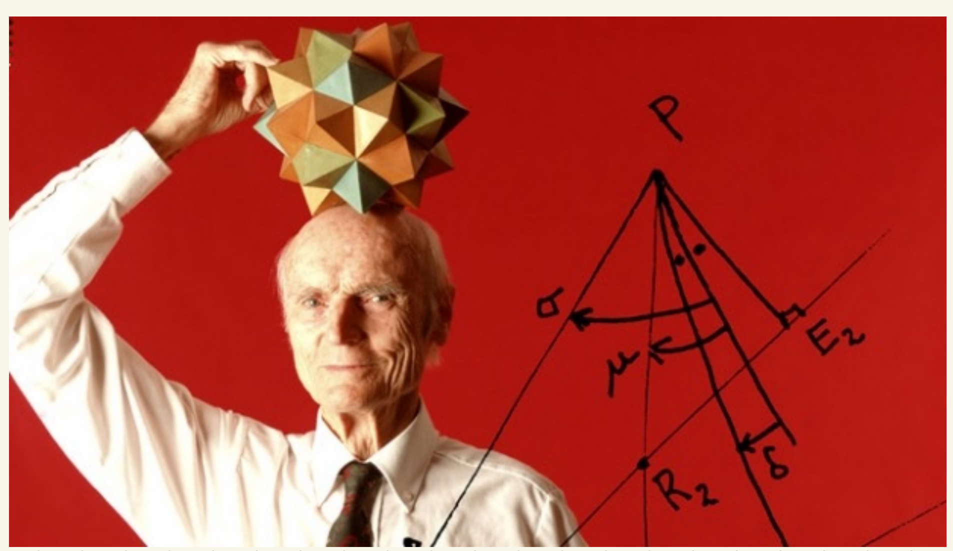
- spherical ✓
- Euclidean ✓
- Hyperbolic ?

Classified by Coxeter in 1934 — No classification

In terms of

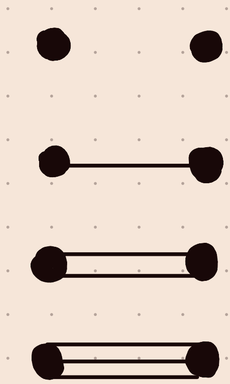


Harold Scott MacDonald Coxeter



Coxeter diagram

- Nodes $\leftrightarrow$ facets f_i of P
- Edges:  if $\angle(f_i f_j) = \frac{\pi}{m_{ij}}$



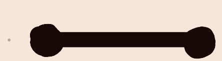
$\pi/2$

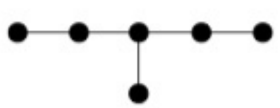
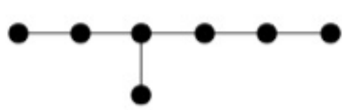
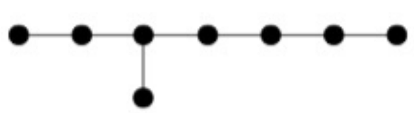
$\pi/3$

$\pi/4$

$\pi/5$

 if $f_i \cap f_j = \emptyset$

 if $f_i \cap f_j \in \partial\mathbb{H}^n$

Connected elliptic diagrams	
A_n $(n \geq 1)$	
$B_n = C_n$ $(n \geq 2)$	
D_n $(n \geq 4)$	
$G_2^{(m)}$	
F_4	
E_6	
E_7	
E_8	
H_3	
H_4	

S^n Spherical Coxeter polytopes | $\mathbb{E}^n$ Euclidean Coxeter polytopes

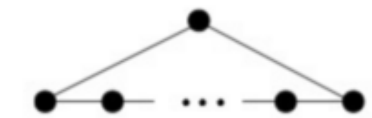
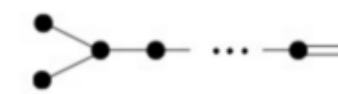
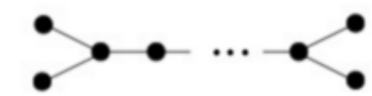
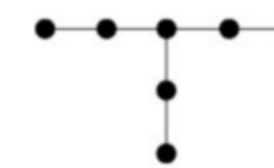
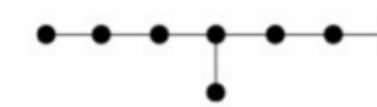
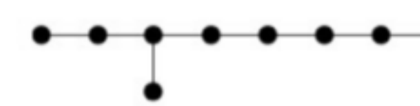
Coxeter diagram = union of connected

elliptic diagrams | parabolic diagrams

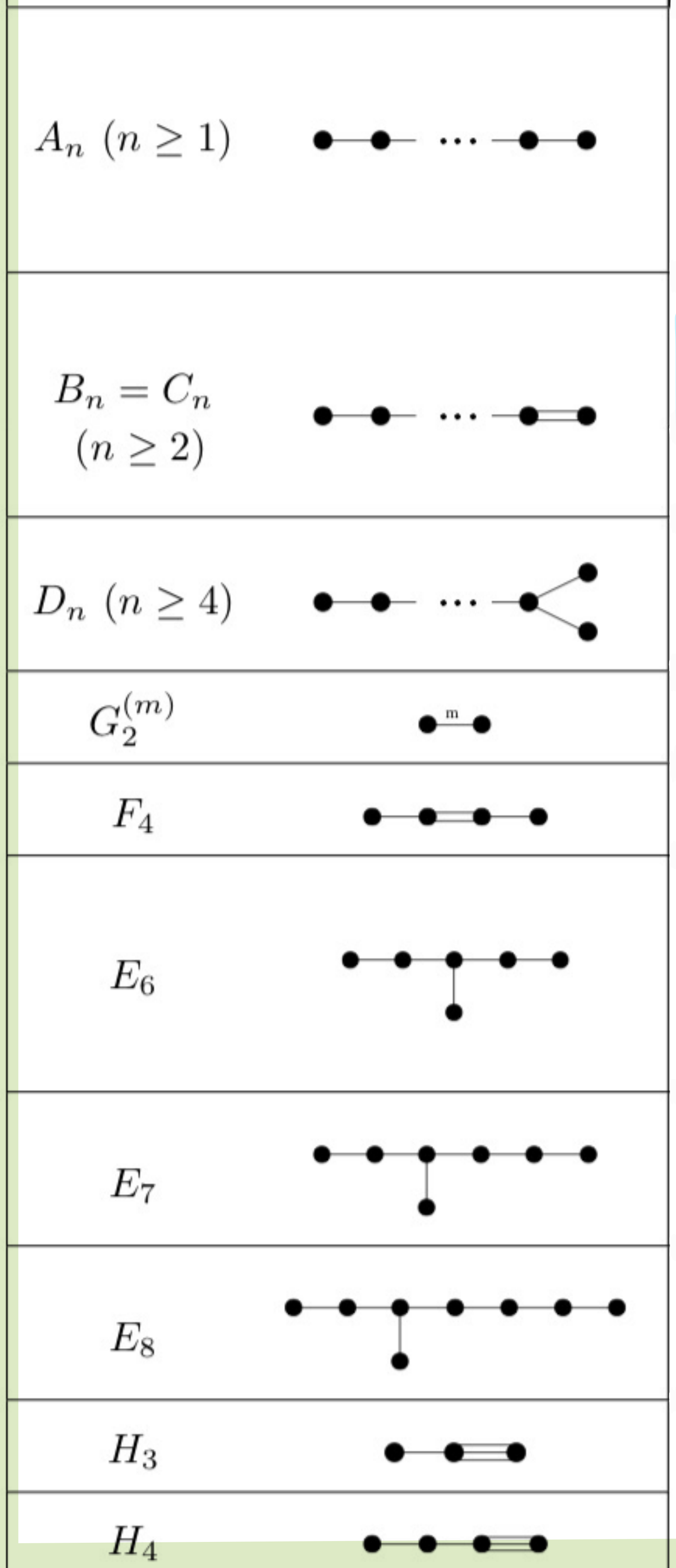
Combinatorics:

- simplices | - products of simplices

Finitely many types in each dimension

Connected parabolic diagrams	
$\widetilde{A}_1$	
$\widetilde{A}_n$ $(n \geq 2)$	
$\widetilde{B}_n$ $(n \geq 3)$	
$\widetilde{C}_n$ $(n \geq 2)$	
$\widetilde{D}_n$ $(n \geq 4)$	
$\widetilde{G}_2$	
$\widetilde{F}_4$	
$\widetilde{E}_6$	
$\widetilde{E}_7$	
$\widetilde{E}_8$	

Connected elliptic diagrams

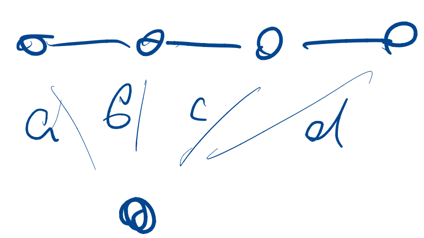


One way to get this:

Induction

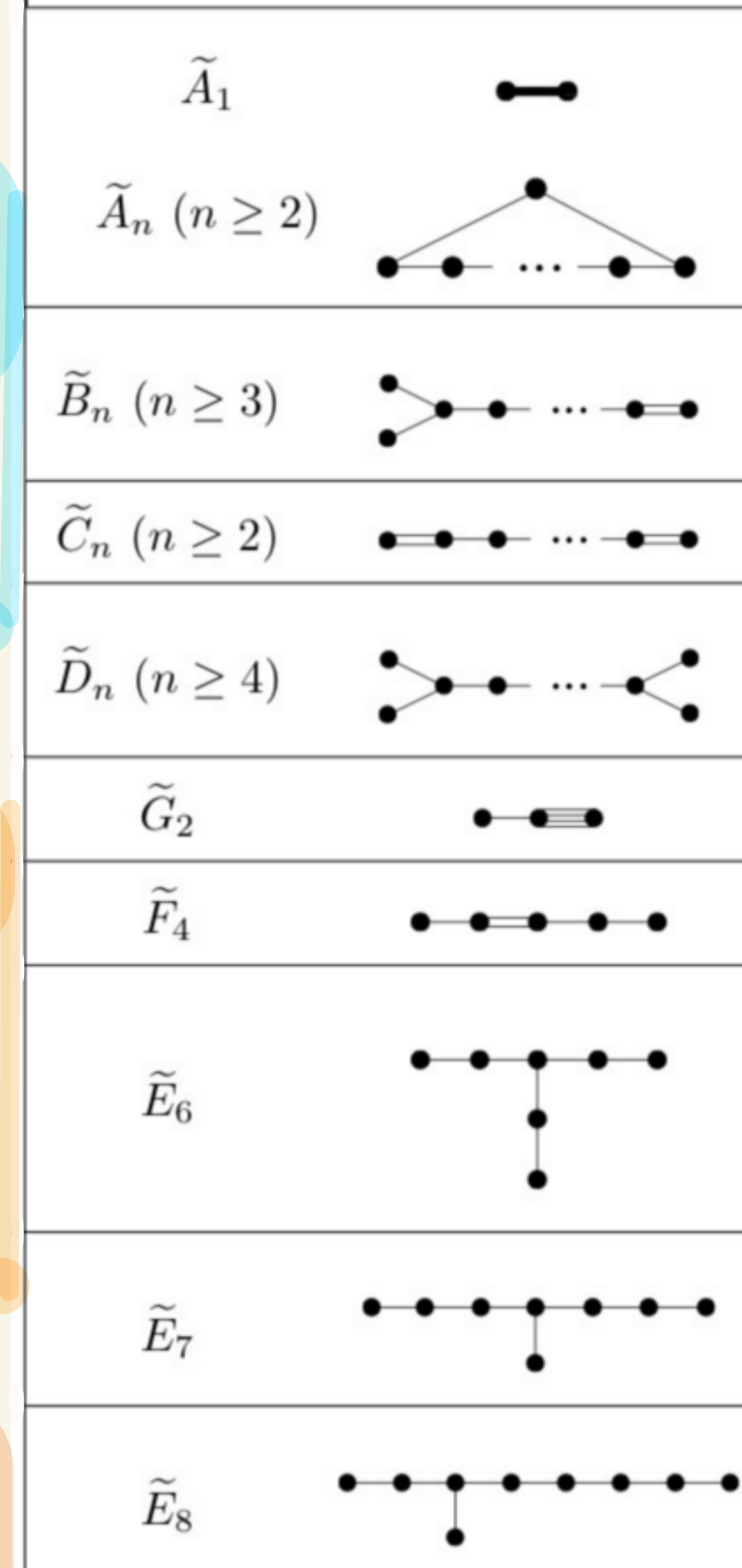
Base: $n = 3$ $\quad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$

Step: – add extra node to a diagram checking that no forbidden subdiagram appear



– then check the signature of Gram matrix

Connected parabolic diagrams



Coxeter polytopes in $\mathbb{S}^n$, $\mathbb{E}^n$ and $\mathbb{H}^n$:

- spherical
  - G positive-definite
- Euclidean
  - G positive-semidefinite
- Hyperbolic
  - G of signature $(n, 1)$

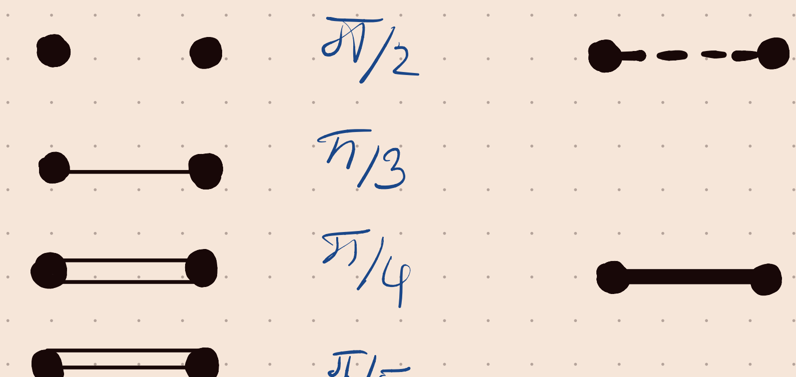
Gram matrix G:

$$g_{ii} = 1,$$

$$g_{ij} = \begin{cases} -\cos\left(\frac{\pi}{m_{ij}}\right), & \text{if } \angle f_i f_j = \frac{\pi}{m_{ij}}; \\ -1, & \text{if } f_i \text{ is parallel to } f_j; \\ -\cosh \rho_{ij}, & \text{if } f_i \text{ and } f_j \text{ diverge} \\ & \text{and lie at distance } \rho_{ij}. \end{cases}$$

Coxeter diagram

- Nodes $\leftrightarrow$ facets f_i of P
- Edges: [edge labelled m_{ij}] if $\angle(f_i f_j) = \frac{\pi}{m_{ij}}$



(dashed edge) if $f_i \cap f_j = \emptyset$

(bold edge) if $f_i \cap f_j \in \partial \mathbb{H}^n$

Coxeter polytopes in $\mathbb{S}^n$, $\mathbb{E}^n$ and $\mathbb{H}^n$:

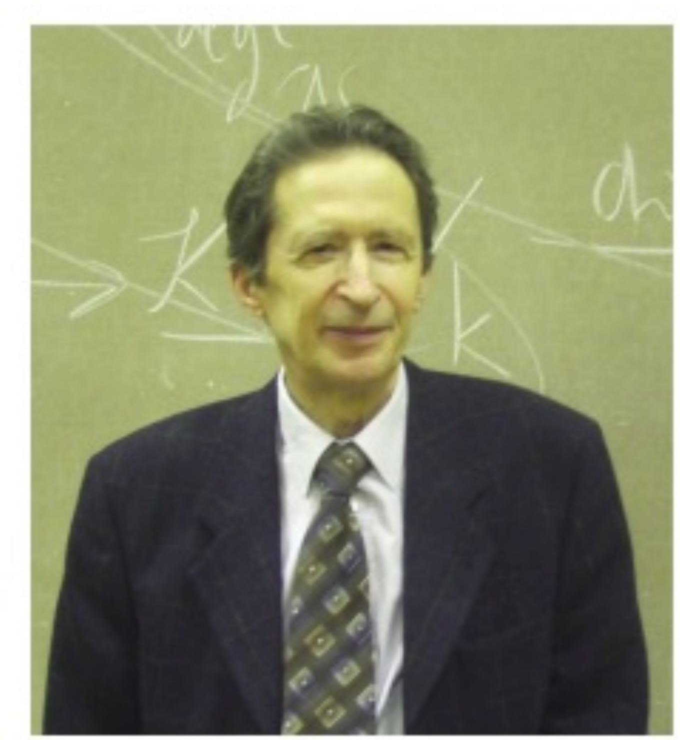
Thm [Vinberg '1985]

Let $G = \{g_{ij}\}$ be an indecomposable symmetric matrix of signature $(d, 1)$ with units on the diagonal and non-positive off-diagonal elements everywhere else. Then there exists a convex polytope P in d-dimensional hyperbolic space $\mathbb{H}^d$ such that the Gram matrix $G(P)$ of P coincides with G. The polytope P is unique up to isometry of $\mathbb{H}^d$.

Gram matrix G:

$$g_{ii} = 1,$$

$$g_{ij} = \begin{cases} -\cos(\frac{\pi}{m_{ij}}), & \text{if } \angle f_i f_j = \frac{\pi}{m_{ij}}; \\ -1, & \text{if } f_i \text{ is parallel to } f_j; \\ -\cosh\ \rho_{ij}, & \text{if } f_i \text{ and } f_j \text{ diverge} \\ & \text{and lie at distance } \rho_{ij}. \end{cases}$$



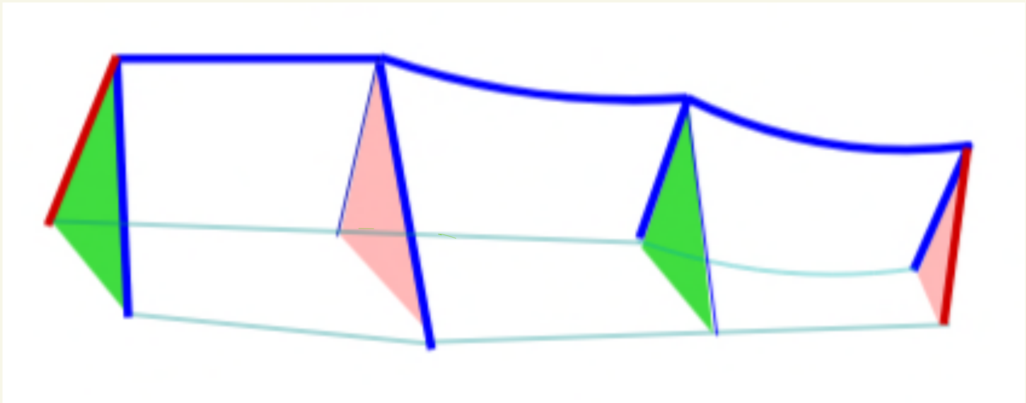
Ernest Borisovich Vinberg

26.07.1937 — 12.05.2020

$\mathbb{H}^d$ Hyperbolic Coxeter polytopes

- Wide variety of compact and finite volume polytopes

Ex:

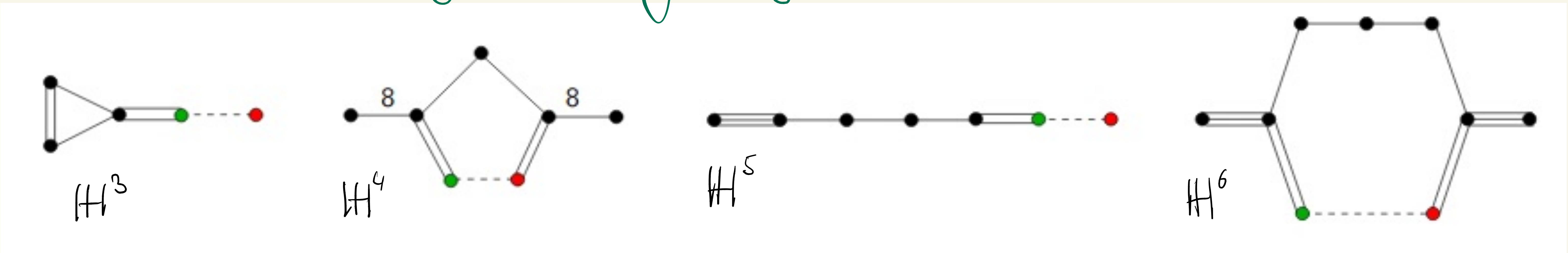


- Any number of facets
- Any complexity of combinatorial types
- Arbitrary small dihedral angles

- Thm [Allcock '05]:

Compact polytopes: infinitely many in $\mathbb{H}^d$ for all $d \leq 6$.
Finite volume polytopes: infinitely many in $\mathbb{H}^d$ for all $d \leq 19$.

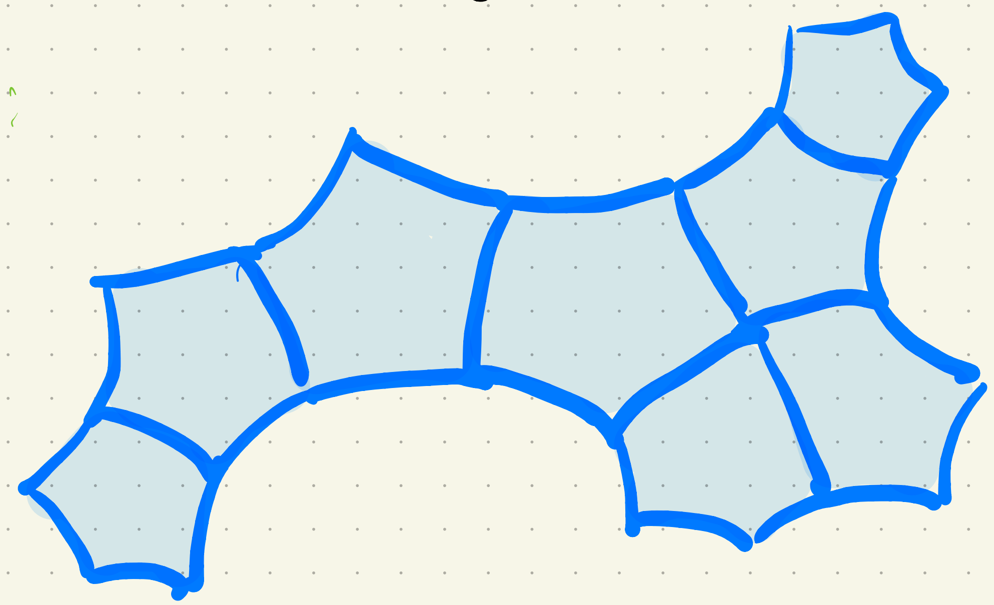
(by building garlands & trees)



$\mathbb{H}^d$ Hyperbolic Coxeter polytopes

- Wide veriety of compact and finite volume polytopes

Ex:



— Any number of facets
— Any complexity of combinatorial types
— Arbitrary small dihedral angles

- Thm [Allcock '05]:

Compact polytopes: infinitely many in $\mathbb{H}^d$ for all $d \leq 6$.
Finite volume polytopes: infinitely many in $\mathbb{H}^d$ for all $d \leq 19$.

- Plan:
  1. How badly we don't know
  2. Small bits we know
  3. Some tools
  4. More recent results

$\mathbb{H}^d$ Hyperbolic Coxeter polytopes

1. How we don't know...

Absence in large dimensions:

If $P \subset \mathbb{H}^d$ is compact, then $d \leq 29$. [Vinberg '84]

Why?

1. Vertices $\Leftrightarrow$ Elliptic subdiagrams
$\Rightarrow$ many right angles

2. Triangular, quadrilateral faces $\rightarrow$
many non-right angles

3. Thm [Nikulin '81]
$\forall$ simple, compact, convex polytope $P \subset \mathbb{E}^d$
$\forall\, i < k \leq [d/2]$ holds

$$\alpha_k^i < \binom{d-i}{d-k} \frac{\binom{[d/2]}{i} + \binom{[(d+1)/2]}{i}}{\binom{[d/2]}{k} + \binom{[(d+1)/2]}{k}}$$

average number of i-faces of a k-face of P

4. $\alpha_2^1 \leq \frac{4(d-\varepsilon)}{(d-1-\varepsilon)}$ $\quad \varepsilon = \begin{cases} 1, & \text{if } d \text{ is even} \\ 0 & \text{otherwise} \end{cases}$
sides of 2-faces

$\Rightarrow$ many triangular/quadr. 2-faces!

$\mathbb{H}^d$ Hyperbolic Coxeter polytopes

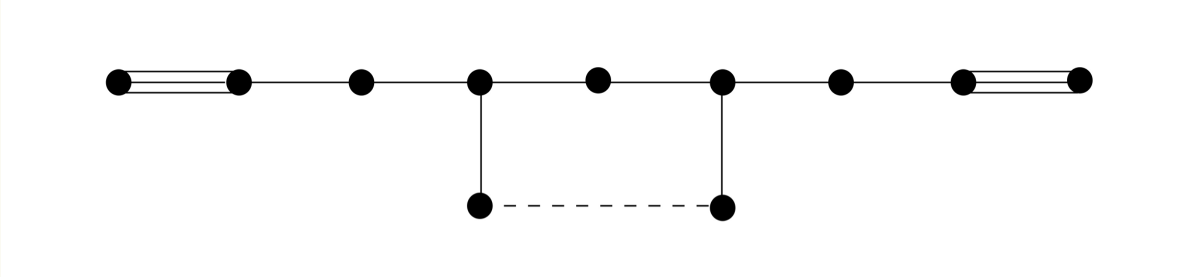
1. How we don't know...

Absence in large dimensions:

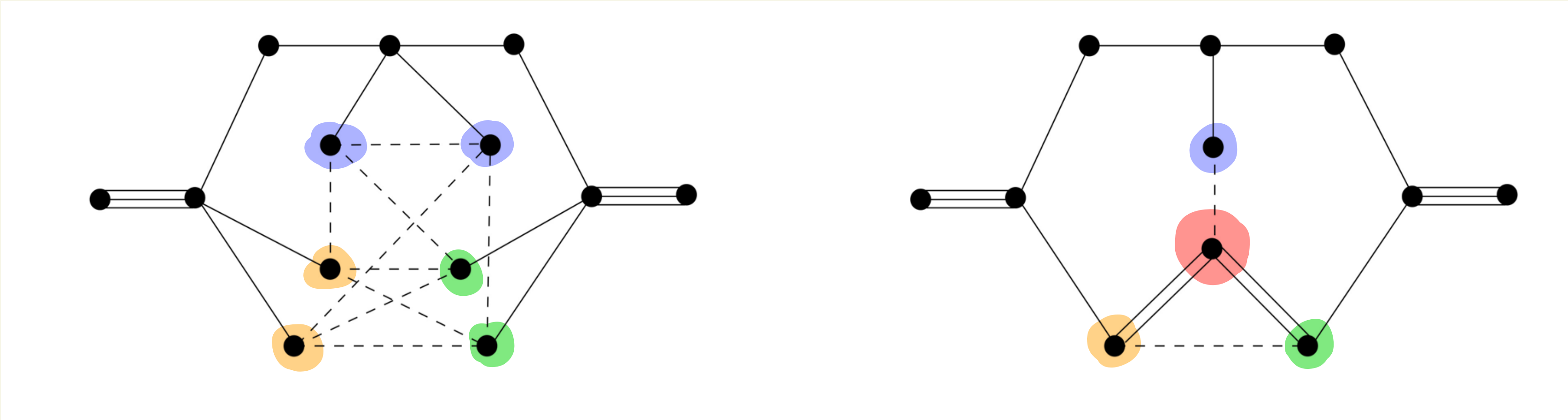
If $P \subset \mathbb{H}^d$ is compact, then $d \le 29$. [Vinberg '84]

Examples known for $d \le 8$

Unique Ex in $d=8$ [Bugaenko '92]



All known Ex in $d=7$ [Bugaenko '84]



If $P \subset \mathbb{H}^d$ is of finite volume, then $d \le 996$
[Prochorov '85, Khovanskiy '86]

Examples known for

$d \le 19$ [Vinberg, Kaplinskaya '78]

$d = 21$ [Borcherds '87]

For compact polytopes

Analogues for finite volume:

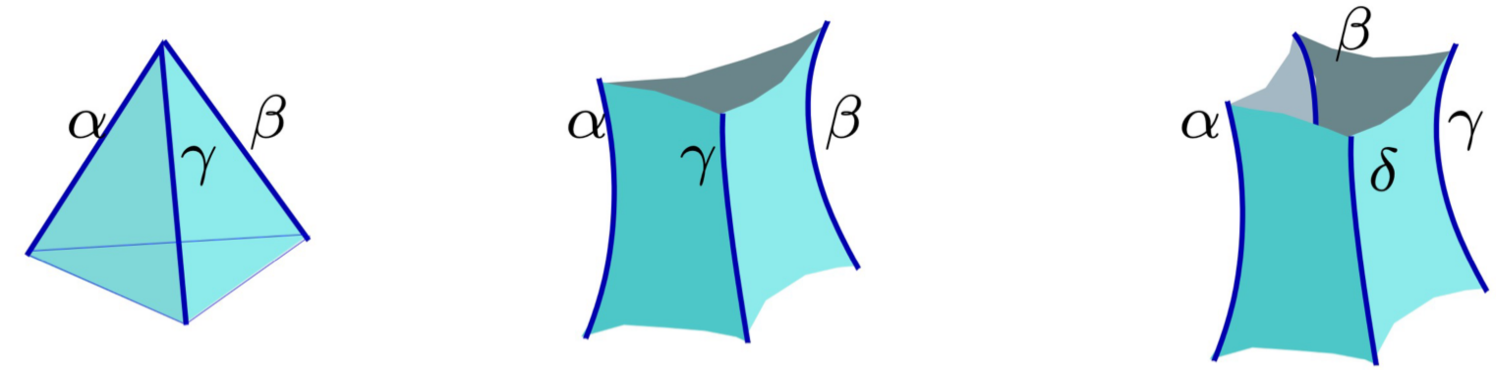
- some exist
- some need more work

$\mathbb{H}^d$ Hyperbolic Coxeter polytopes

2. Bits we know...

1. By dimension.

- dim = 2 [Poincaré '1882]: $\sum \alpha_i \leq \pi(n-2)$
- dim = 3 [Andreev '70] necessary and sufficient condition for dihedral angles



$\alpha + \beta + \gamma > \pi$ $\alpha + \beta + \gamma < \pi$ $\alpha + \beta + \gamma + \delta < 2\pi$

- dim = 4 ?

$\mathbb{H}^d$ Hyperbolic Coxeter polytopes

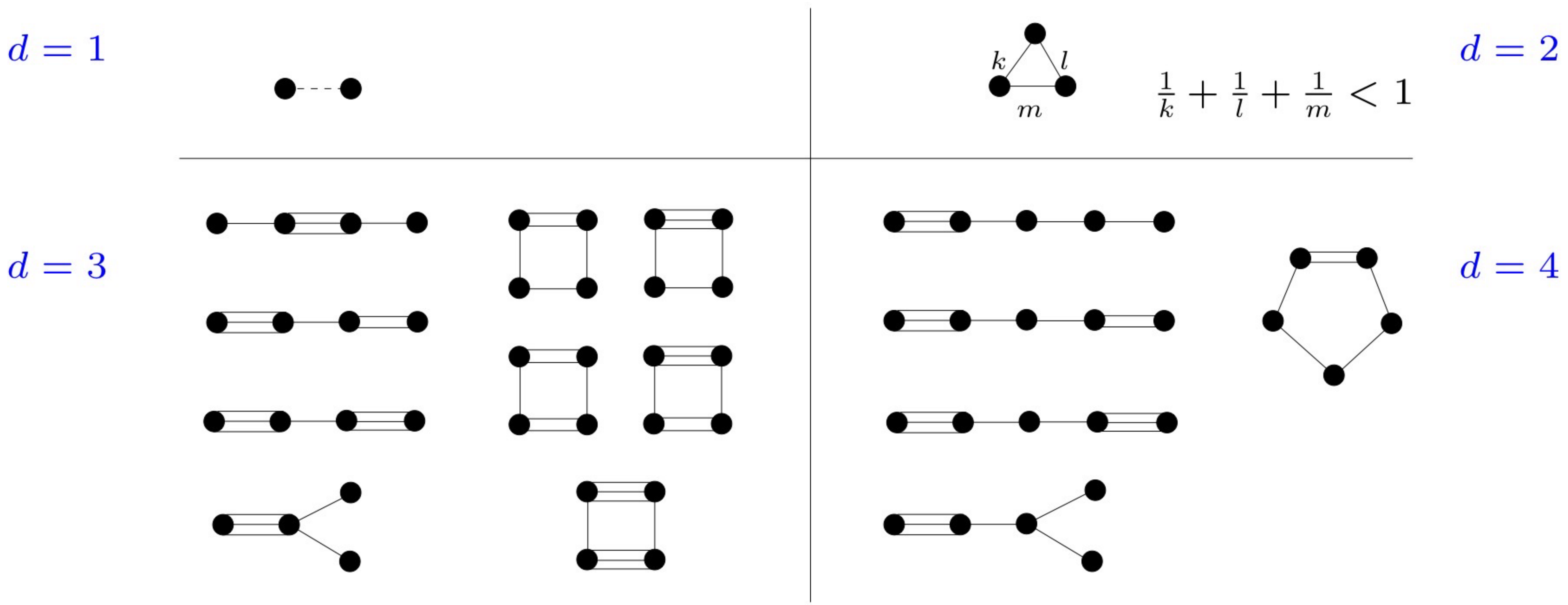
2. Bits we know...

1. By dimension.

2. By number of facets

How to do this? This are minimal non-elliptic diagrams

- $n = d + 1$, simplices [Lanner'50]: $d \leq 4$, fin. many for $d > 2$.



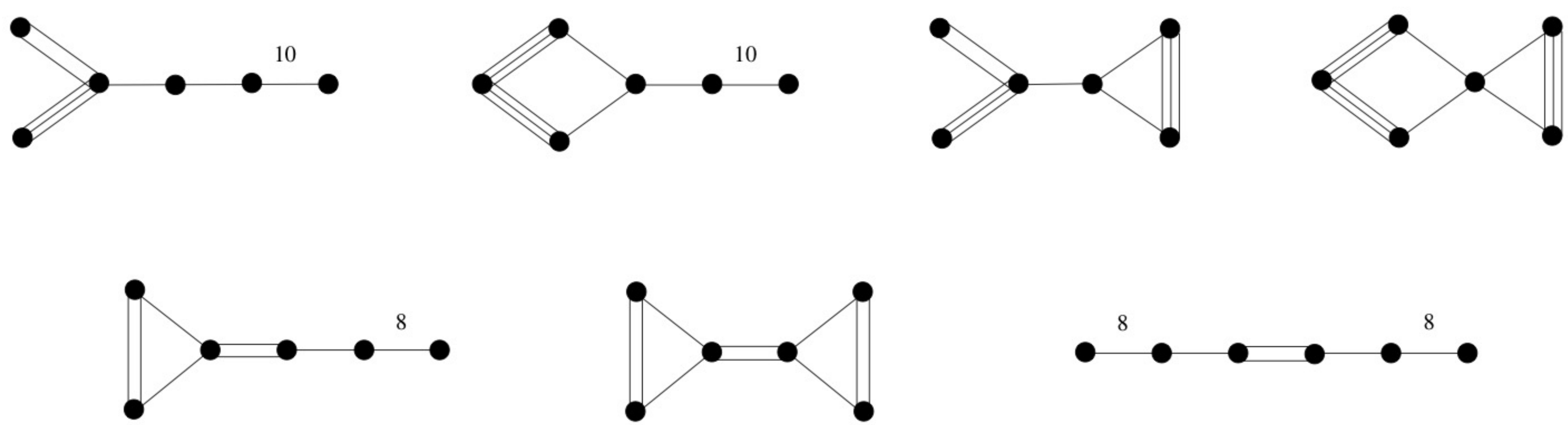
$\mathbb{H}^d$ Hyperbolic Coxeter polytopes

2. Bits we know...

1. By dimension.

2. By number of facets

- $n = d + 1$, simplices [Lanner'50]: $d \leq 4$, fin. many for $d > 2$.
- $n = d + 2$, $\Delta^k \times \Delta^l$
  - prisms [Kaplinskaya'74]: $d \leq 5$, fin. many for $d > 3$.
  - others [Esselmann'96]: $d = 4$, $\Delta^2 \times \Delta^2$, 7 items:



$\mathbb{H}^d$ Hyperbolic Coxeter polytopes

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  - prisms [Kaplinskaya'74]: $d \le 5$, fin. many for $d > 3$.
  - others [Esselmann'96]: $d = 4$, $\Delta^2 \times \Delta^2$, 7 items:
- $n = d+3$, many combinatorial types
  [Tumarkin'03]: $d \le 6$ or $d = 8$, fin. many for $d > 3$.
- $n = d+4$, really many combinatorial types...
  [F,T'05]: $d \le 7$.

  [Burcroft'22]: $d=4 \to 348$ polytopes

  [Ma, Zheng'22] $d=5 \to 51$ polytopes

  $d=6 \to ???$

  $d=7 \to$! (Bugaenko)
- $n = d+5$, [F,T' "06"]: $d \le 8$.

$\mathbb{H}^d$ Hyperbolic Coxeter polytopes

2. Bits we know...

1. By dimension.

2. By number of facets

	1	2	3	4	5	6	7	8	… 29
dim+1	+	+	+	+	—	—	—	—	—
dim+2		+	+	+	+	—	—	—	—
dim+3		+	+	+	+	+	–	+	—
dim+4		+	+	+	+	+	+	—	—
dim+5		+	+	+	+	+			—

dim

? ? ? ? ? ? ? ? ? ? ? ? ...

	1	2	3	4	5	6	7	8
d+1	+	+	f	f	—	—	—	—
d+2		+	+	f	f	—	—	—
d+3		+	+	f	f	f	–	!
d+4		+	+	+	f	f	!	–
d+5		+	+	+	+	+		

1) proofs are similar
2) use previous cases

???

$\mathbb{H}^d$ Hyperbolic Coxeter polytopes

2. Bits we know...

1. By dimension.
2. By number of facets
3. By largest denominator

- Right-angled polytopes [Potyagailo, Vinberg' 05]:
  $d \leq 4$, examples for $d = 2, 3, 4$.
- (Some) polytopes with angles $\pi/2$ and $\pi/3$ [Prokhorov' 88].

$\mathbb{H}^d$ Hyperbolic Coxeter polytopes

2. Bits we know...

1. By dimension.
2. By number of facets
3. By largest denominator
4. By number of dotted edges

Hope: • everything can be glued from them

"Evidence": all but one known "building blocks" satisfy $p \leq n-d-2$

Potential counter-evidence: Bugaenko's 6-polytope with 34 facets

- $p = 0$, [F,T'06]: Simplices and Esselmann's polytopes only.
  $d \leq 4$, $n \leq d+2$.
- $p = 1$, [F,T'07]: Only polytopes with $n \leq d+3$.
  $d \leq 6$ and $d = 8$.
- $p \leq n-d-2$, [F,T,'07]: finitely many polytopes. Algorithm.
  - Implemented the algorithm for $d = 4$:
    nothing new.

$\mathbb{H}^d$ Hyperbolic Coxeter polytopes

2. Bits we know...

1. By dimension.
2. By number of facets
3. By largest denominator
4. By number of dotted edges
5. By combinatorial types
   - Pyramids [Tumarkin '02, 04; Mcleod '13]
   - Cubes [Jacquemet '17; Jacquemet, Tschantz '18]
     (exist up to dim=5)
   - Products of simplices [Alexandrov '22]
     (is either a cube or a product of at most 3 simplices)

- Reading combinatorics from Coxeter diagram

diagrams of spherical simplices

faces of P	⟷	elliptic subdiagrams in the Coxeter diagram

diagrams of Euclidean polytopes

ideal vertices of P	⟷	parabolic subdiagrams

diagrams of hyperbolic simplices

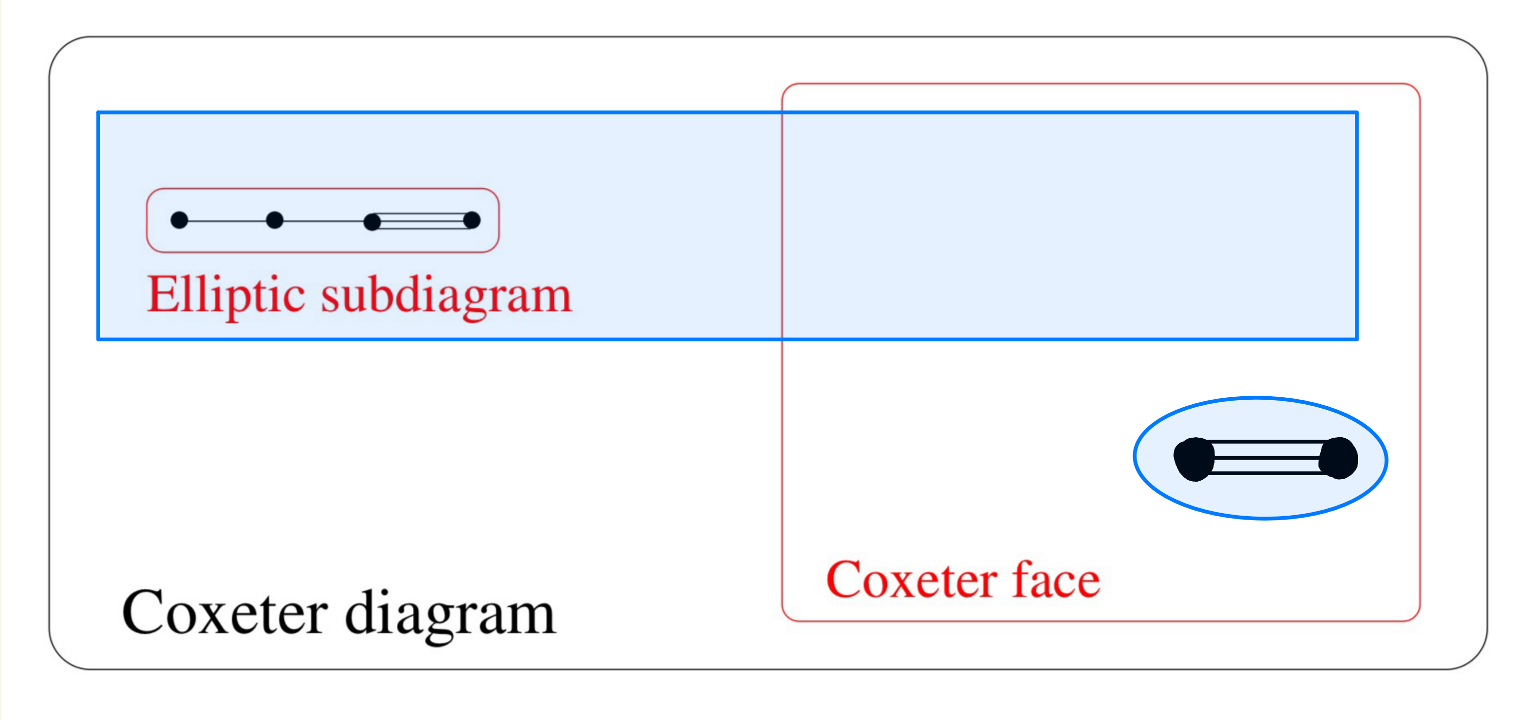
"missing faces" minimal sets, not giving a face	⟷	Lanner subdiagrams (or quasi-Lanner)

- Reading combinatorics from Coxeter diagram
- Coxeter faces

  - [Borcherds'98]: Elliptic subdiagram without A_n and D_5 $\rightarrow$ Coxeter face
  - [Allcock'05]: Angles of this face are easy to find.

- "Upside down" techique:

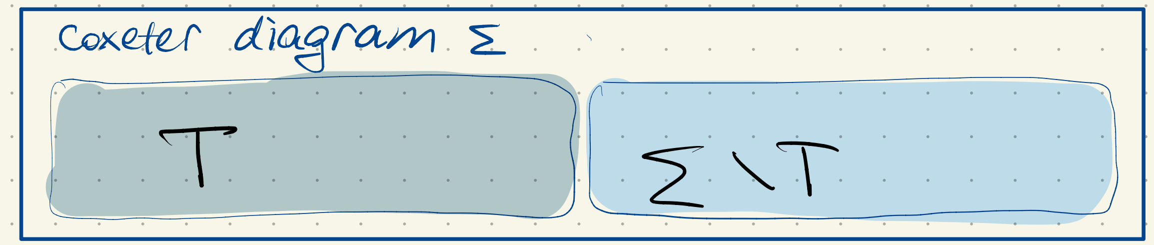
  [reducing dim]



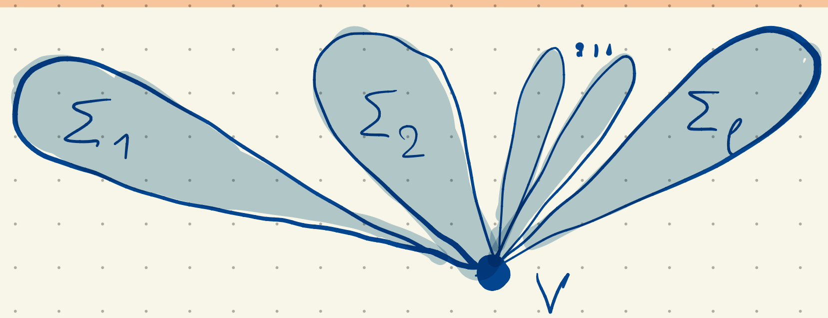
$\mathbb{H}^d$ Hyperbolic Coxeter polytopes

3. Some tools

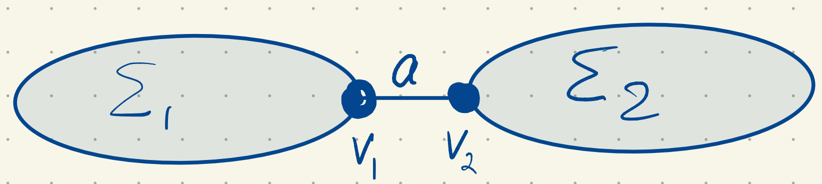
- Reading combinatorics from Coxeter diagram
- Coxeter faces
- Local determinants,
  [Vinberg '1985]



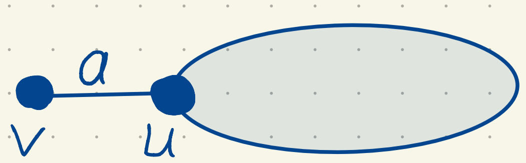
$$\det(\Sigma, T) = \frac{\det \Sigma}{\det(\Sigma \setminus T)}$$



$$\det(\Sigma, v) - 1 = \sum_{i=1}^{l} (\det(\Sigma_i, v) - 1).$$

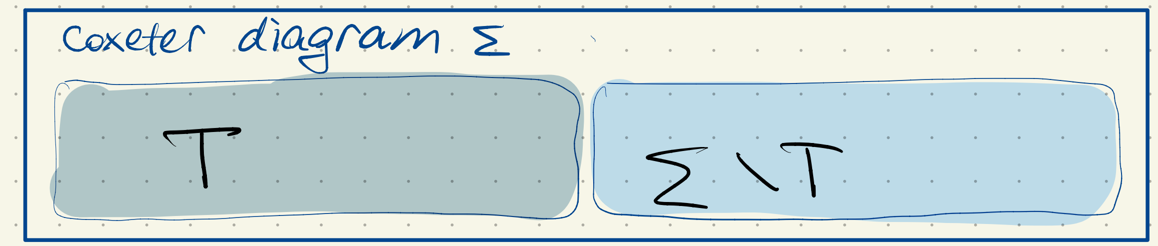


$$\det(\Sigma, \langle v_1, v_2 \rangle) = \det(\Sigma_1, v_1)\det(\Sigma_2, v_2) - a^2,$$



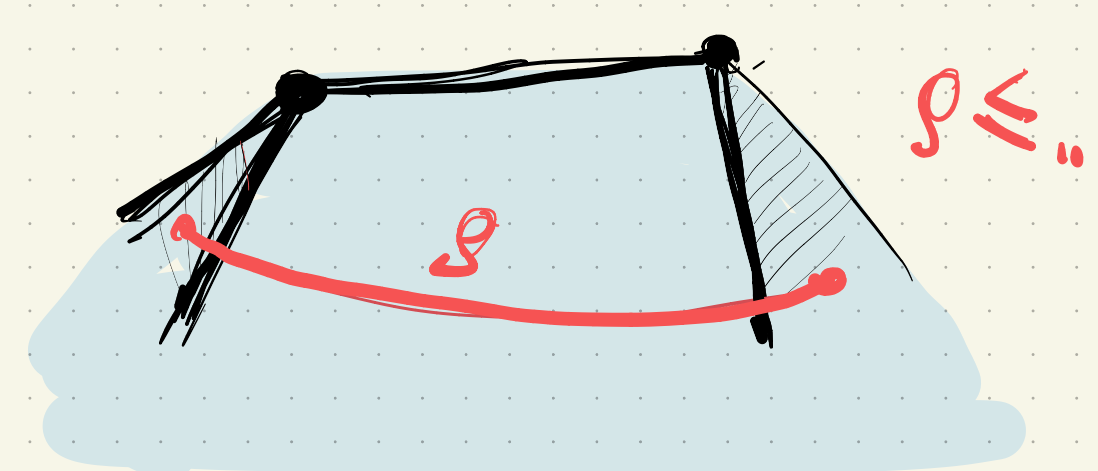
$$\det(\Sigma, v) = 1 - \frac{a^2}{\det(\Sigma \setminus v, u)},$$

- Reading combinatorics from Coxeter diagram
- Coxeter faces



$$\det(\Sigma, T) = \frac{\det \Sigma}{\det(\Sigma \setminus T)}$$

- Local determinants, [Vinberg '1985]
- Geometric constrains on ridges [Bogachev '22]



New tools needed!

$\mathbb{H}^d$ Hyperbolic Coxeter polytopes

4. More recent

Naomi Bredon:

(Still, with the same tools)

$d \leq 996$

Prop [2025] If $P \subset \mathbb{H}^d$, finite volume, then for any dihedral angle π/m of P

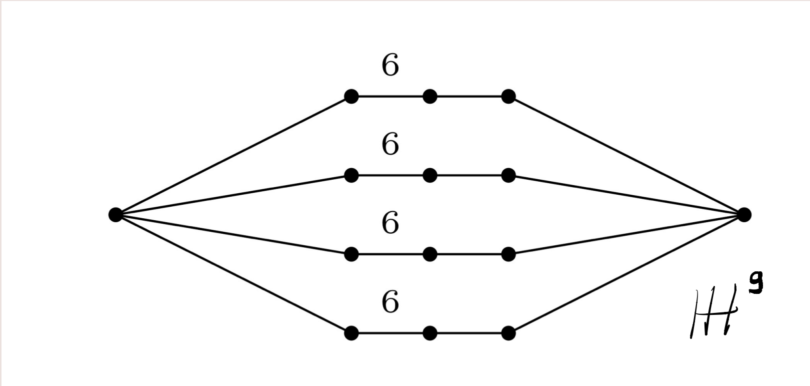
- if $d \geq 32$, then $m \leq 6$
- if $d \geq 7$, and P has mutually intersecting faces then $m \leq 6$
- if $d \geq 4$ and P is ideal, then $m \leq 5$.

Naomi Bredon; PhD 2024

Thm [2025] If $P \subset \mathbb{H}^d$ has mutually intersecting faces and $m_{ij} \leq 6$, Then P is one of 24 polytopes, and $d \leq 11$

It is one of: simplex
truncated simplex
pyramid

or



www.maths.dur.ac.uk/users/anna.felikson/Polytopes/polytopes.html 67%

Hyperbolic Coxeter polytopes

- **Disclaimer:**
  - This is an attempt to collect some results concerning classification and properties of hyperbolic Coxeter polytopes.
  - This page is under construction. Any corrections, suggestions or other comments are very welcome.
- **Arithmetic groups:**
  - For a detailed discussion of advances in the arithmetic case see the recent survey by M. Belolipetsky [Bel].
  - See also the webpage on arithmetic hyperbolic reflection groups by Nikolay Bogachev.
- **Why "hyperbolic":** spherical and Euclidean Coxeter polytopes are classified by H.S.M.Coxeter in 1934 [Cox].

Basic definitions (see [Vin1], [Vin3], [Vin6], [Vin7]):

- Definitions of Coxeter polytope, Gram matrix, Coxeter diagram.
- Faces of Coxeter polytopes.
- Existence and Uniqueness of a polytope with given Gram matrix.

Absence in large dimensions:

- Compact hyperbolic Coxeter polytopes:
  - do not exist in dimensions dim>29 [Vin2];
  - examples are known only up to dim=8, the unique known example in dim=8 and both known examples in dim=7 are due to Bugaenko [Bug1].
- Finite volume hyperbolic Coxeter polytopes:
  - do not exist in dimensions dim>995 [Pr1], [Khov];
  - examples are known in dimensions dim≤19 [Vin4], [KV] and dim=21 [Bor].

Some known classifications:

By dimension (dim):

- dim=2: there exists a n-gon with given angles if and only if the sum of angles is less than π(n-2) [Po].
- dim=3: see Andreev's theorem [And1], [And2], [RHD]. See also [Pog].

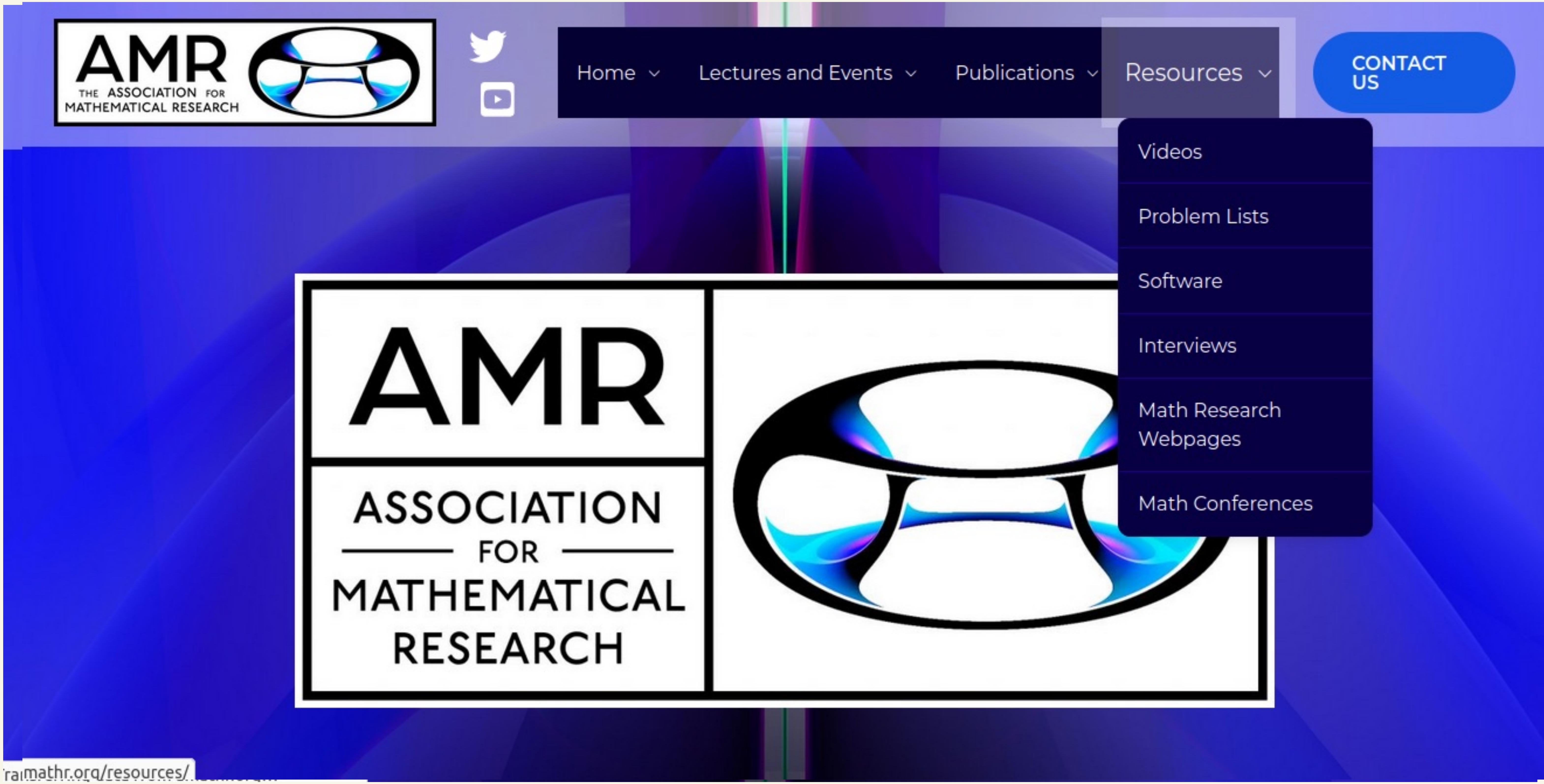
- dim>31: non-zero dihedral angles are of the form π/m with m<=6 [Br].

By number of facets (n):

- n=dim+1: compact simplices (Lannér diagrams [Lan], dim=2,3,4) and non-compact simplices (quasi-Lannér diagrams [Ch], [Ko], [Ch],[Vin7], [Bou], dim=2,...,9).
- n=dim+2:
  - Products of two simplices:
    - Simplicial prisms exist in dim=3,4,5 [Kap], see also [Vin3].
    - Other products of two simplices (exist in dim=4 only): Esselmann polytopes [Ess1],[Ess2] and the unique non-compact polytope [Tum1].
  - Pyramids over a product of two simplices [Tum1], dim=3,...,13, 17.

More info, tables, links on the webpage!

4. One recent result



4. One recent result

AMR THE ASSOCIATION FOR MATHEMATICAL RESEARCH

Home Lectures and Events Publications Resources Contact

Webpages for Research in Mathematics

General Mathematics

- Conference Listings/Mathematics
- Research seminars
- MathMeetings.net, list for research mathematics conferences, workshops, summer schools
- AMS Mathematics Calendar
- The On-Line Encyclopedia of Integer Sequences® (OEIS®)
- Cut the Knot Interactive Mathematic Miscellany and Puzzles, by Alexander Bogomolny
- Online Mathematics Textbooks list by George Cain
- Research Guides, by University of Michgan Library
- The Erdös Number Project, by Jerry Grossman
- Math Stack Exchange
- MathOverflow
- Mathematics Genealogy Project
- MathWorld from Wolfram
- DLMFNIST Digital Library of Mathematical Functions NIST Digital Library of Mathematical Functions
- Mathematics and Areas of Mathematics on Wikipedia

Digital and digitalised publications

- arXiv, Mathematics
- ICM Proceedings 1893-2018 and ICM Videos 1998-2014, maintained by IMU
- EuDML, European Digital Math Library
- World Digital Mathematics Library (WDML), by IMU
- AMS Digital Mathematics Registry
- MathSciNet, AMS Mathematical REviews
- Numdum, the French digital mathematics library, set up by Cellule MathDoc
- DML: Digital Mathematics Library, by Ulf Rehmann (Uni Bielefeld)

Logic

- Directory of links on logic and foundations of mathematics by Sylvain Poirier

Machine learning

- Machine Learning Mastery by Jason Brownlee
- Analytics Vidhya Data Science community
- Network Flows Bibliographies by Joseph Malkevitch

Mathematical Biology

- Mathematical Biology Bibliographies by Joseph Malkevitch

Mathematical Physics

- String Theory Wiki
- Solitons at Work, informal network
- International Society of Nonlinear Mathematical Physics, non-profit Learned Society

Metric Geometry

- Hyperbolic Coxeter Polytopes by Anna Felikson
- Arithmetic Hyperbolic Reflection Groups by Nikolay V. Bogachev
- Geometry by Anna Felikson
- Ptolemy Relation and Friends by Anna Felikson

Number Theory

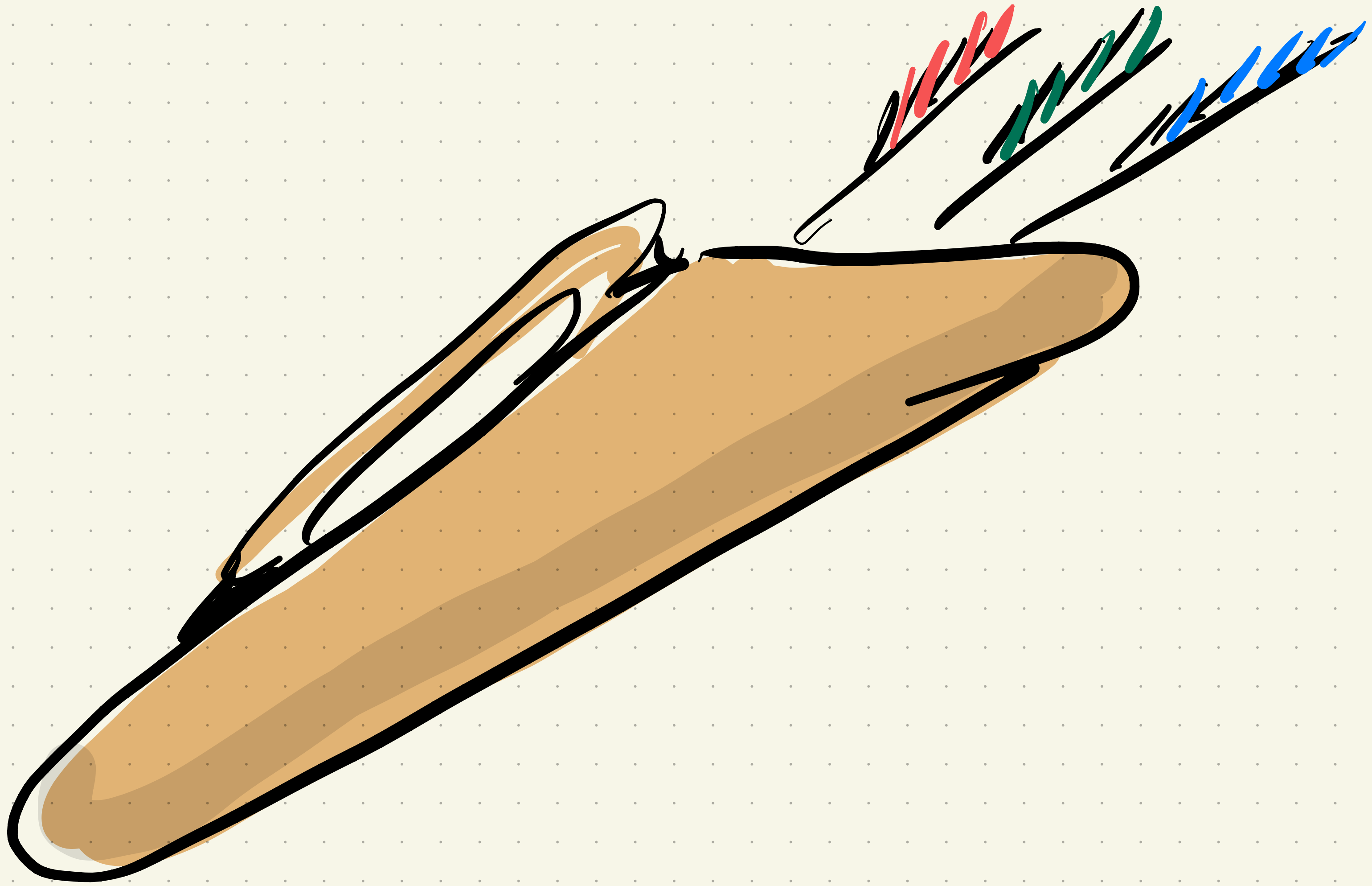
- Number Theory Web by Keith Matthews
- The L-functions and modular forms database (LMFDB)
- Markov numbers by Anna Felikson
- Continued fractions by Anna Felikson
- Conferences in arithmetic geometry by Kiran Kedlaya
- Conferences in number theory by Johann Birnick
- Motivic stuff Cohomology, homotopy theory, and arithmetic geometry, by Andreas Holmstrom

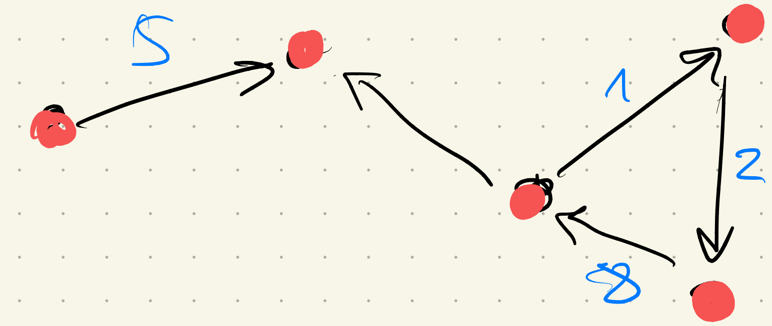
Numerical Analysis

- Numerical Analysis Digest, a weekly newsletter

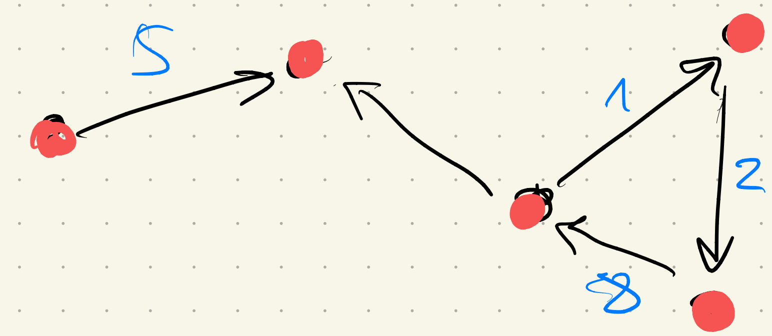
Operator Algebras

II Quivers of finite mutation type



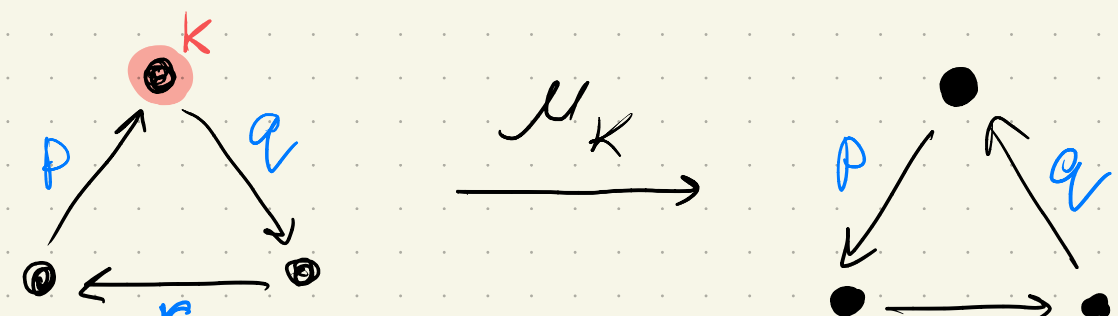


II Quivers of finite mutation type



- **Quiver** = oriented graph
  finitely many vertices,

- **Quiver mutation**: μ_k
  - reverse all arrows incident to vertex k
  -



$p, q > 0$ $\qquad$ $r' = pq - r$

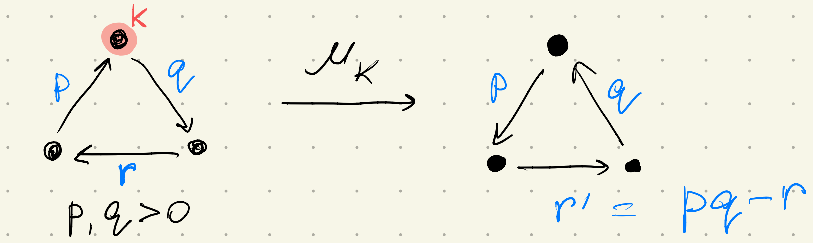
[Fomin, Zelevinsky '2000]

Quiver Q is of **finite mutation type** if there are finitely many quivers obtained from Q by iterated mutations.

Ex: $\bullet \xrightarrow{p} \bullet$

How to classify?

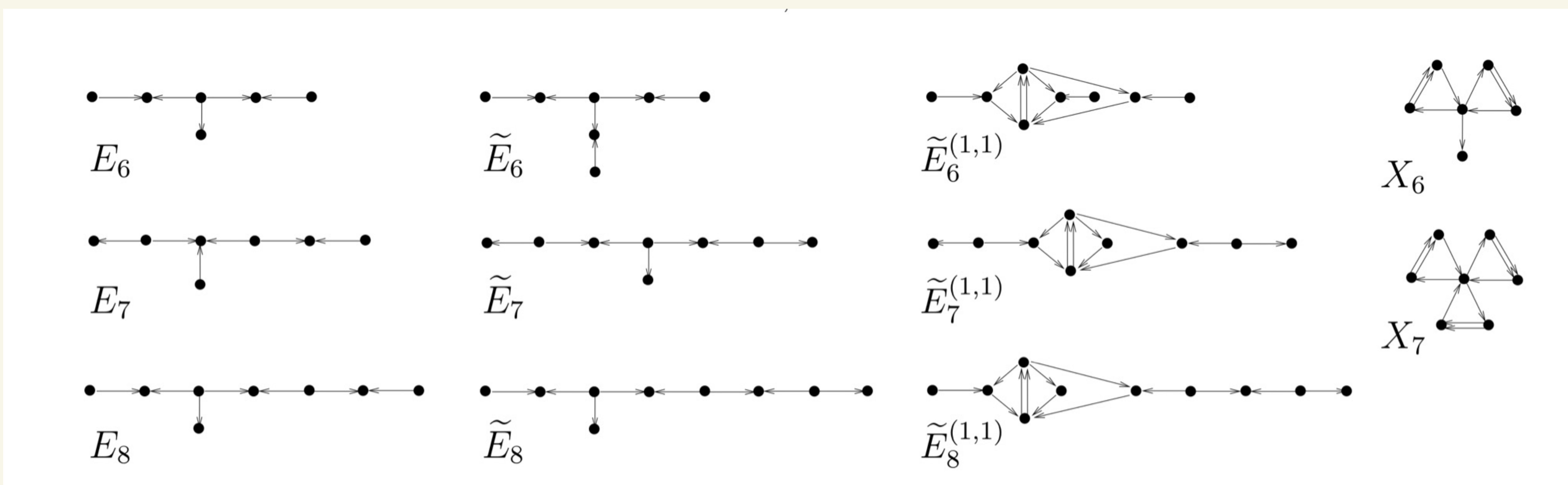
II Quivers of finite mutation type



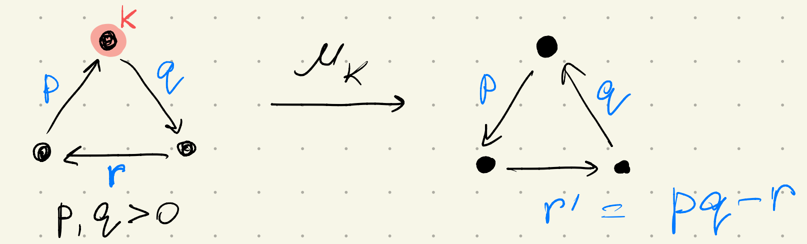
Thm [F, Shapiro, Tumarkin '08]

Let Q be a quiver of finite mutation type.
Then either Q has 2 vertices,
or Q comes from triangulated surfaces
or Q is one of the following 11 quivers:

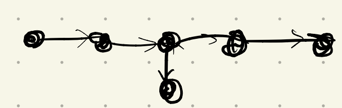
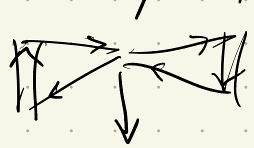
Infinitely many quivers
built from 5 blocks
associated
to small surfaces



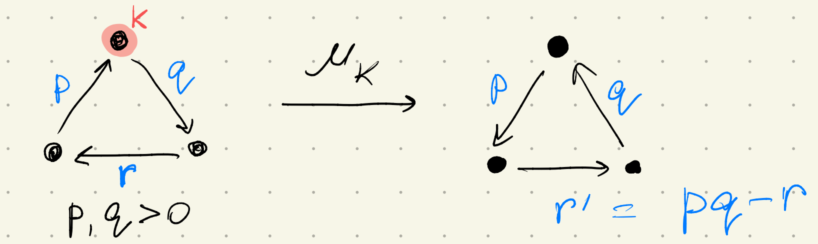
II Quivers of finite mutation type



Compare:

Coxeter diagram [signature (n,1)]	Quiver [mut. finite]
? polytopes to build ∞ly many examples	5 blocks to build ∞ly many examples
Lanner subdiagrams [minimal non-elliptic]	Minimal subquivers not decomposable into blocks
Upside-down technique with Coxeter faces	Upside-down technique to list min. non-decomp quivers (E_6  ; X_6 )
computer search for adding vertices one by one	computer search for adding vertices one by one

II Quivers of finite mutation type



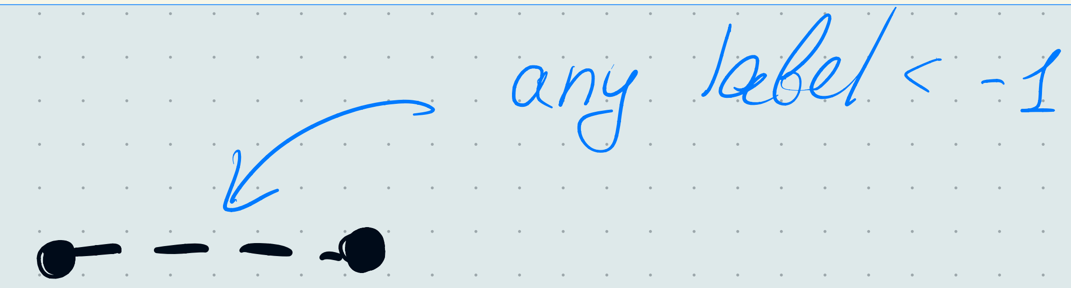
Classification of quivers of fin mut. type

worked

classification of hyperbolic Coxeter polytopes

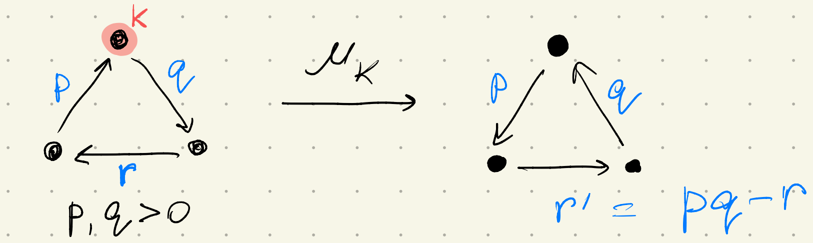
did not

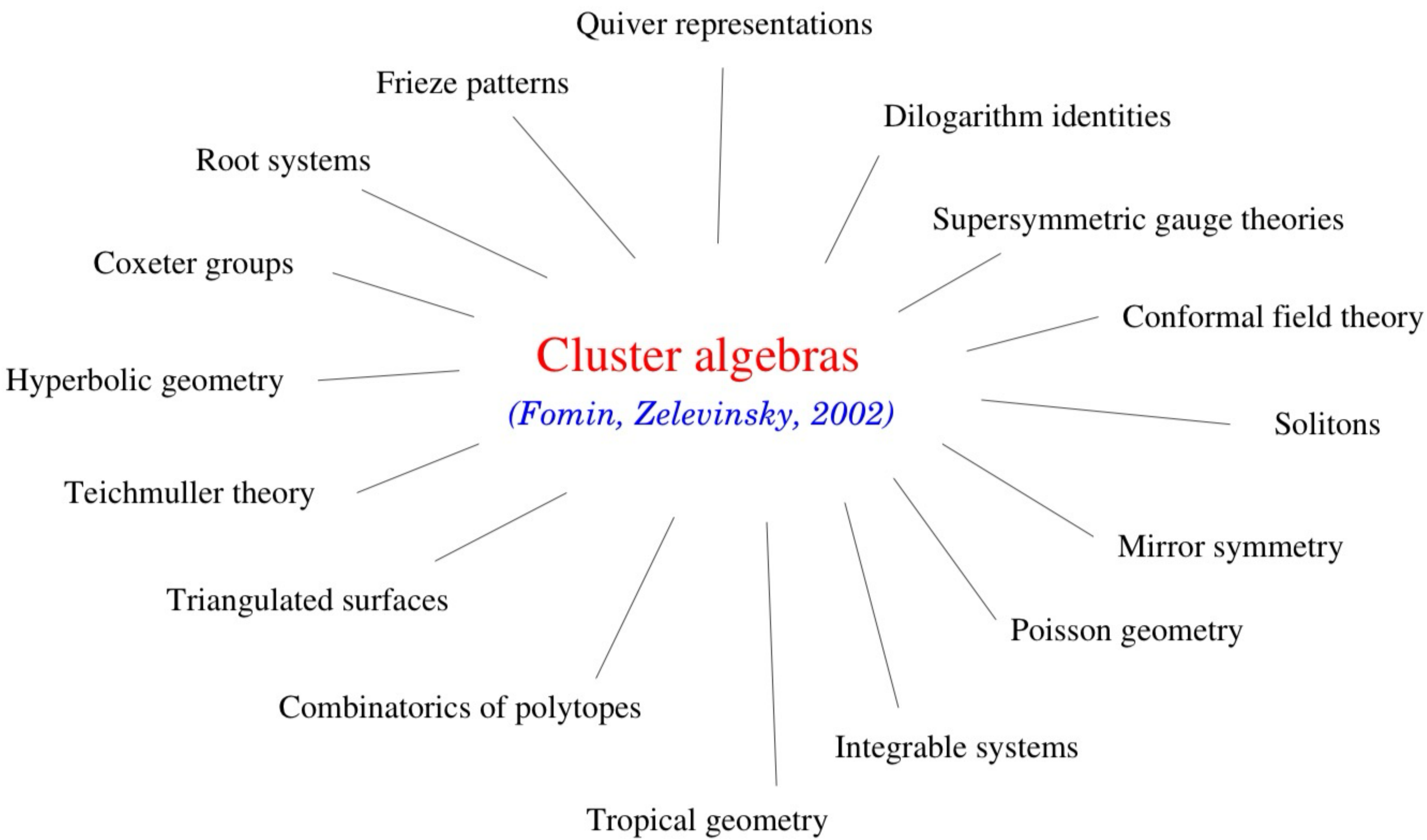
Why?

- Integer labels only on arrows ⟷ 

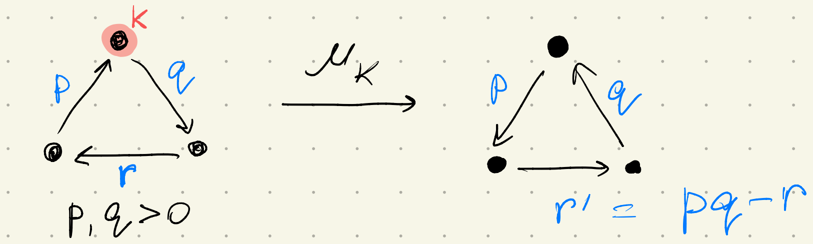
- Small number of blocks from surfaces (classified) ⟷ Many basic polytopes to glue from (not classified)

II Quivers of finite mutation type





II Quivers of finite mutation type



Cluster Algebras Portal

By Sergey Fomin

This is a collection of links on **cluster algebras** and related topics.

Papers

- arXiv
- arXiv full text search (math)
- Google Scholar
- MathSciNet (requires subscription)
- zbMATH
- Book chapters, lecture notes, surveys, and expository articles

Conferences, summer schools, and lecture series

- ▸ **2003-2010** (49 events)
- ▸ **2011-2020** (89 events)
- ▸ **2021 and beyond**

▸ **Seminars and working groups**

▸ **Courses**

▸ **Software and data**

▸ **Thematic programs and research labs**

▸ **Publicity and awards**

▸ **Andrei Zelevinsky, 1953-2013**

▸ **Other**

- IACR International Association for Cryptologic Research
- SIAM Society for Industrial and Applied Mathematicsociety for Industrial and Applie

Category Theory

- Logic Matters by Peter Smith
- nLab collaborative project
- The n-Category Cafe A group blog on math, physics and philosophy [Category The

Classical Analysis and ODEs

Combinatorics

- Catalan Numbers by Igor Pak
- Cluster Algebra Portal by Sergey Fomin
- Associahedra and Noncrossing Partitions by Jon McCammond
- Frieze patterns by Anna Felikson
- Combinatorics of polytopes by Anna Felikson
- Squaring the square, by Stuart Anderson
- Encyclopedia of Combinatorial Polytope Sequences..., by Stefan Forcey
- Graph Theory and Combinatorics Bibliographies by Joseph Malkevitch
- Links to Combinatorial Conferences, by Douglas B. West

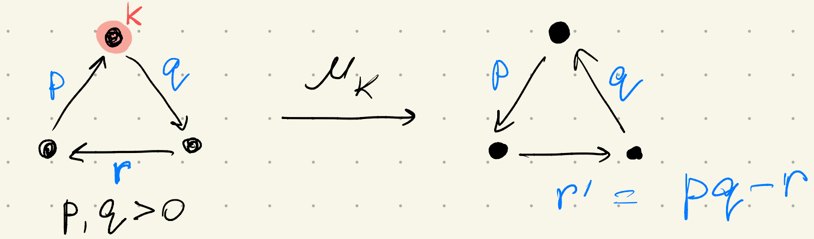
Commutative Algebra

- commalg.org, the website for the commutative algebra community

Computational Geometry

- Computational Geometry Pages, by by Jeff Erickson
- Geometry in Action, by David Eppstein

III Hyperbolic manifolds from quiver mutations



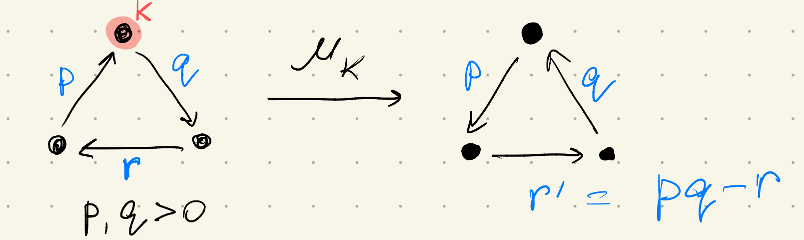
Idea

- Take a quiver Q
- Forget directions of arrows
- Get a Coxeter diagram of a Coxeter group G

* mutate Q – what happens to G?

[It changes, but one can fix this by taking a quotient]

III Hyperbolic manifolds from quiver mutations



Construction [Barot-Marsh '2011]

- Let Q be a quiver mutation-equivalent to A_n, D_n or $E_{6,7,8}$
- quiver $Q \to$ group $G(Q)$:

- Generators of G – nodes of Q.
- Relations of G – (R1) $s_i^2 = e$

  (R2) $(s_i s_j)^{m_{ij}} = e$,

$$m_{ij} = \begin{cases} 2, & \bullet \quad \bullet \\ 3, & \bullet\!-\!\!-\!\!-\!\bullet \\ \infty, & otherwise. \end{cases}$$

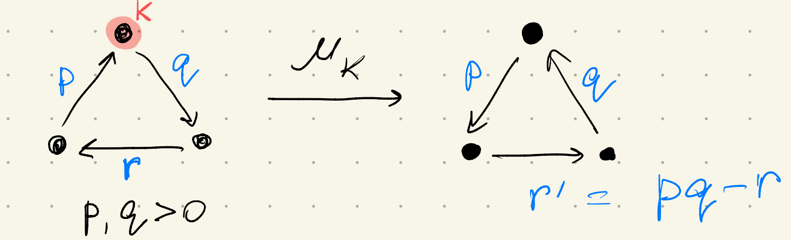
  (R3) Cycle relation:

  for each chordless cycle $1 \to 2 \to \cdots \to n \to 1$

  $(s_1 \ s_2 s_3 \ldots s_n \ldots s_3 s_2)^2 = e.$

- If $Q = A_n, D_n$ or E_n then G is the corresponding Coxeter group (as Q has no oriented cycles).

III Hyperbolic manifolds from quiver mutations



Construction [Barot-Marsh '2011]

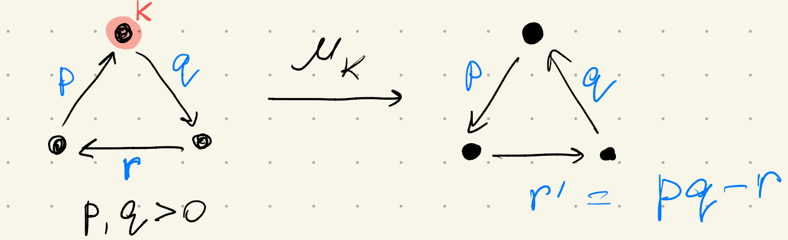
- Let Q be a quiver mutation-equivalent to A_n, D_n or $E_{6,7,8}$

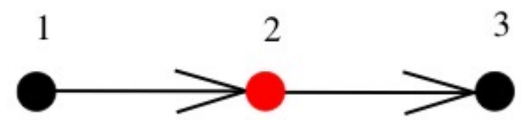
Theorem. [Barot-Marsh '2011] Given a quiver Q of finite type, $G(Q)$ is invariant under mutations of Q, i.e. $G(Q) = G(\mu_k(Q))$.

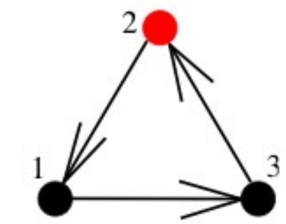
- In particular, $G(Q)$ is a finite Coxeter group.
- If $Q_2 = \mu_k(Q_1)$, s_i - generators of $G(Q_1)$, t_i generators of $G(Q_2)$, then

$$t_i = \begin{cases} s_k s_i s_k, & i \longrightarrow k \text{ in } Q_1 \\ s_i, & \text{otherwise} \end{cases}$$

III Hyperbolic manifolds from quiver mutations

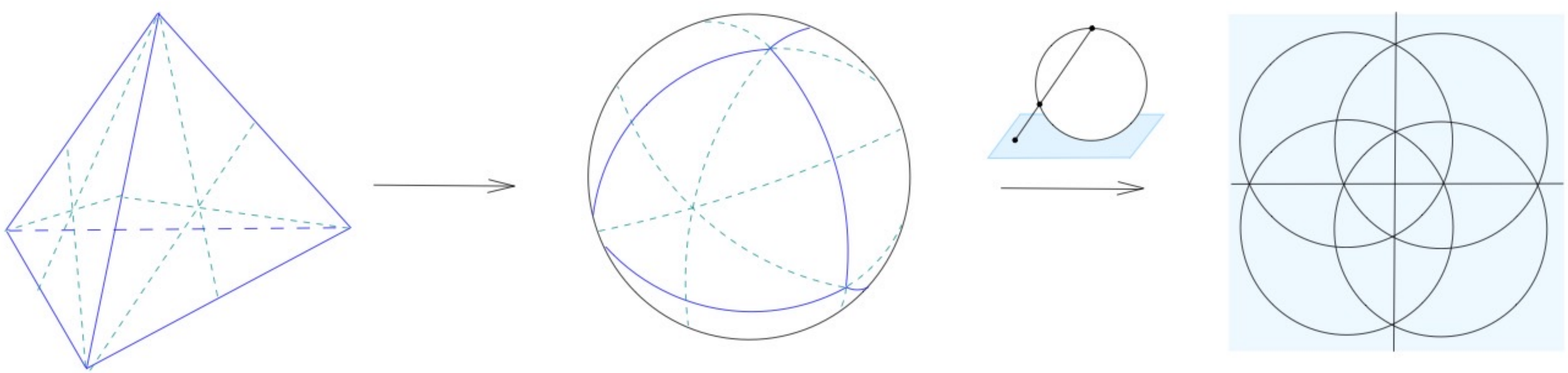


Example: $Q_1 = A_3 =$ 

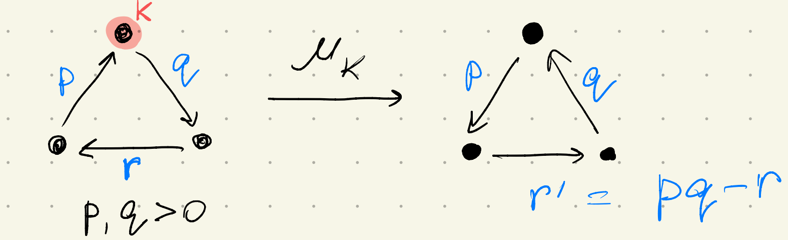
$\xrightarrow{\mu_2}$ $Q_2 =$ 

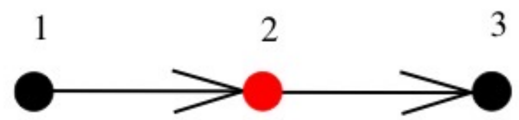
$$G(Q_1) = \langle s_1, s_2, s_3 \mid s_i^2 = (s_1s_2)^3 = (s_2s_3)^3 = (s_1s_2)^2 = e\rangle$$

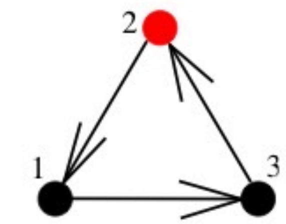
finite Coxeter group A_3, acts on S^2 by reflections, 24 elements:



III Hyperbolic manifolds from quiver mutations



Example: $Q_1 = A_3 =$ 

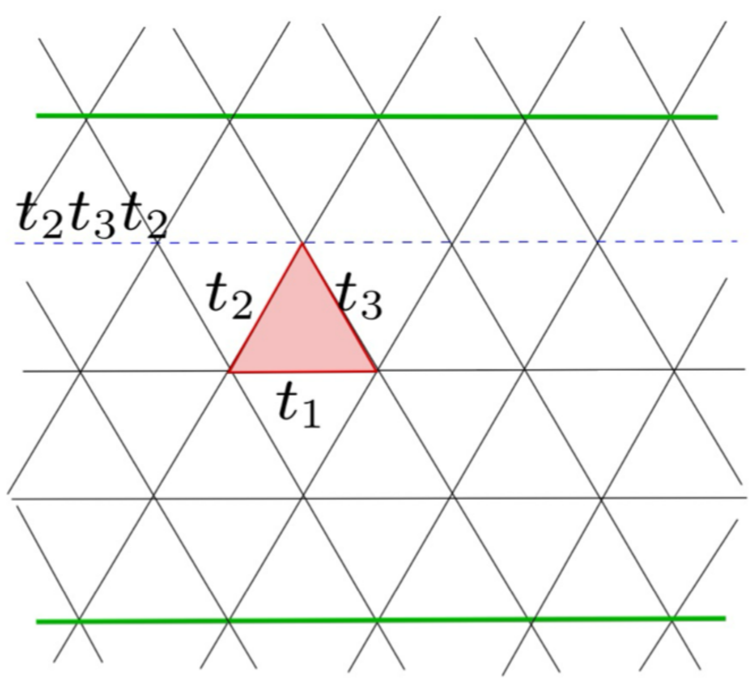
$\xrightarrow{\mu_2}$ $Q_2 =$ 

$$G(Q_1) = \langle s_1, s_2, s_3 \mid s_i^2 = (s_1 s_2)^3 = (s_2 s_3)^3 = (s_1 s_2)^2 = e \rangle$$

finite Coxeter group A_3, acts on S^2 by reflections, 24 elements:

$$G(Q_2) = \langle \underbrace{t_1, t_2, t_3 \mid t_i^2 = (t_i t_j)^3}_{} = (t_1 \; t_2 t_3 t_2)^2 = e \rangle$$

G_0 – affine Coxeter group $\widetilde{A}_2$, acts on $\mathbb{E}^2$ by reflections.

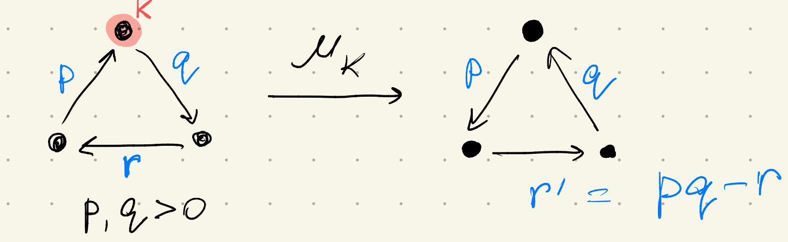


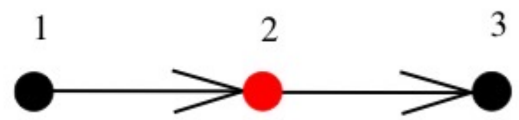
$(t_1 \; t_2 t_3 t_2)^2 = ?$

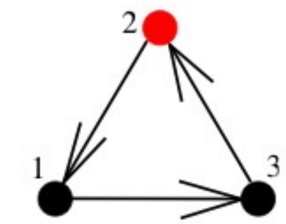
$t_1 \; t_2 t_3 t_2$ - translation by 2 levels

$(t_1 \; t_2 t_3 t_2)^2$ - translation by 4 levels

III Hyperbolic manifolds from quiver mutations



Example: $Q_1 = A_3 =$ 

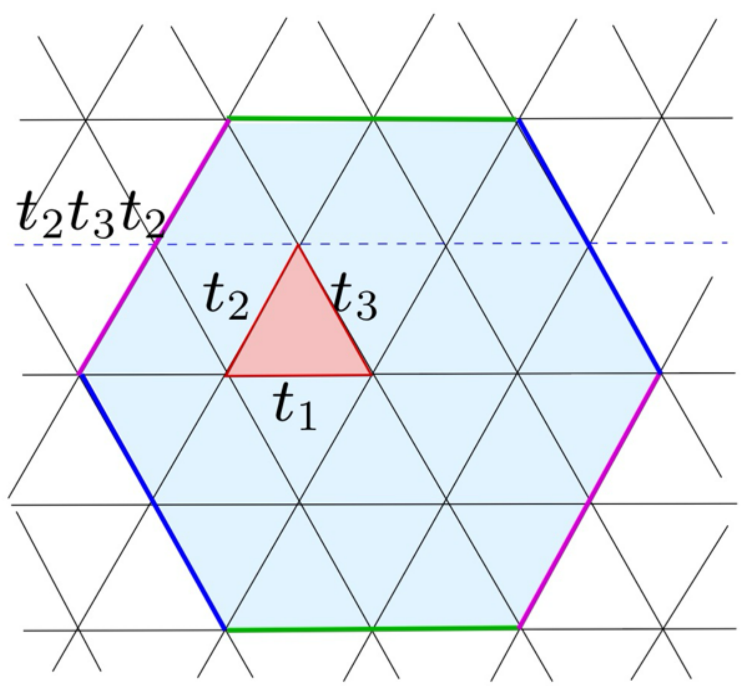
$\xrightarrow{\mu_2}$ $Q_2 =$ 

$$G(Q_1) = \langle s_1, s_2, s_3 \mid s_i^2 = (s_1 s_2)^3 = (s_2 s_3)^3 = (s_1 s_2)^2 = e \rangle$$

finite Coxeter group A_3, acts on S^2 by reflections, 24 elements:

$$G(Q_2) = \langle \underbrace{t_1, t_2, t_3 \mid t_i^2 = (t_i t_j)^3}_{} = (t_1 \; t_2 t_3 t_2)^2 = e \rangle$$

G_0 – affine Coxeter group $\widetilde{A}_2$, acts on $\mathbb{E}^2$ by reflections.

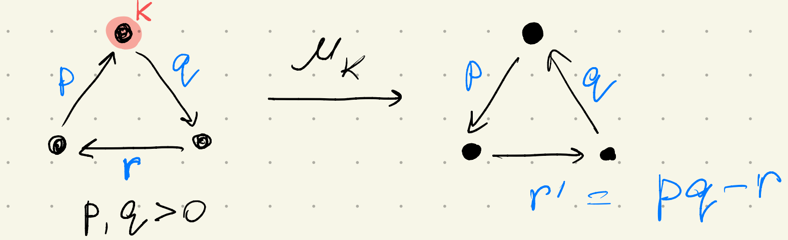


$(t_1 \; t_2 t_3 t_2)^2 = e =$ transl. by 4 levels – Identify!

$G = G_0 / NCl((t_1 \; t_2 t_3 t_2)^2)$ – Identify! Identify!

$G = G(Q_2)$ acts on a torus T^2.

III Hyperbolic manifolds from quiver mutations



Thm [F–Tumarkin '14] **(Manifold property)**
Taking the quotient does not introduce singularities
(i.e. if the Coxeter group G_0 acts on a manifolds then $G(Q)$ also does)

Corollary: can cook hyperbolic manifolds with large symmetry groups.

Example: $A_4 =$

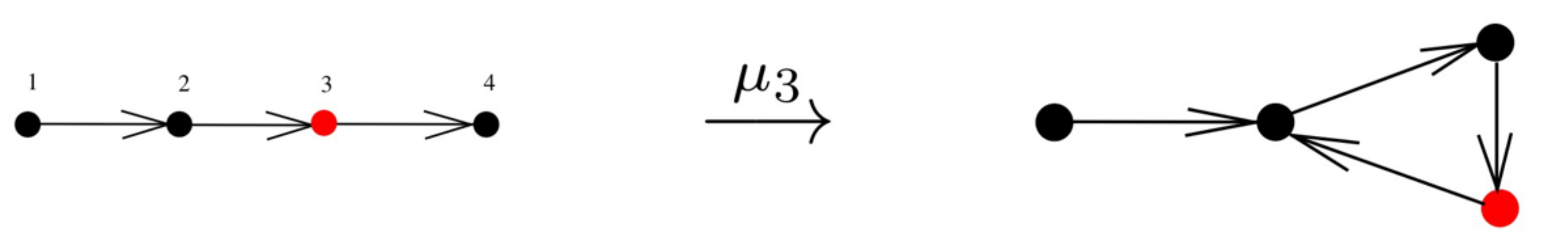


diagram of hyperbolic simplex

$\Rightarrow$ Hyperbolic 3-manifold with action of the group A_4.

III Hyperbolic manifolds from quiver mutations

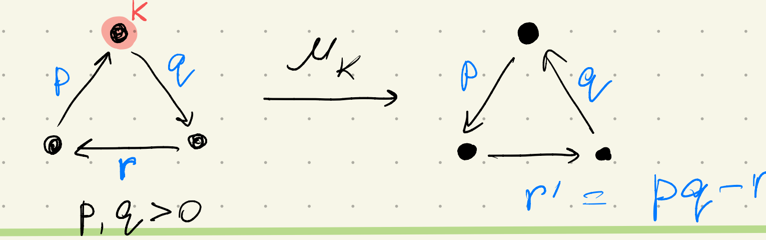
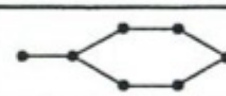
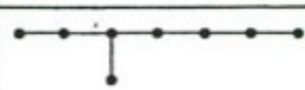
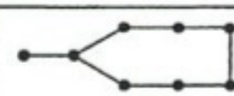
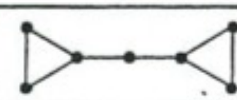
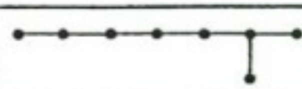


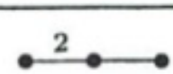
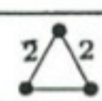
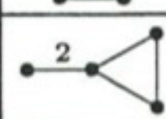
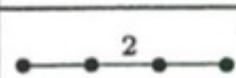
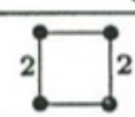
TABLE 5.1. Actions on hyperbolic manifolds.

W	Q	Q_1	$\lvert W\rvert$	$\dim X$	vol X approx.	number of cusps	$\chi(X)$
A_4			5!	3	$\lvert W\rvert \cdot 0.084578$	5	
D_4			$2^3 \cdot 4!$	3	$\lvert W\rvert \cdot 0.422892$	16	
D_5			$2^4 \cdot 5!$	4	$\lvert W\rvert \cdot 0.013707$	10	2
E_6			$2^7 \cdot 3^4 \cdot 5$	5	$\lvert W\rvert \cdot 0.002074$	27	
E_7			$2^{10} \cdot 3^4 \cdot 5 \cdot 7$	6	$\lvert W\rvert \cdot 2.962092 \times 10^{-4}$	126	-52
E_8			$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$	7	$\lvert W\rvert \cdot 4.110677 \times 10^{-5}$	2160	
A_7			8!	5		70	
D_8			$2^7 \cdot 8!$	6	$\lvert W\rvert \cdot 0.002665$	1120	-832

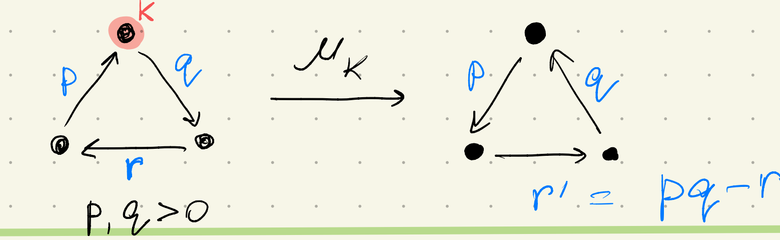
simplices (A_4 – E_8)

pyramids over a product of 2 simplices (A_7, D_8)

TABLE 7.1. Actions on hyperbolic manifolds, non-simply-laced case.

W	$\mathcal{G}$	$\mathcal{G}_1$	$\lvert W\rvert$	$\dim(X)$	vol X approx.	number of cusps	$\chi(X)$ ($\dim X$ even)
B_3			$2^3 \cdot 3!$	2	8π	compact	-4
B_4			$2^4 \cdot 4!$	3	$\lvert W\rvert \cdot 0.211446$	16	
F_4			$2^7 \cdot 3^2$	3	$\lvert W\rvert \cdot 0.222228$	compact	

III Hyperbolic manifolds from quiver mutations

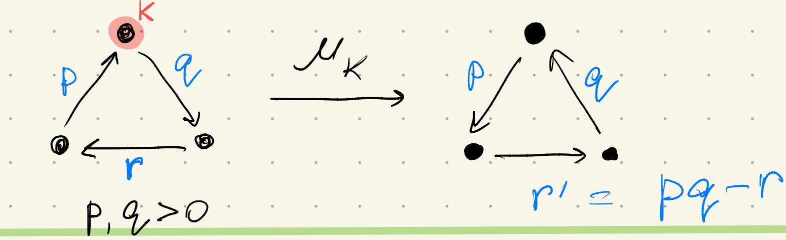


- One can build similar groups from other quivers of finite-mutation type (not only from Dynkin quivers)

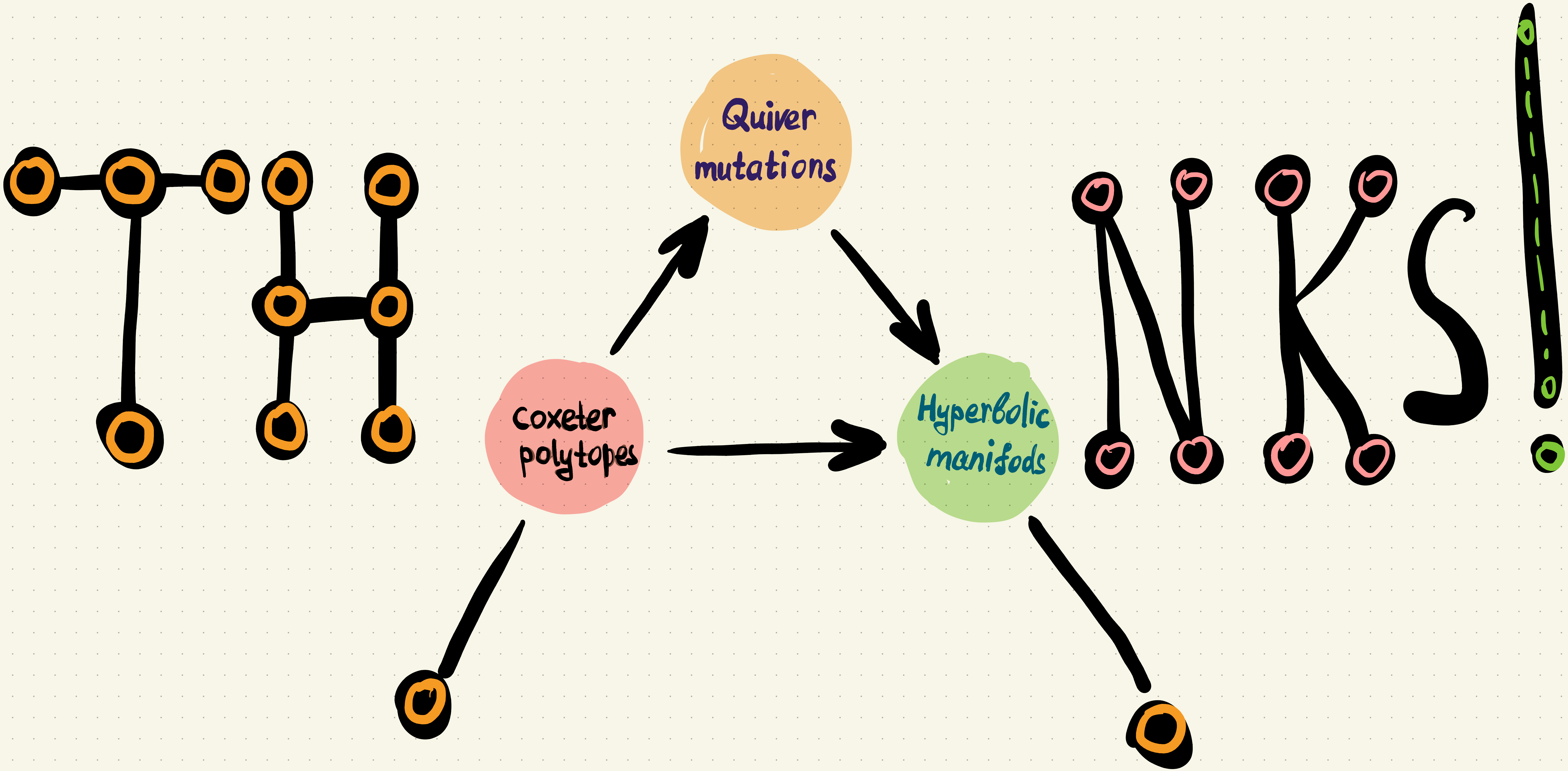
- So that the groups are mutation-invariant

- However – we don't know what groups will arise (after taking quotient)
  – we can NOT prove the Manifold Property

III Hyperbolic manifolds from quiver mutations



Is there a webpage collecting hyperbolic manifolds?


Quiver mutations
Coxeter polytopes
Hyperbolic manifods