Quiver mutations, reflection groups and curves on punctured disc

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(joint with Pavel Tumarkin)

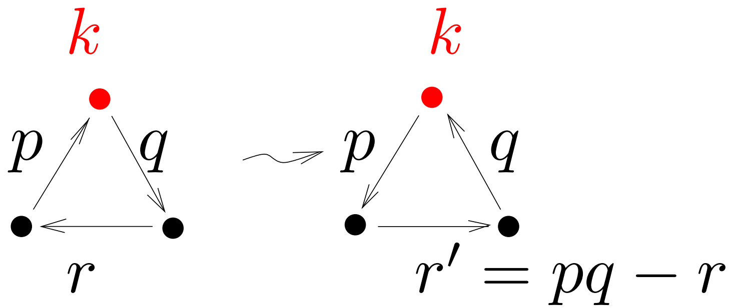
Cluster Algebras: Twenty Years On

CIRM, 19-23 March, 2018

- Quiver is a directed graph without loops and 2-cycles.

Agreement: $\bullet \xrightarrow{p} \bullet \quad = \quad \bullet \xleftarrow{-p} \bullet$

- Mutation μ_k of quivers:
  - reverse all arrows incident to k;
  - for every oriented path through k (i.e. $p, q > 0$, r - any) do



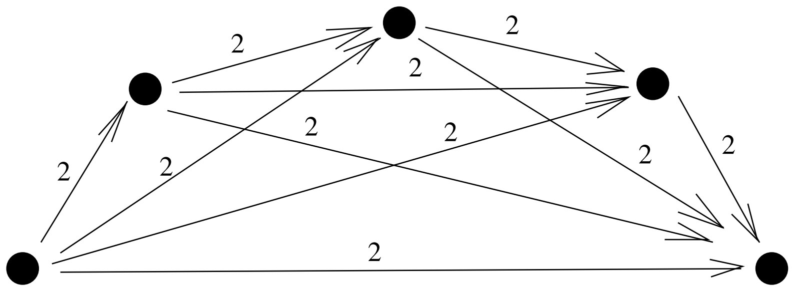
Notation: Q quiver, $\quad b_{ij}$ arrows $i \to j \quad (b_{ij} = -b_{ji})$.

$n = \#($ vertices of $Q)$.

Settings: • Q is acyclic quiver: no oriented cycles in Q

after reordering of vertices, $b_{ij} \geq 0$ for $i < j$.

• Q is 2-complete: $b_{ij} \geq 2$.



1. Acyclic mutation classes via reflection groups

- $Q = (b_{ij}) \quad \rightsquigarrow \quad M = \begin{pmatrix} 2 & & -|b_{ij}| \\ & 2 & \\ -|b_{ij}| & & 2 \end{pmatrix} = \langle v_i, v_j \rangle$

  $(v_1, \ldots, v_n)$ - basis of quadratic space V of same signature as M.

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$$r_v(u) = u - \langle u, v \rangle v.$$

- Let $G = \langle s_1, \ldots, s_n \rangle$ where $s_i = r_{v_i}$.

  G acts discretely in a cone $C \subset V$ with fundamental domain

$$F = \bigcap_{i=1}^{n} \Pi_i^-, \quad \text{where } \Pi_i^- = \{u \in V \mid \langle u, v_i \rangle < 0\}.$$

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Mutation $\rightsquigarrow$ Partial reflection

$$\mu_k(v_i) = \begin{cases} v_i - \langle v_i, v_k \rangle v_k, & \text{if } k \to i \text{ in } Q \\ -v_k, & \text{if } i = k \\ v_i, & \text{otherwise} \end{cases}$$

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new set of generators in $G = \langle s'_1, \dots, s'_n \rangle$:

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Theorem. (Barot, Geiss, Zelevinsky'06; Seven'15)
The values $\langle v_i, v_j \rangle$ change under mutations in the same way as the weights of the arrows in Q.

1. Acyclic mutation classes via reflection groups

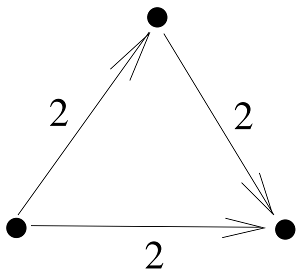
Remark: $\mathbf{c}$-vectors and Y-seeds

- If $(v_1^0, \ldots, v_n^0)$ are the initial vectors, then vectors $(v_1, \ldots, v_n)$ (written in the basis $(v_1^0, \ldots, v_n^0)$) are $\mathbf{c}$-vectors.

- The collection $(v_1, \ldots, v_n)$ is a Y-seed.

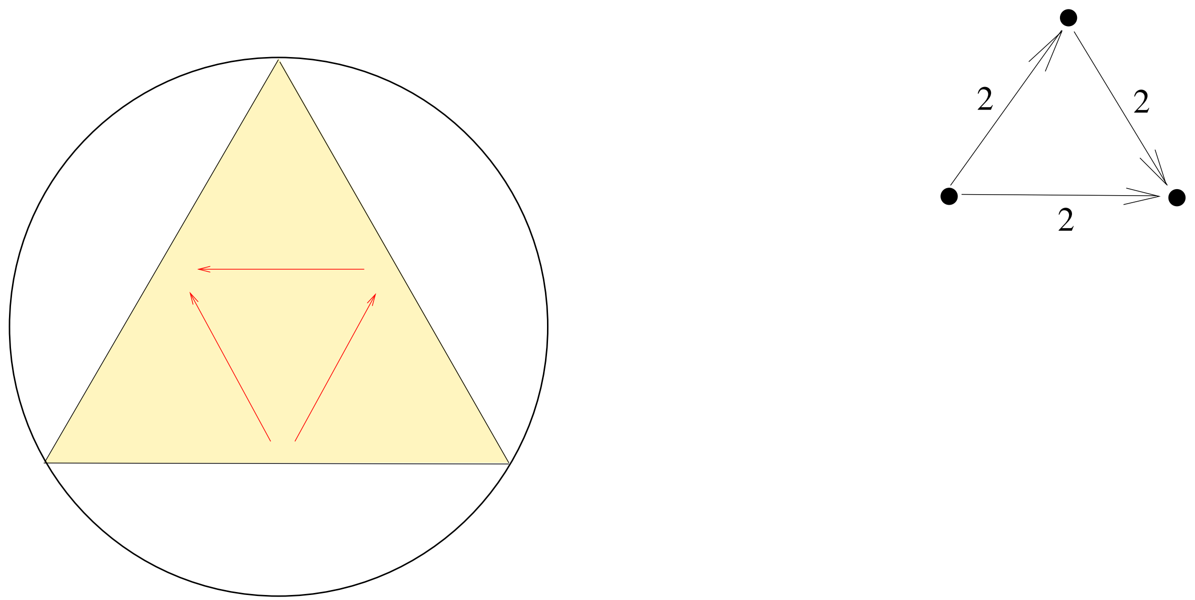
1. Acyclic mutation classes via reflection groups

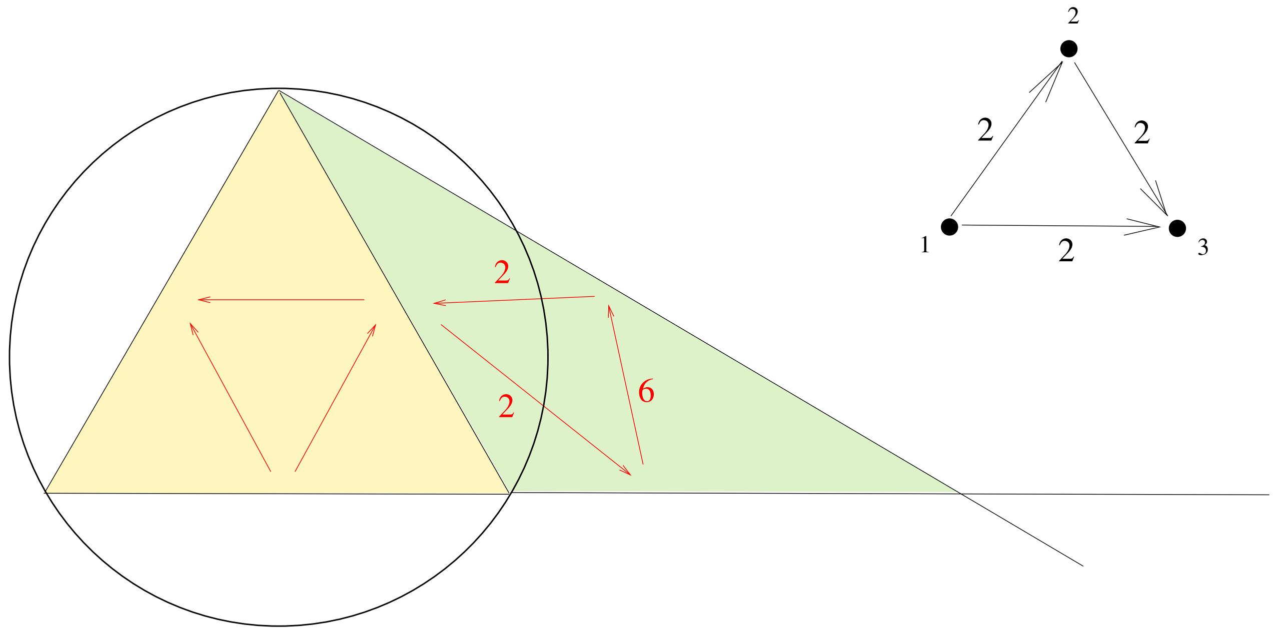
Example:



Then $V = \langle v_1, v_2, v_3 \rangle = \mathbb{H}^2$ $\qquad |\langle u, v\rangle| = \begin{cases} 2\cosh d, & \text{if } \langle v, u\rangle > 2, \\ 2\cos\alpha, & \text{otherwise} \end{cases}$

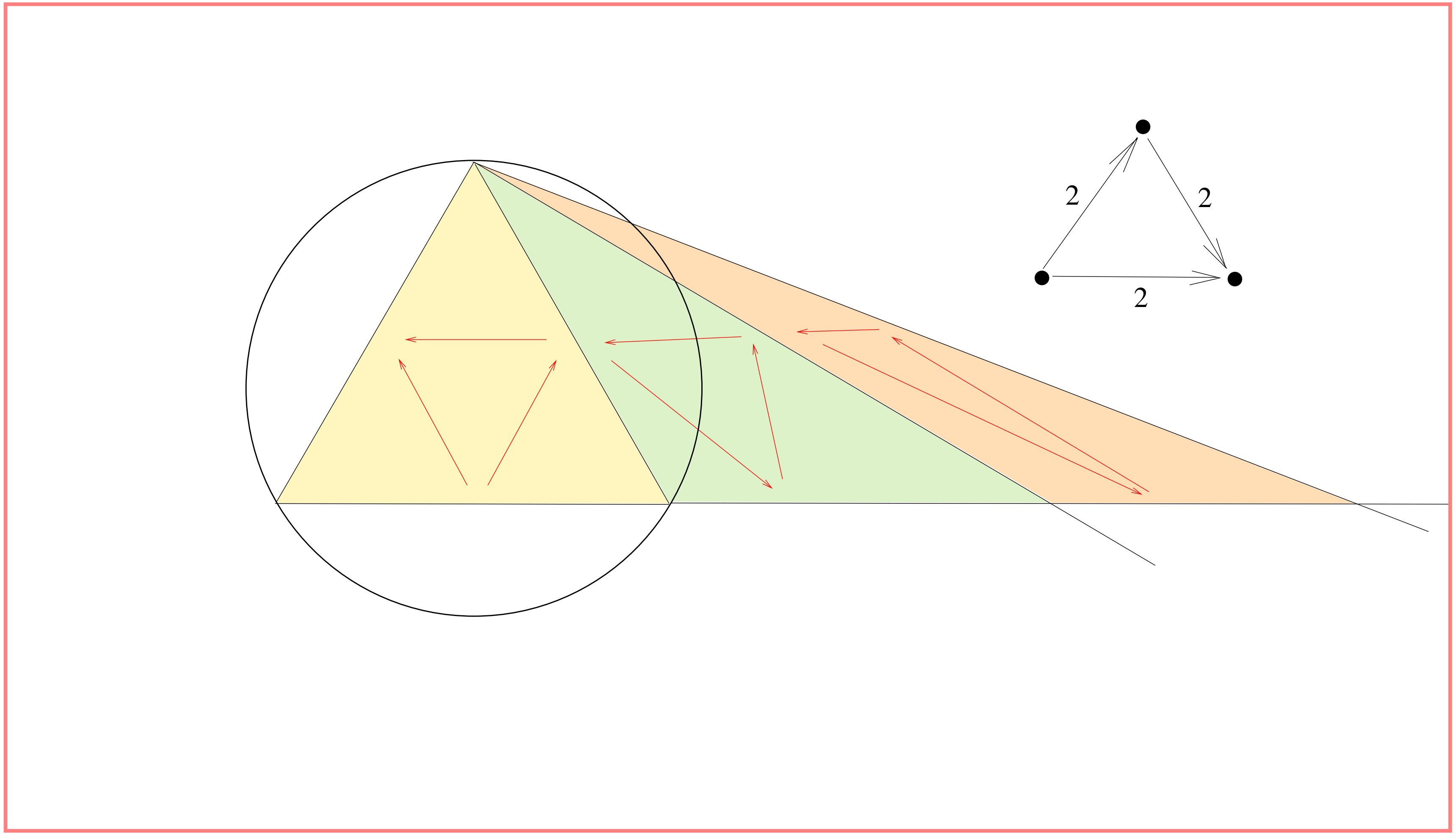
$\langle v_i, v_j \rangle = 2 \quad \Rightarrow \quad \Pi_i$ is parallel to Π_j.

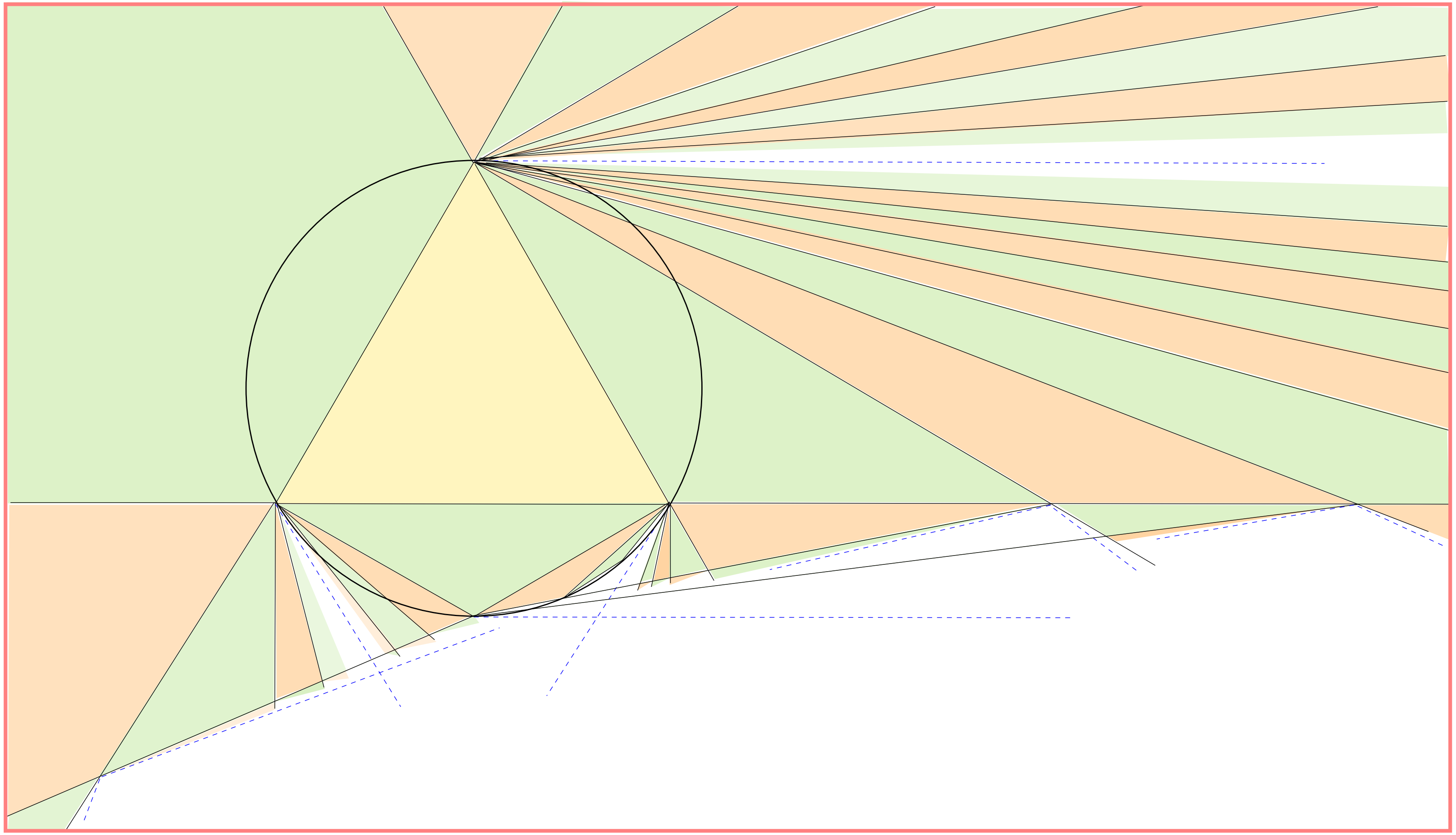

2
2
2



$$v_3' = \mu_2(v_3) = v_3 - \langle v_3, v_2\rangle v_2 = v_3 + 2v_2$$

$$\langle v_3', v_1\rangle = \langle v_3', v_1\rangle + 2\langle v_2, v_1\rangle = -6$$


2
2
2



Corollaries from this picture (examples):

- All quivers in the mutation class of Q are 2-complete.

- All acyclic quiver in this mutation class look “similar” (only differ by permutations and directions of arrows).

- One can move from one acyclic representative to any other via sink/source mutations only.

- Exchange graph for this mutation class is a tree.

Less known:

- How to describe seeds ($=$ sets of walls in one domain)?

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Then:

> If reflections $r_1, \dots, r_n \in G$ form a seed then
> one can reorder them so that $r_1 r_2 \dots r_n = s_1 s_2 \dots s_n$.

- How to describe seeds ($=$ sets of walls in one domain)?

Consider the ordering of the vertices of Q from source to sink (so that $b_{ij} > 0$).
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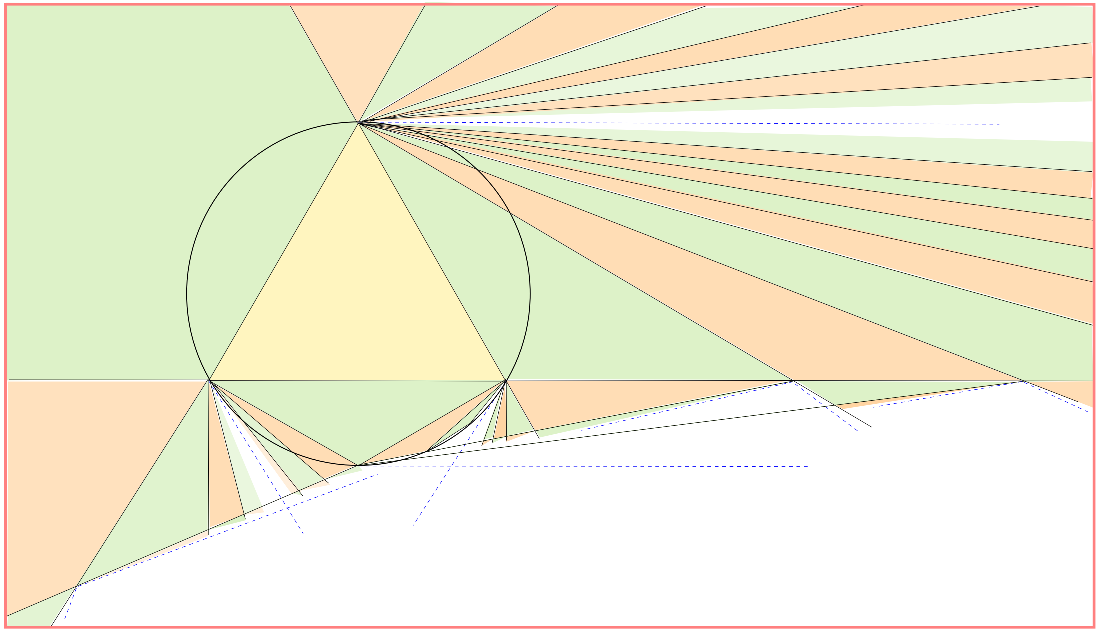
Theorem (Speyer, Thomas' 10)
A collection of roots $u_1, \ldots, u_n$ forms a seed iff

1) If u_i and u_j are both positive roots (or both negative) then $\langle u_i, u_j \rangle \leq 0$;
2) Up to renumbering of $u_1, \ldots, u_n$, the positive roots precede the negative roots and $r_1 r_2 \ldots r_{n-1} r_n = s_1 s_2 \ldots s_n$.

Another question:

- Which reflections appear in the picture?



Or, in other words: How to characterise $\mathbf{c}$-vectors?

Another question:

- Which reflections appear in the picture?

Answer: (“$\Rightarrow$” Nagao’13, “$\Leftarrow$” Nájera Chávez’14)
$r \in G$ appears in the picture iff
the corresponding root u is a real Schur root (or its opposite).

(real Schur roots are
dimension vectors of indecomposable rigid modules
over the path algebra of Q).

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Conjecture: (Kyungyong Lee – Kyu-Hwan Lee'17)
Schur roots are in bijection with
simple curves in some surfaces.

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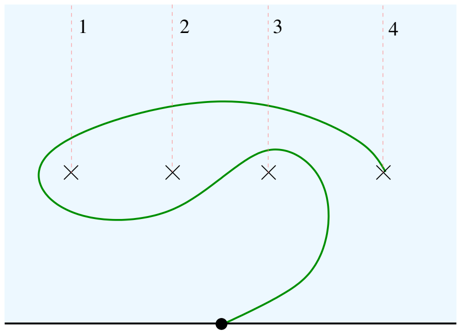
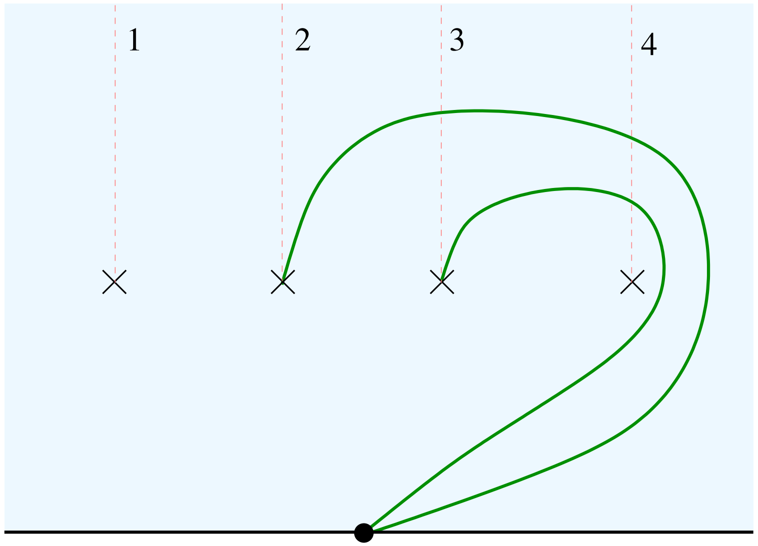
Our answer:

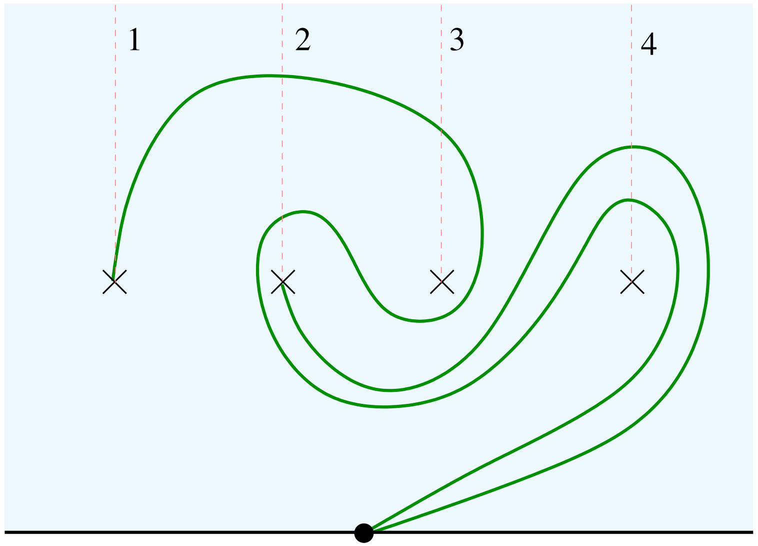
Real Schur roots $=$
arcs in a disc



$s_3 s_1 s_2 s_3 s_4 s_3 s_2 s_1 s_3$

Two arcs form a bad pair if one is a prefix for another:





$\underline{s_4 s_3} s_4$

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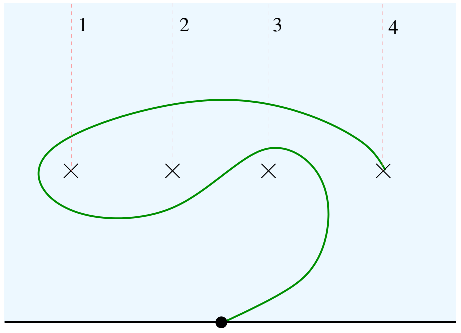
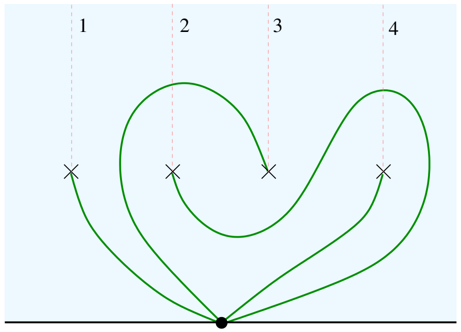
Theorem. (F., Tumarkin'17)

- Real Schur roots $=$ arcs in a disc



- Seeds $=$ collections of non-intersecting arcs with at most one consecutive bad pair



2. Seeds on the Cayley graph

Reflection group G constructed above is a presentation of the universal Coxeter group

$$\langle s_1, \ldots, s_n \mid s_i^2 = e \rangle.$$

(This does not depend on Q, if Q is acyclic and 2-complete).

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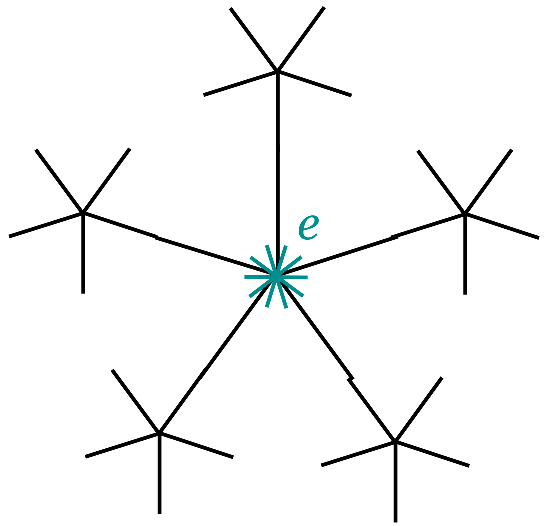
n-regular tree:



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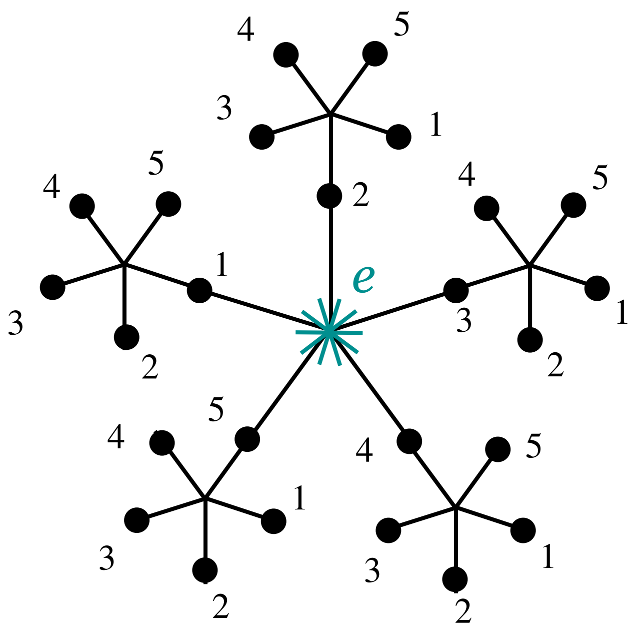

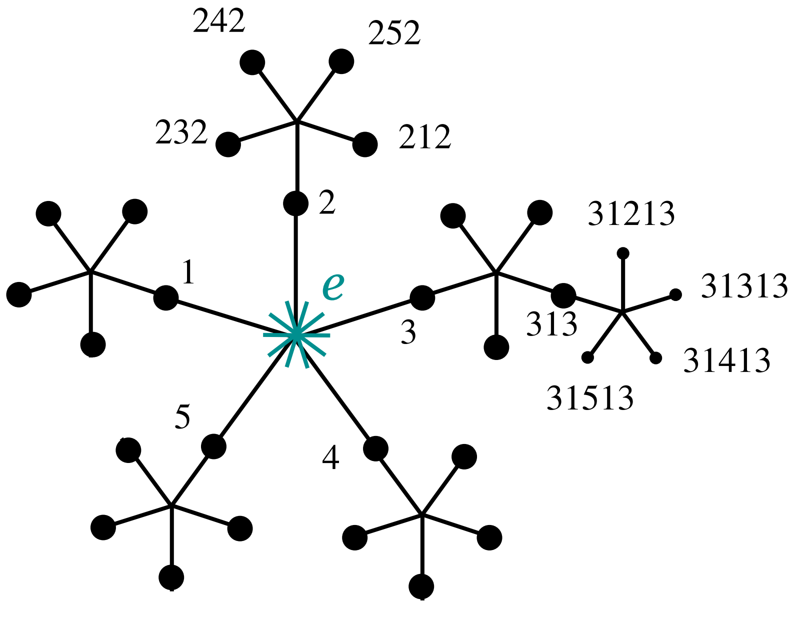

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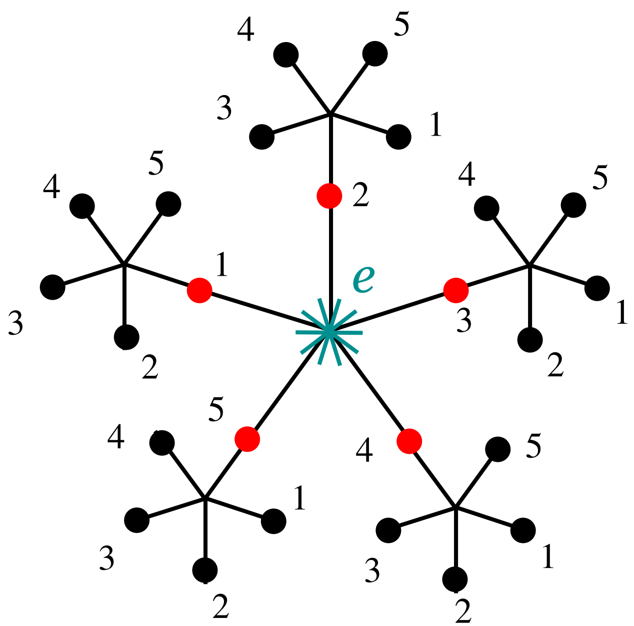
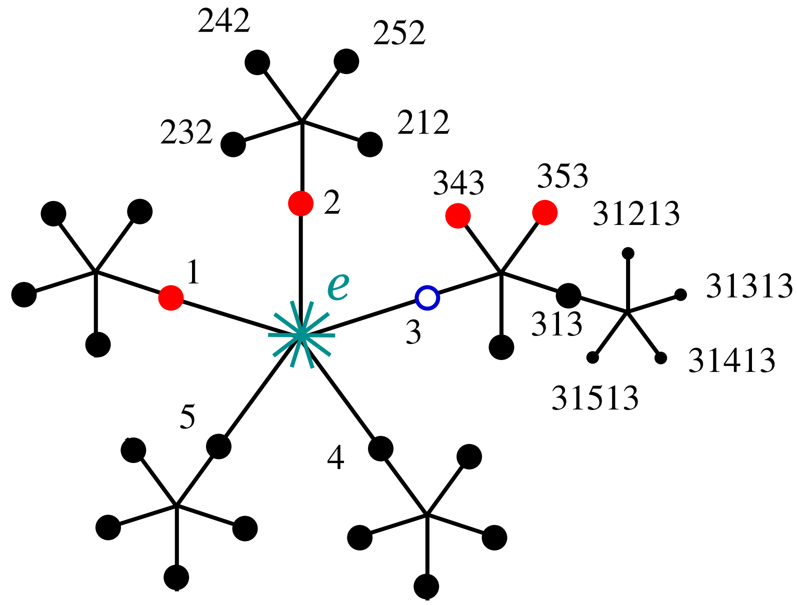


Initial seed



After mutation μ_3

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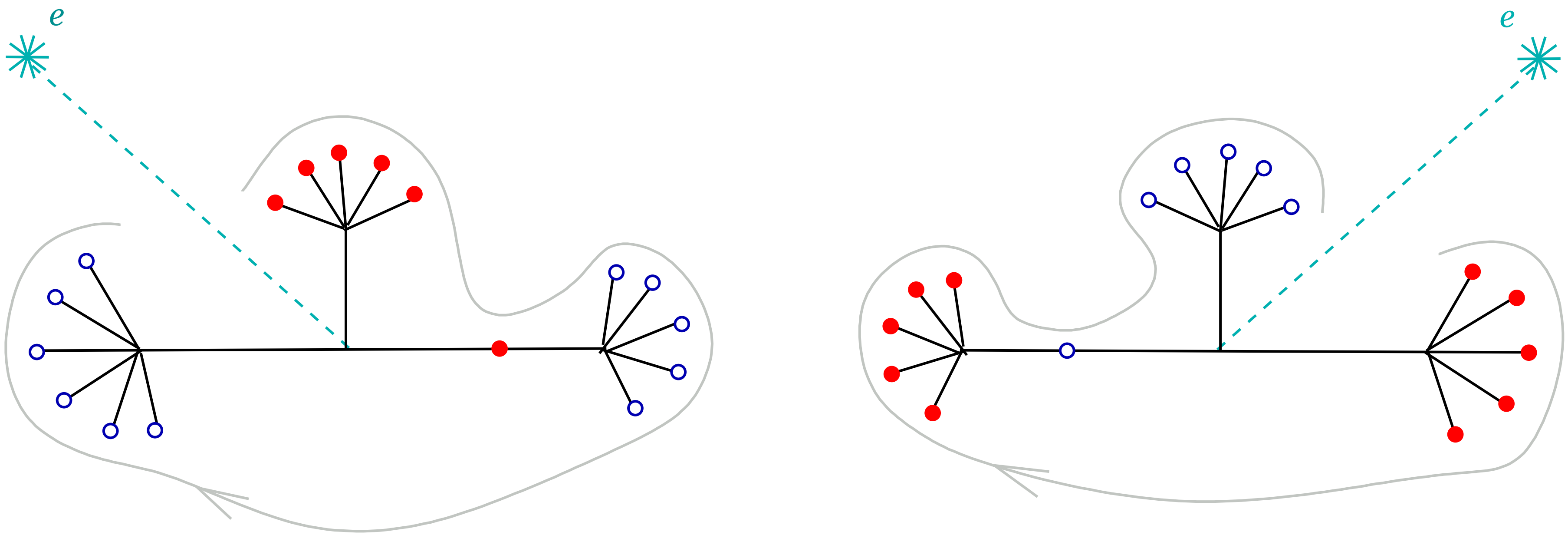
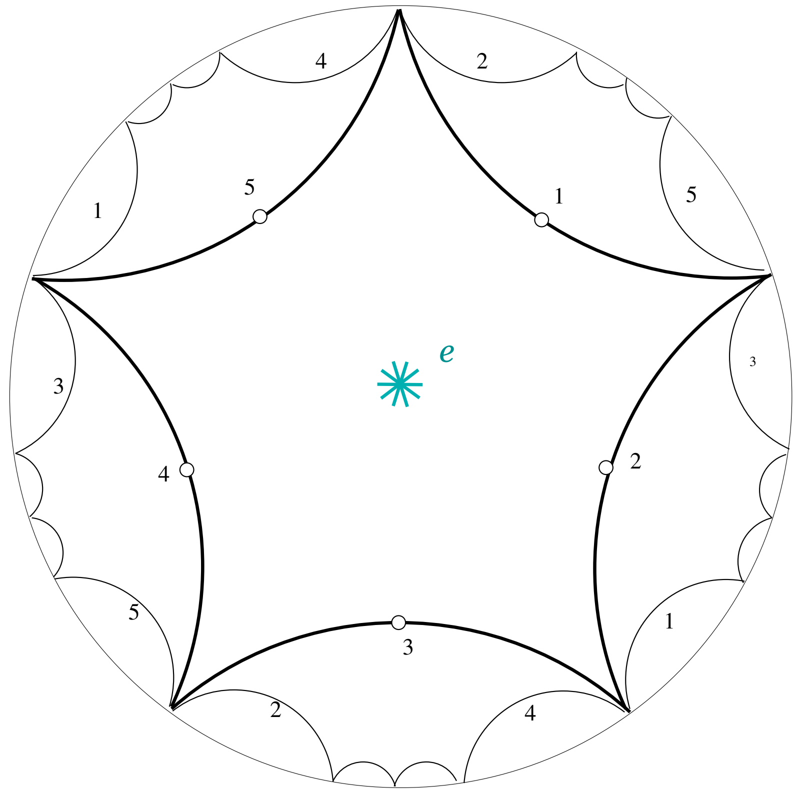


$$r_1 r_2 \dots r_{n-1} r_n = s_1 \dots s_n$$

Proof: induction on the number of mutations.

3. Cayley graph in the hyperbolic plane

- G is isomorphic to a group generated by π-rotations. Denote it by G_{rot}.



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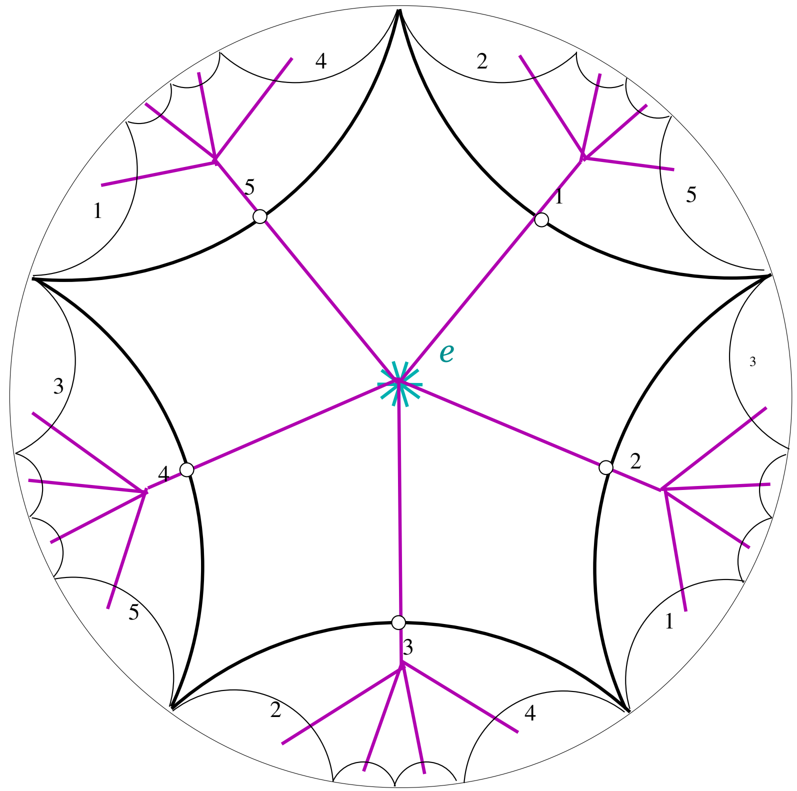

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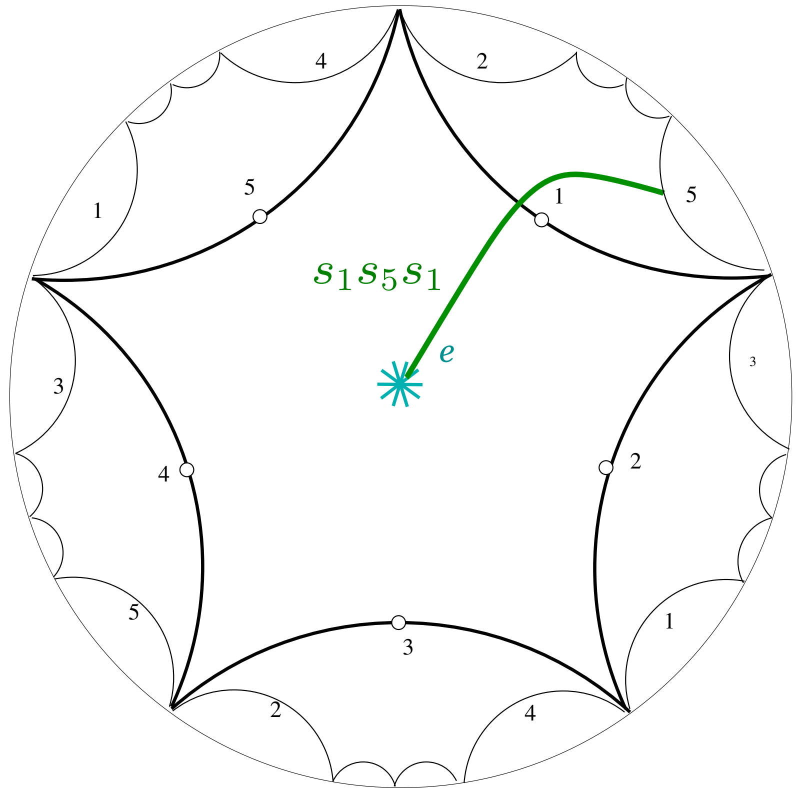
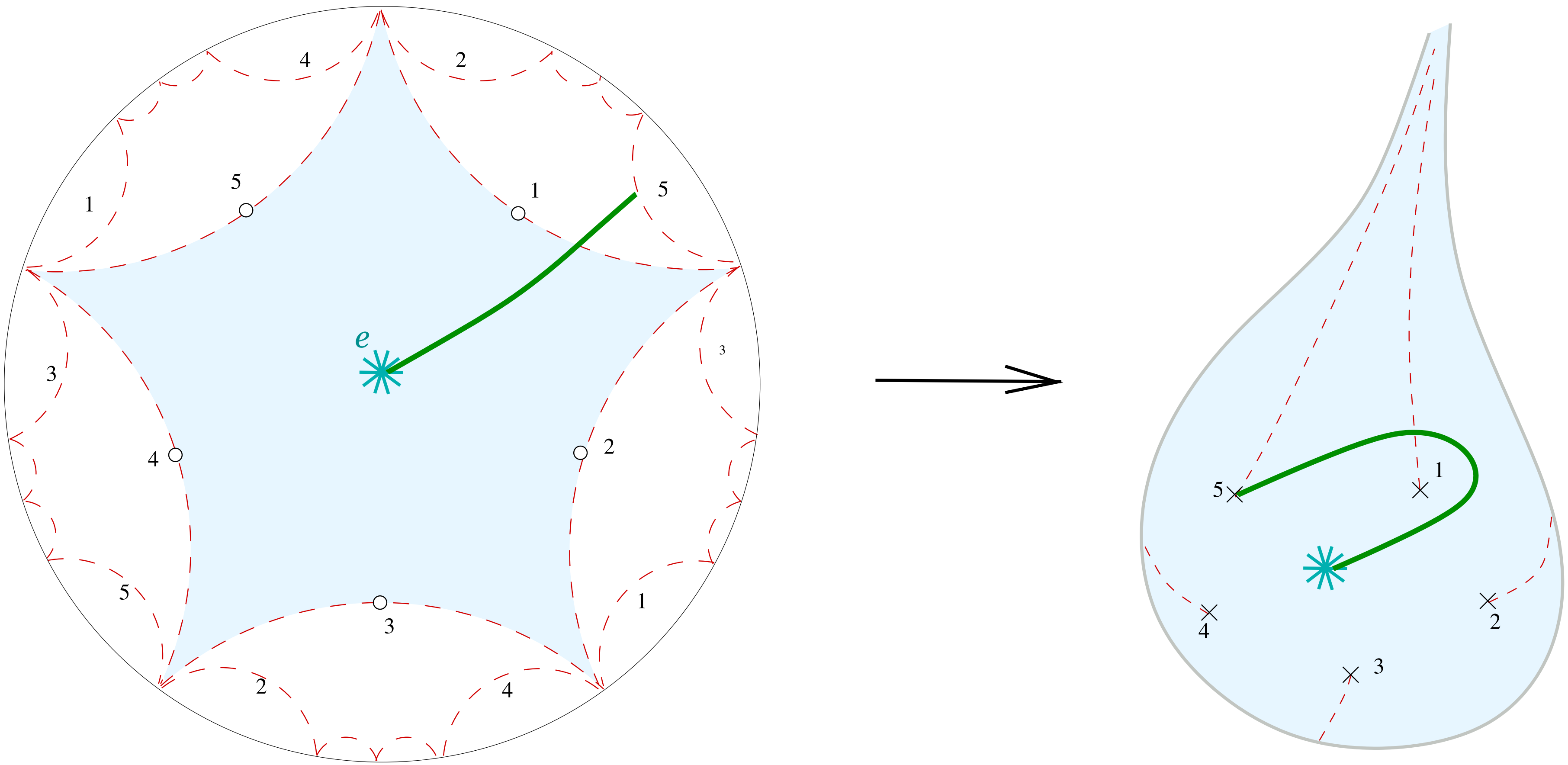
- reflection $r \in G$ may be represented by a path.



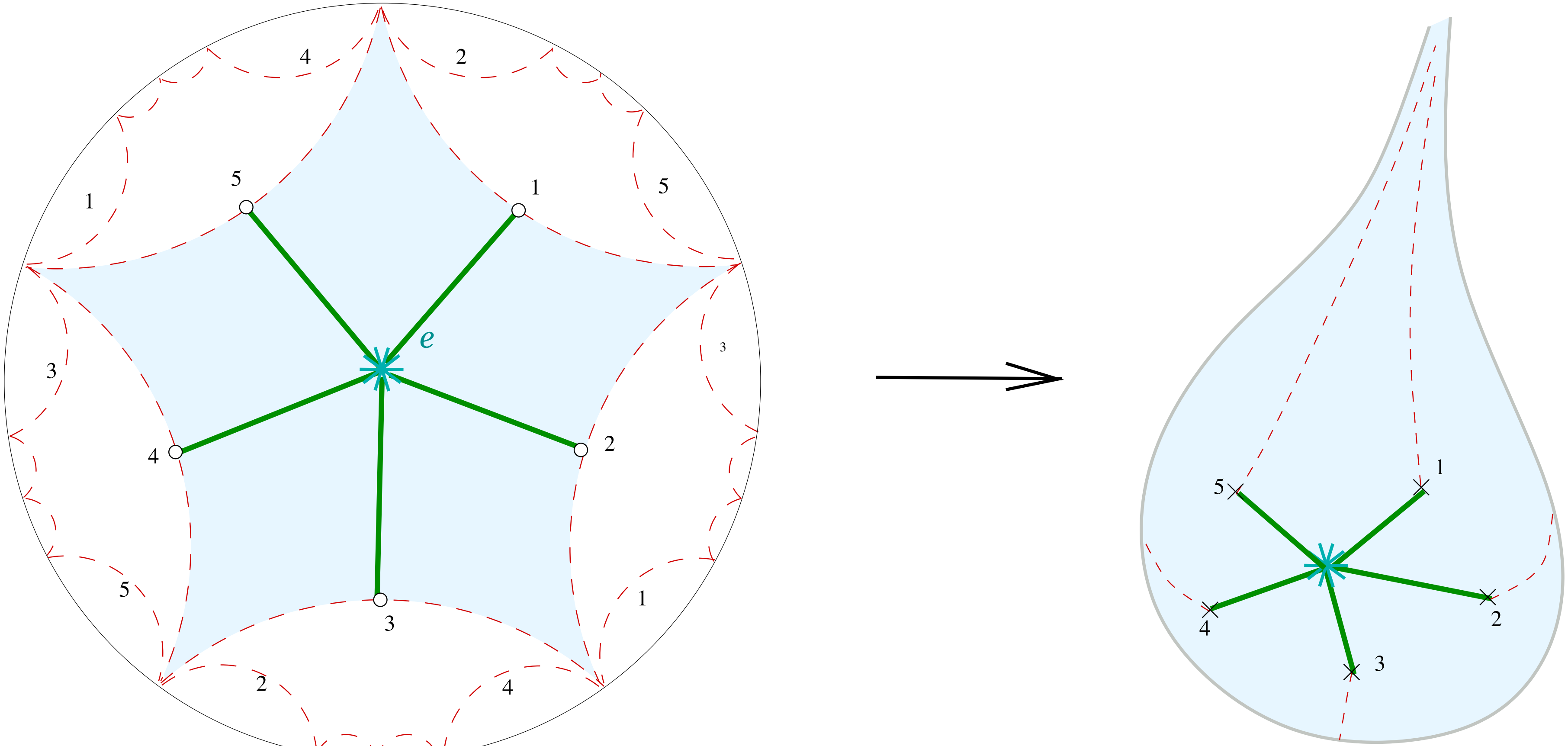
3. Orbifold: from $\mathbb{H}^2$ to an orbifold

Consider $\mathbb{H}^2/G_{rot}$:



3. Orbifold: seeds on the orbifold

Initial seed:



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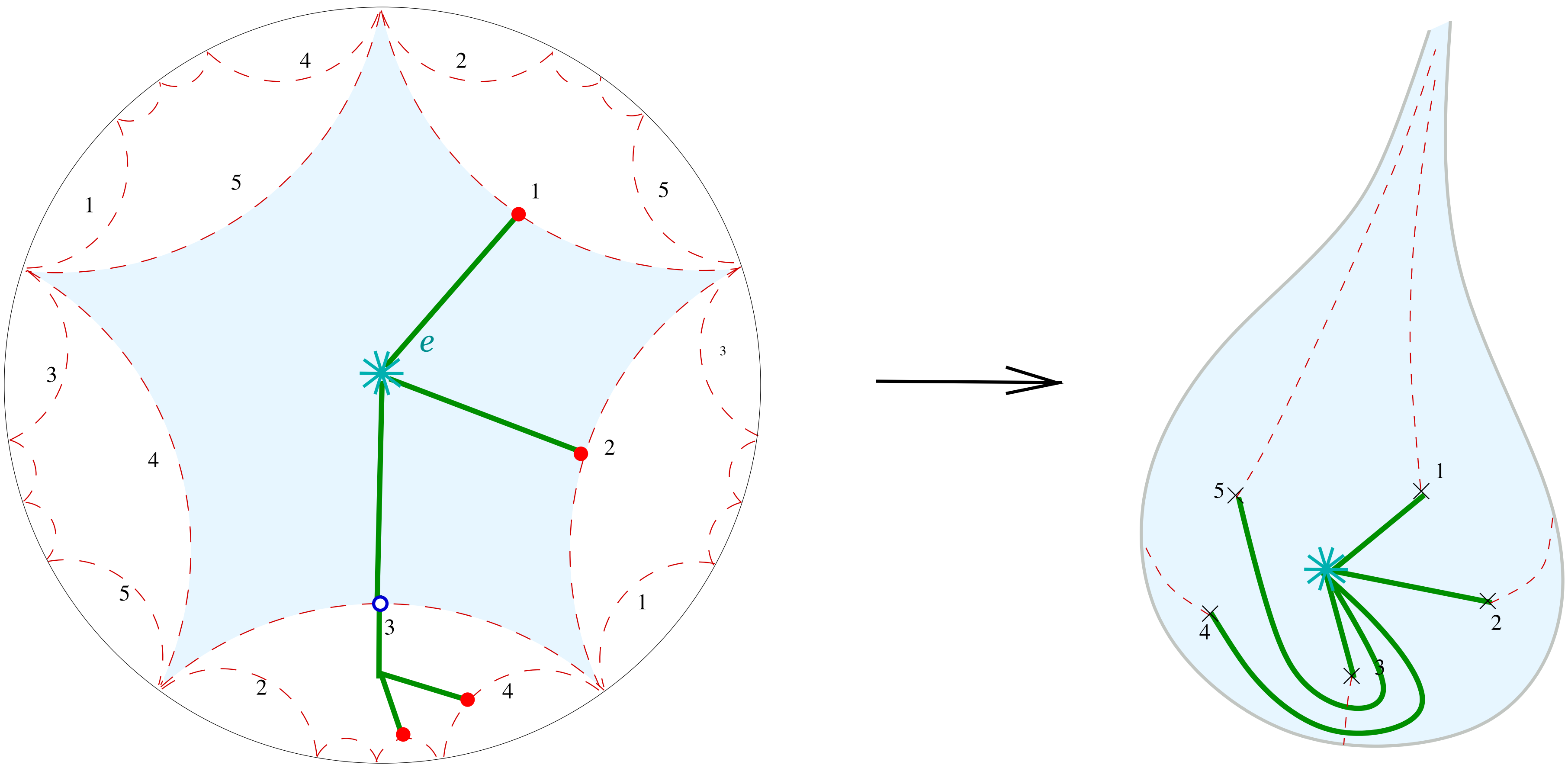
After mutation μ_3:



3. Orbifold: seeds on the orbifold

Let $s \in G$ be a reflection, let u_s be the corresponding root u.
Let $\hat{\gamma}_s$ be the arc in $\mathbb{H}^2$,
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Claim.

- If u_s is a Schur root then γ_s is simple.
- If $u_1, \dots, u_n$ is a seed then $\gamma_{u_1}, \dots, \gamma_{u_n}$ are non-intersecting.
- If $u_1, \dots, u_n$ is a seed then there exists a geodesic ray $l \in \mathcal{O}$ such that no of γ_{u_i} intersects l.

Proof: induction by the number of mutations.

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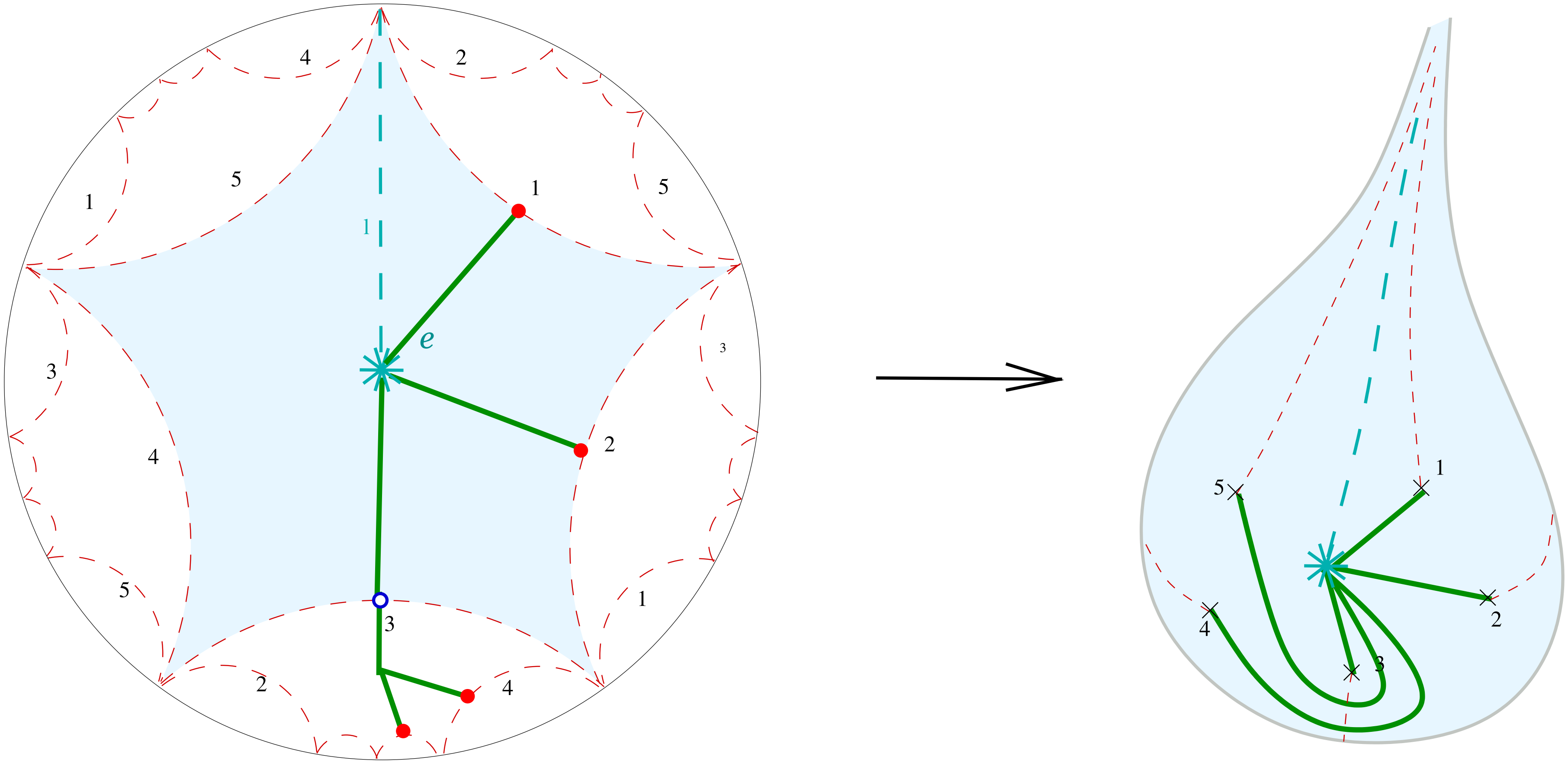
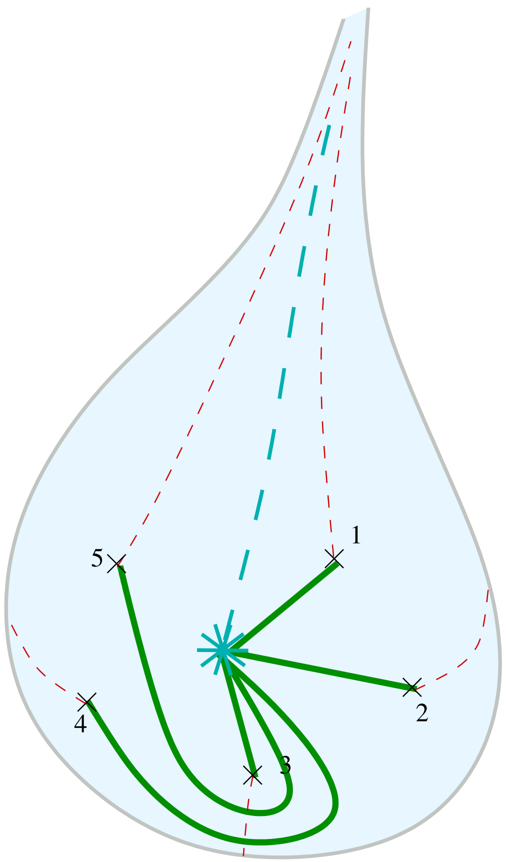
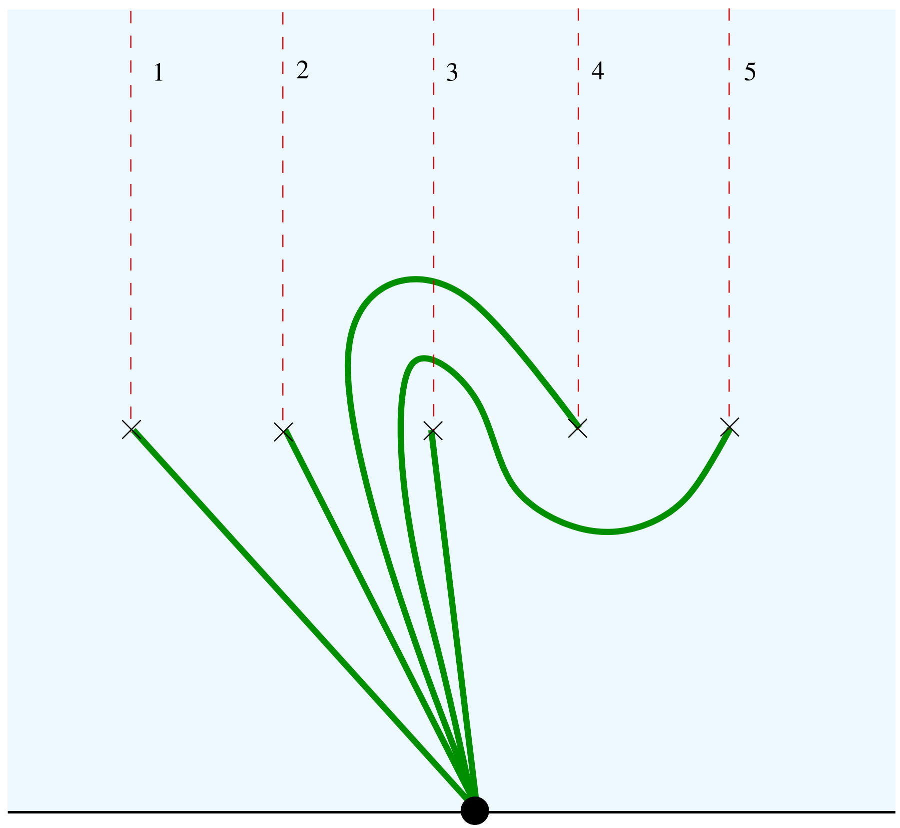


4. From orbifold to disc

Cut along l:





Remarks

- This explains how to map Schur roots to arcs in the disc.
  Why do we get all arcs?

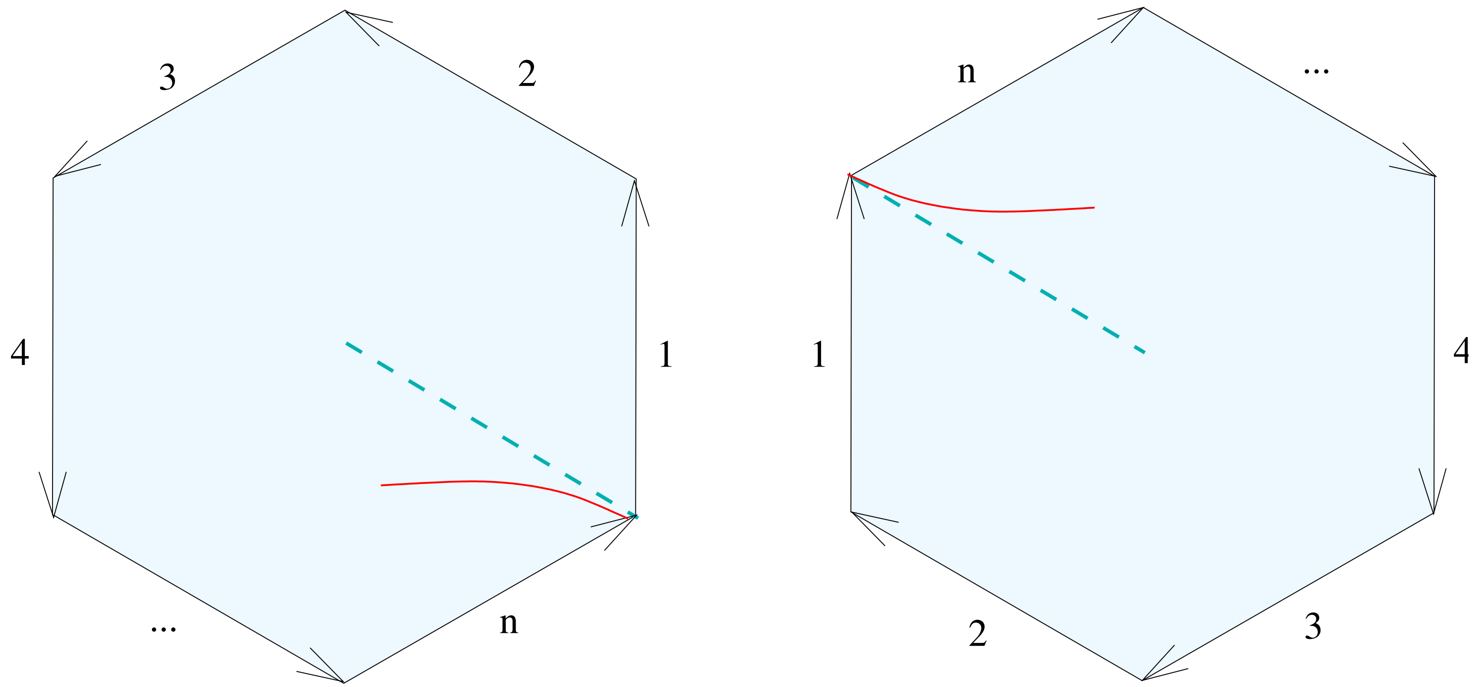
  (a) every (good) set of arcs corresponds to a seed;
  (use the braid grup $\mathbb{B}_n = Aut(D)$
  to verify conditions given by Speyer and Thomas)

  (b) every arc can be included into a (good) set of arcs.
  (induction on n)

- The " Schur roots" part of our theorem
  implies Lee – Lee conjecture.
  (after taking a double cover of the orbifold $\mathcal{O}$)

Lee-Lee conjecture:

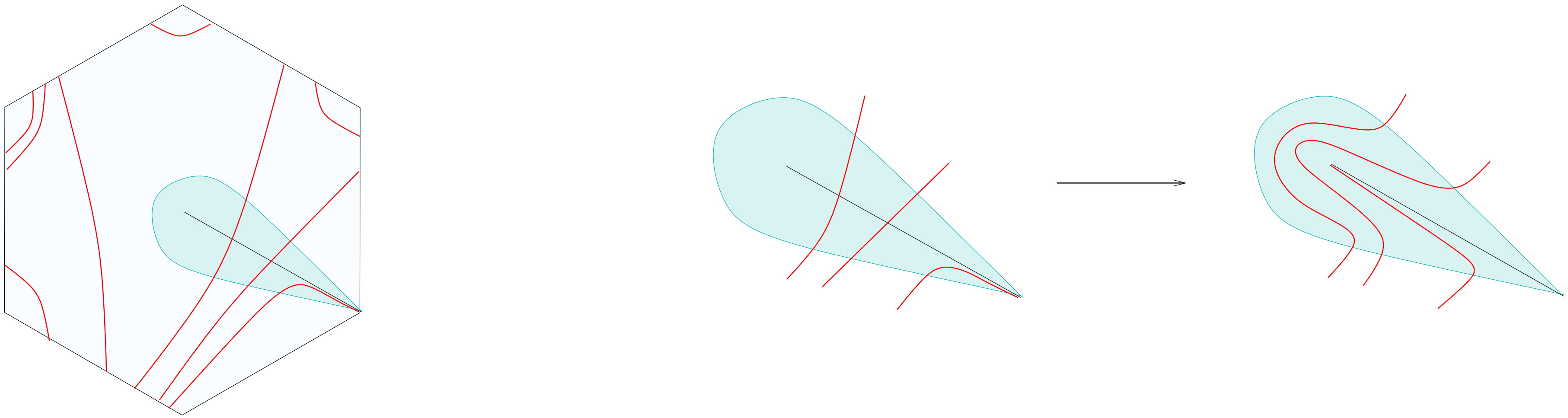
Schur roots are in bijection with arcs on the following surface S:



- Conjectured for all acyclic quivers (not necessarily 2-complete).
- Proved for 2-complete quivers of rank 3.

Lee-Lee conjecture $\Leftrightarrow$ our theorem:
for 2-complete Q

Surface S is a double cover of the orbifold $\mathcal{O}$.



Curves on S $\longrightarrow$ arcs on the disc.

Open questions:

- General (not necessirily 2-complete) acyclic quivers?
- When are two roots compatible?
  (i.e. when there exists a seed containing them both?).
- Is a collection of mutually compatible roots compatible itself?

Thanks!