Coxeter groups and cluster algebras

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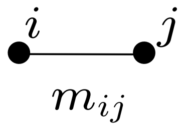
1. Coxeter groups

$$(G,S)=\langle s_1,\dots s_n\in S \mid s_i^2=(s_is_j)^{m_{ij}}=1\ \rangle$$

Coxeter diagram 

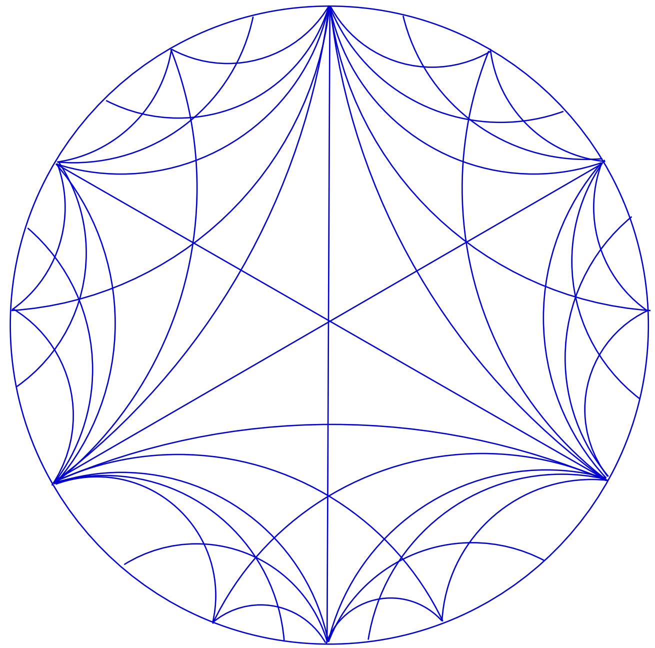
reflections: s_i and their conjugates

Coxeter polytopes = chambers =

=polytopes with angles π/m_{ij}

reflection subgroups =

subgroups generated by reflections



1. Coxeter groups: some results. (joined with P. Tumarkin)

a. Reflection subgroups:

a1. Classification of reflection subgroups in Coxeter groups generated by hyperbolic simplices.

a2. If W is indecomposable infinite Coxeter group and $V \subset W$ is a finite index reflection subgroup then $rank\, V \geq rank\, W$.

a3. Classification of odd-angled Coxeter groups containing finite index reflection subgroups.

b. Hyperbolic Coxeter polytopes:

Classification of compact polytopes with $n \leq d + 4$ or with $p \leq n - d - 2$, where $n = rank\, G$, $d =$ dimension, $p = \#\{\ (i, j) \mid m_{ij} = \infty\}$.

c. Applications to Kac-Moody algebras.

Cluster algebras

(introduced by S. Fomin and A. Zelevinsky in 2000)

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A seed is a pair (B, x) (exchange matrix, cluster)

$n \times n$ integer , $x = (x_1, \ldots, x_n)$

$B^t = -B$ rational functions

cluster variables

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Seed mutation: $\mu_k(B, x) = (B', x')$

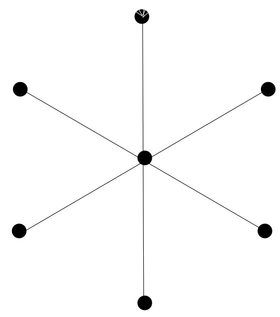
$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } i = k \text{ or } j = k; \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2}, & \text{otherwise.} \end{cases}$$

$$x'_i = x_i \quad \text{for } i \neq k; \qquad x_k x'_k = \prod_{b_{kj}>0} x_j^{b_{kj}} + \prod_{b_{kj}<0} x_j^{-b_{kj}}$$

Property: $\mu_k^2 = Id$

Iterated mutations $\longrightarrow$ many other seeds

$B \longrightarrow$ its mutation class



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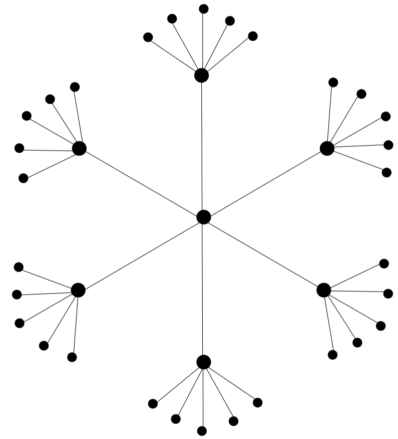

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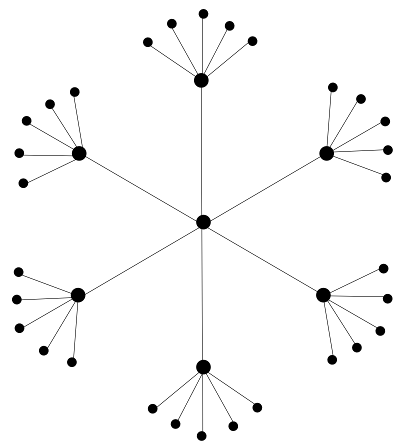


Iterated mutations $\longrightarrow$ many other seeds

$B \longrightarrow$ its mutation class

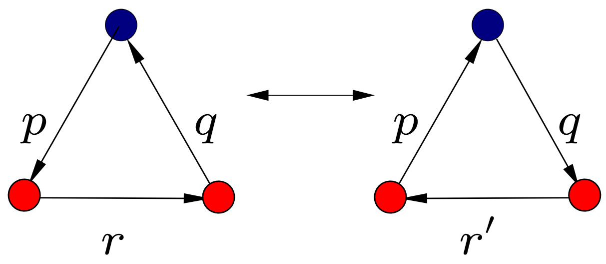
Definition. Cluster algebra $A(B)$ is a $\mathbb{Q}$-subalgebra of $\mathbb{Q}(x_1, \dots, x_n)$ generated by all cluster variables.

More generally: one starts from skew-symmetrizable matrix B (i.e. $\hat{B} = BD$, $\hat{B}^t = -\hat{B}$ for some positive integer diagonal D).

Skew-symmetric exchange matrix B $\longleftrightarrow$ quiver Q

$$i \xrightarrow{b_{ij}} j \qquad \text{if } b_{ij} > 0$$

Quiver mutation μ_k:



$$r + r' = pq$$

Java applet for quiver mutation by Bernhard Keller.

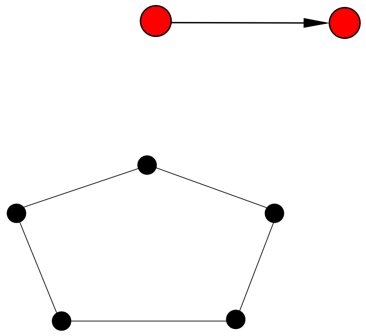
Cluster algebra is of finite type if the number of cluster variables is finite.

Theorem. (Fomin, Zelevinsky' 2002).

Cluster algebras of finite type	$\Leftrightarrow$	finite arithmetic Coxeter groups (types $A_n, B_n = C_n, D_n, E_6, E_7, E_8, F_4, G_2$)

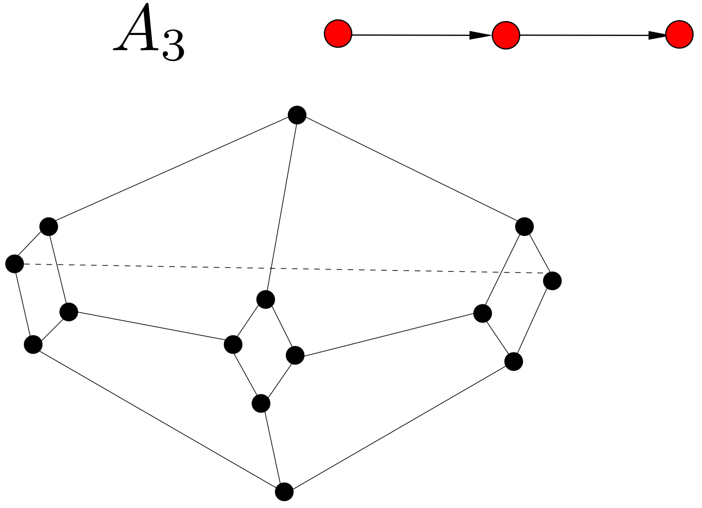
Example:

A_2



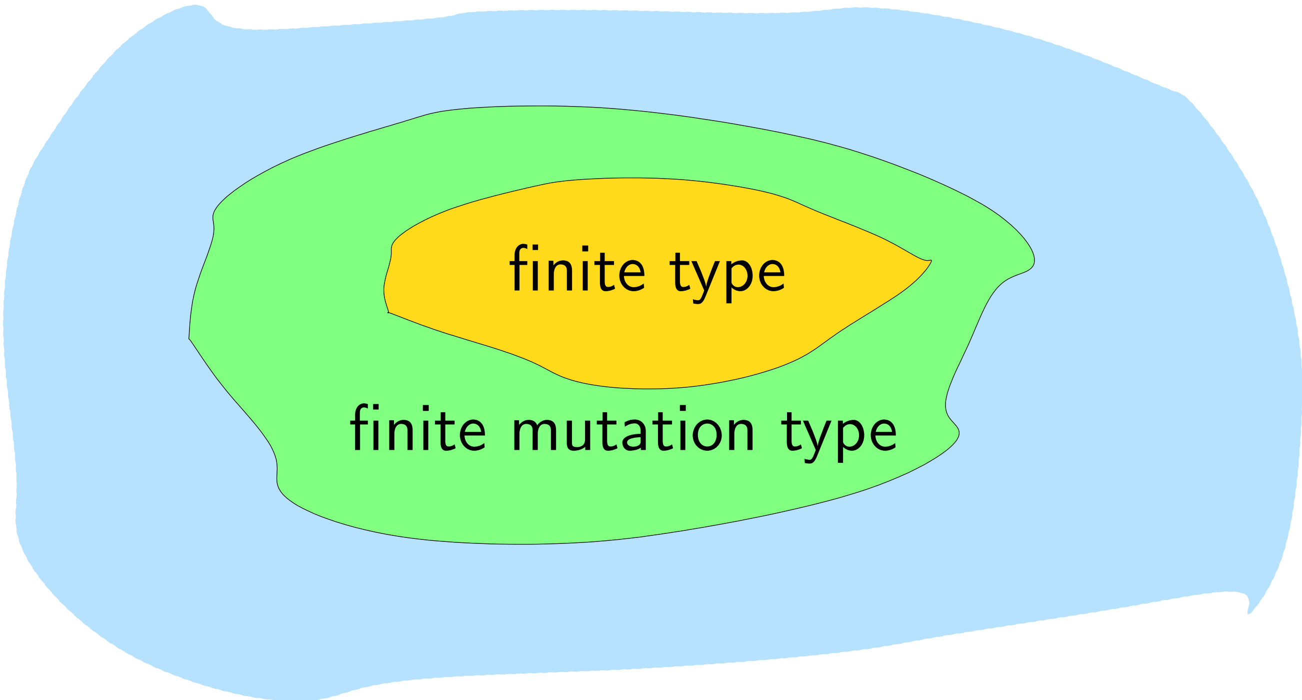
$$x_1, x_2, \frac{1+x_1}{x_2}, \frac{1+x_2}{x_1}, \frac{x_1+x_2+1}{x_1x_2}$$

A_3



Cluster algebra is of finite mutation type if mutation class of B consists of finitely many matrices.

- If $n > 2$ and $|b_{ij}| > 4$ for some i, j then mut. class of B is infinite.



mutation class	cluster variables
$< \infty$	$< \infty$
$< \infty$	∞
∞	∞
$\cancel{\infty}$	$\cancel{< \infty}$

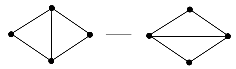
Problem. Classify algebras of finite mutation type.

In skew-symmetric case ($B^t = -B$):
classify quivers of finite mutation type.

Examples:

1. $n = 2$.
2. Quivers arising from triangulated surfaces.
3. (conjectured by Fomin, Shapiro, Thurston)
   Finitely many are left.

Quivers from triangulated surfaces

Triangulated surface	$\longrightarrow$	Quiver
edge of triangle		vertex of quiver
two edges of one triangle		arrow in quiver
flip of triangulation		mutation of quiver

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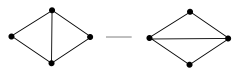
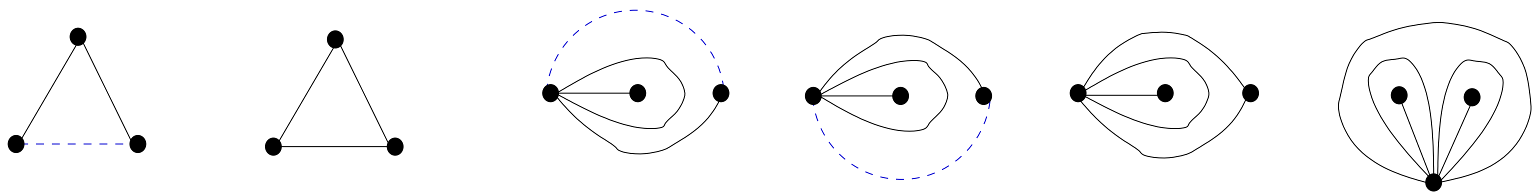
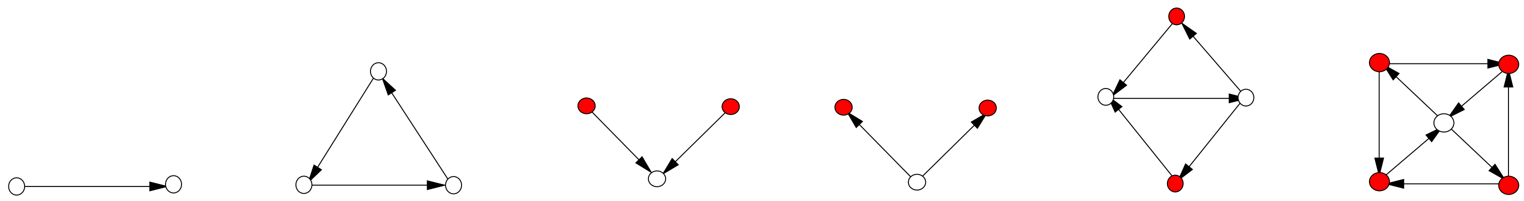
flip of triangulation 		mutation of quiver

Surface can be glued from small pieces:



Quiver may be obtained from blocks ("is block-decomposable"):



When Q is mutation-finite but is not block-decomposable?

Analogy from Coxeter groups:

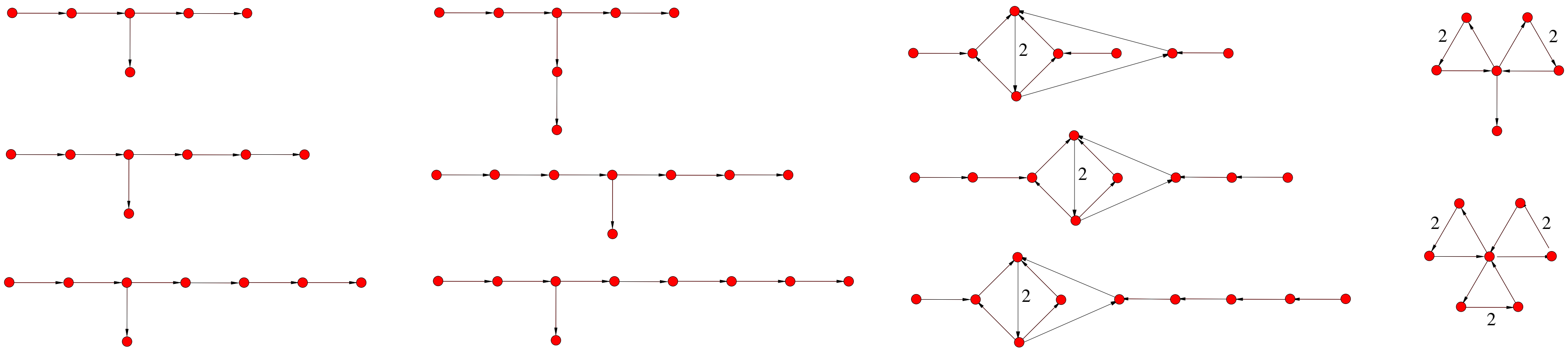
Combinatorics of a compact hyperbolic Coxeter polytope is determined by subdiagrams corresponding to

- Finite Coxeter groups
- Minimal infinite Coxeter groups
  (diagrams of hyperbolic simplices, classified).

Idea: find minimal quivers which are not block-decomposable.

Results (with M. Shapiro and P. Tumarkin):

Theorem 1. A connected mutation-finite quiver of order $n > 2$ is either built from a triangulated surface or is mutation-equivalent to one of the following 11 exclusions:



Theorem 2. If Q is minimal mutation-infinite quiver then $n \leq 10$.

Corollary: criterion for quiver of finite mutation type.

Theorem 1'. Let $A(B)$ be cluster algebra of finite mutation type. Then either $n \le 2$, or B is obtained from triangulated orbifold, or a diagram of B is mutation-equivalent to one of 11+7 exclusions.

Theorem 2'. If $A(B)$ is minimal mutation-infinite then $n \le 10$.

Corollary: criterion for cluster algebra of finite mutation type.

Theorem 3'. Let S be a Coxeter diagram of a finite volume arithmetic hyperbolic simplex. Then there exists an orientation of S corresponding to a minimal mutation-infinite cluster algebra.


THANKS!