Bases for cluster algebras from orbifolds

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joint work with P. Tumarkin

4th workshop on combinatorics of moduli spaces, cluster algebras, and topological recursion

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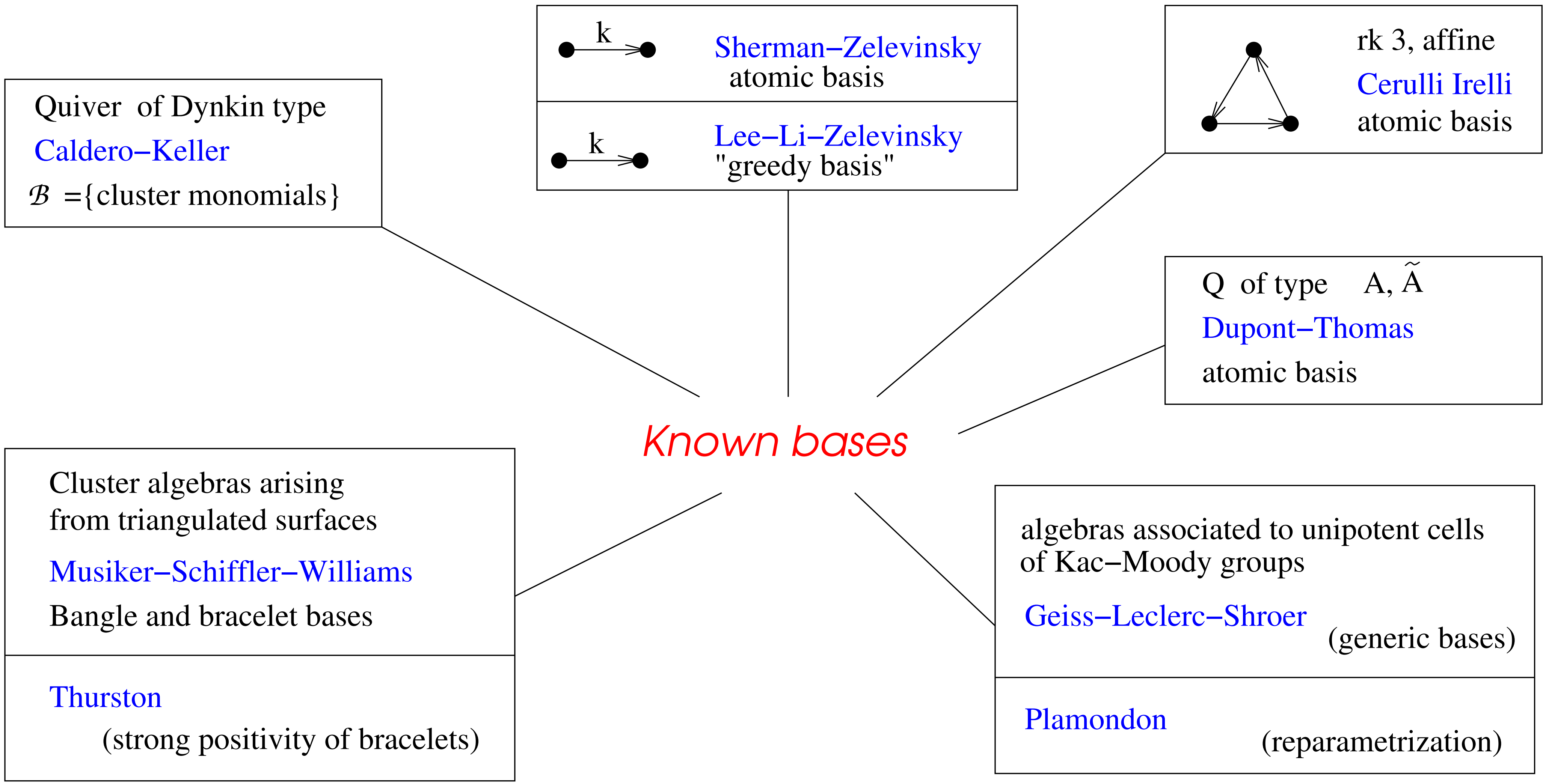
Introduction

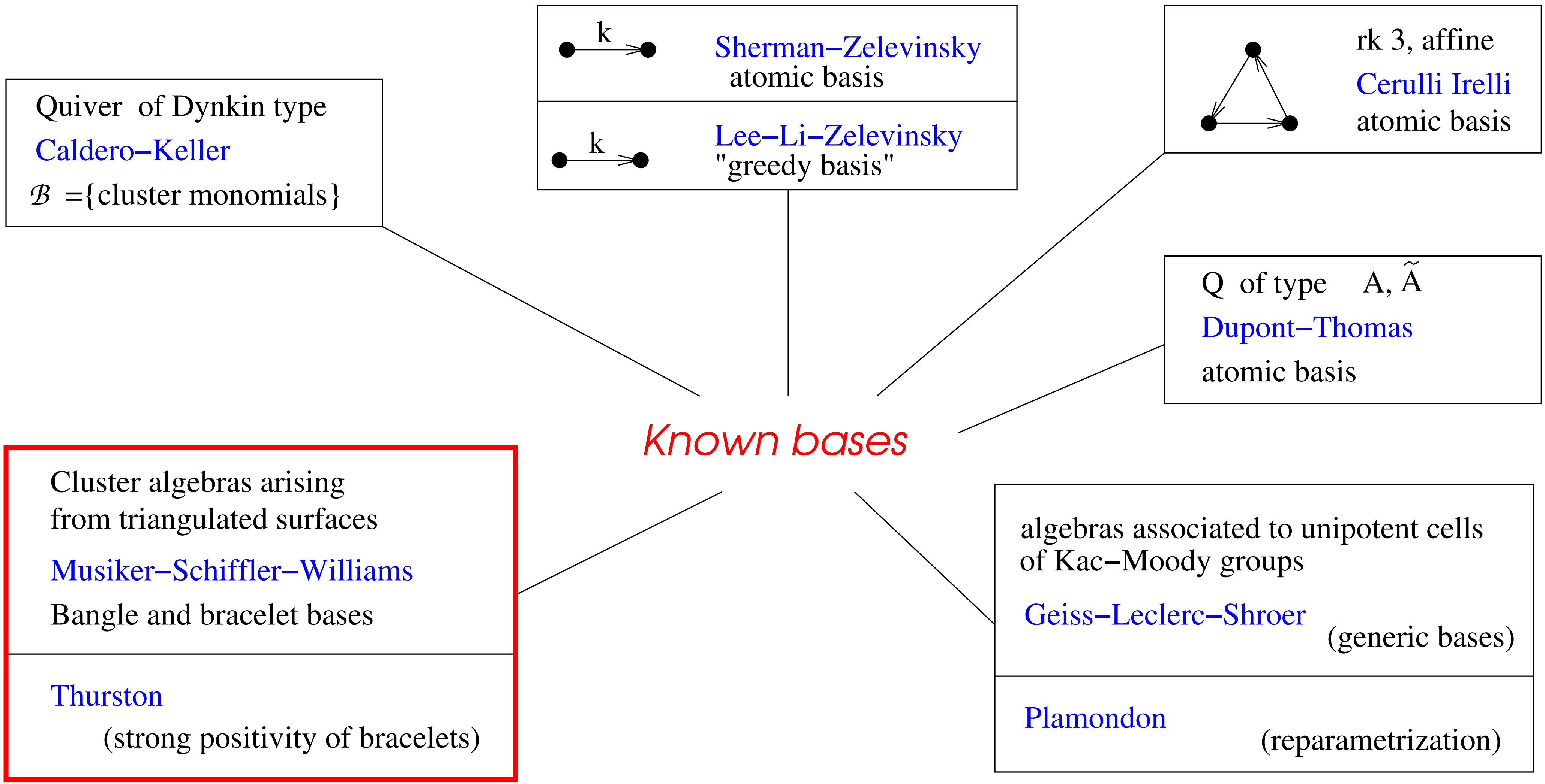
B - skew-symmetrizable $n \times n$ integer matrix,
$\mathcal{A}_B$ cluster algebra $\subseteq \mathbb{Q}(x_1, \ldots, x_n)$.

As an algebra $\mathcal{A}_B$ is generated by all cluster variables.

Problem: what are the "good" sets to span $\mathcal{A}_B$ as a vector space?

[FZ]: "good" means "containing all cluster monomials"
(i.e. monomials in variables sitting in any given cluster).


Known bases
Quiver of Dynkin type
Caldero–Keller
$\mathcal{B}$ ={cluster monomials}
k
Sherman–Zelevinsky
atomic basis
k
Lee–Li–Zelevinsky
"greedy basis"
rk 3, affine
Cerulli Irelli
atomic basis
Q of type $A, \tilde{A}$
Dupont–Thomas
atomic basis
Cluster algebras arising
from triangulated surfaces
Musiker–Schiffler–Williams
Bangle and bracelet bases
Thurston
(strong positivity of bracelets)
algebras associated to unipotent cells
of Kac–Moody groups
Geiss–Leclerc–Shroer
(generic bases)
Plamondon
(reparametrization)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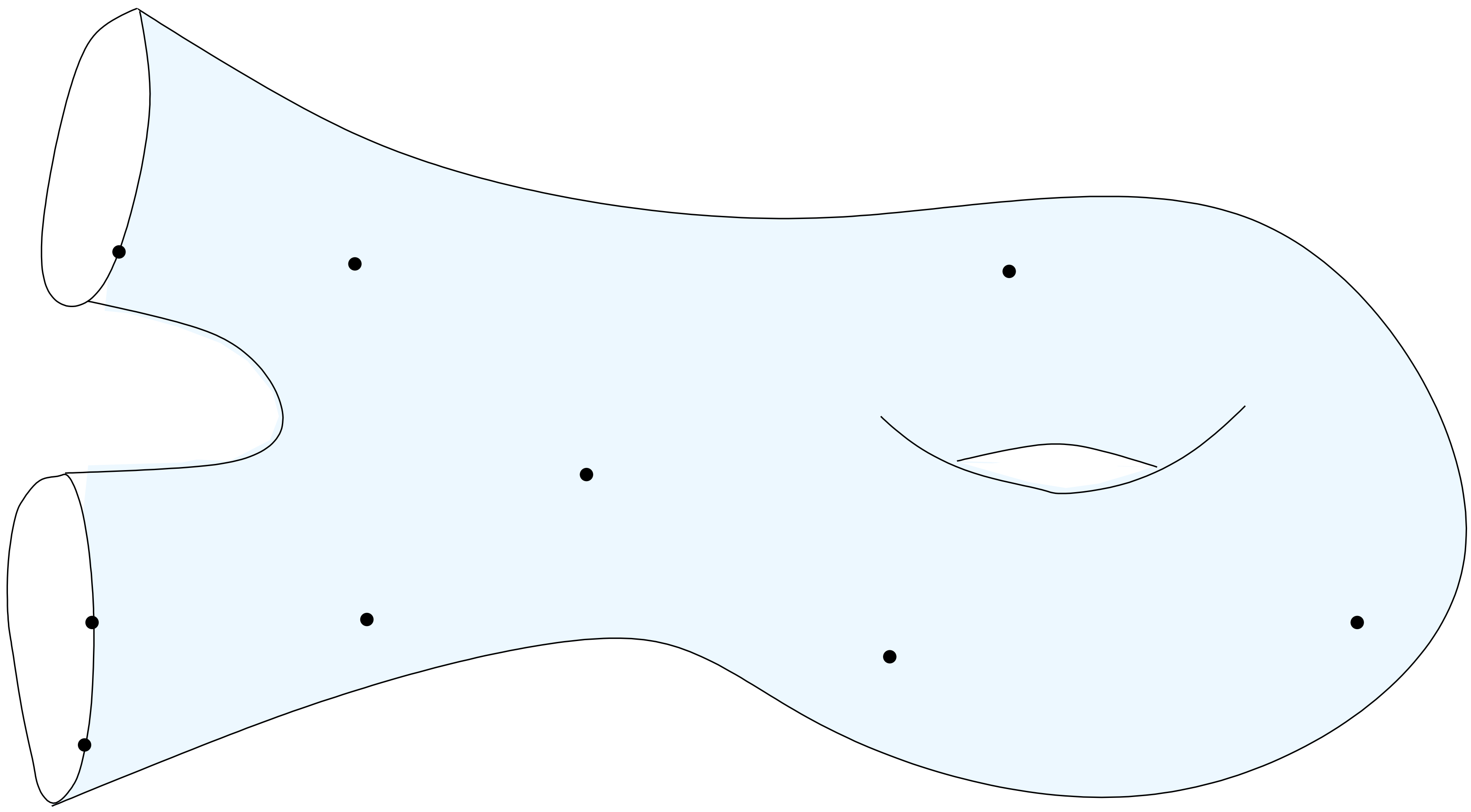
Plan:

1. Cluster algebras from surfaces and their bases
2. Cluster algebras from orbifolds and their bases
3. Why the construction works?

1.1. Cluster algebras from surfaces

[Fomin, Shapiro, Thurston' 2008]



1.1. Cluster algebras from surfaces

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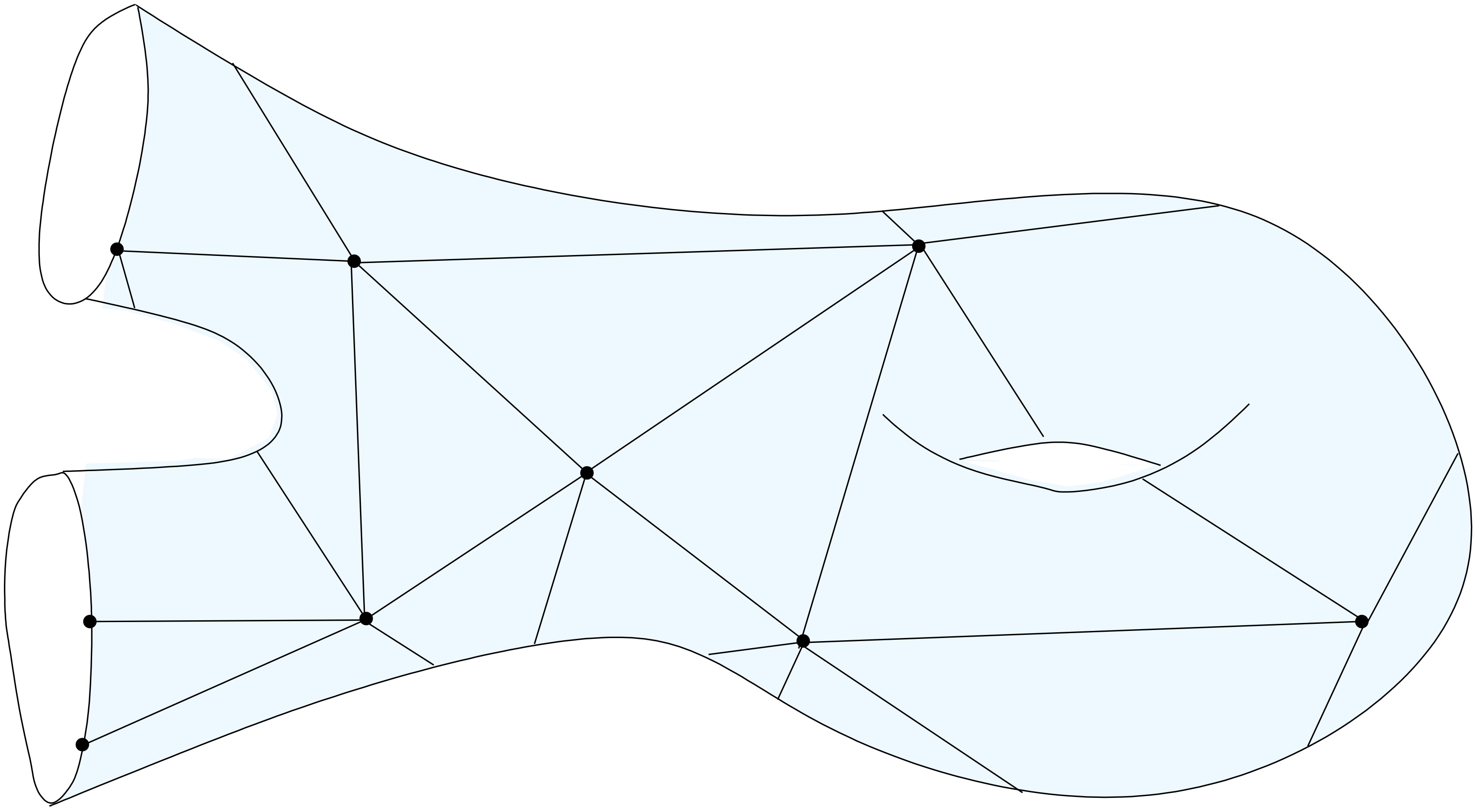

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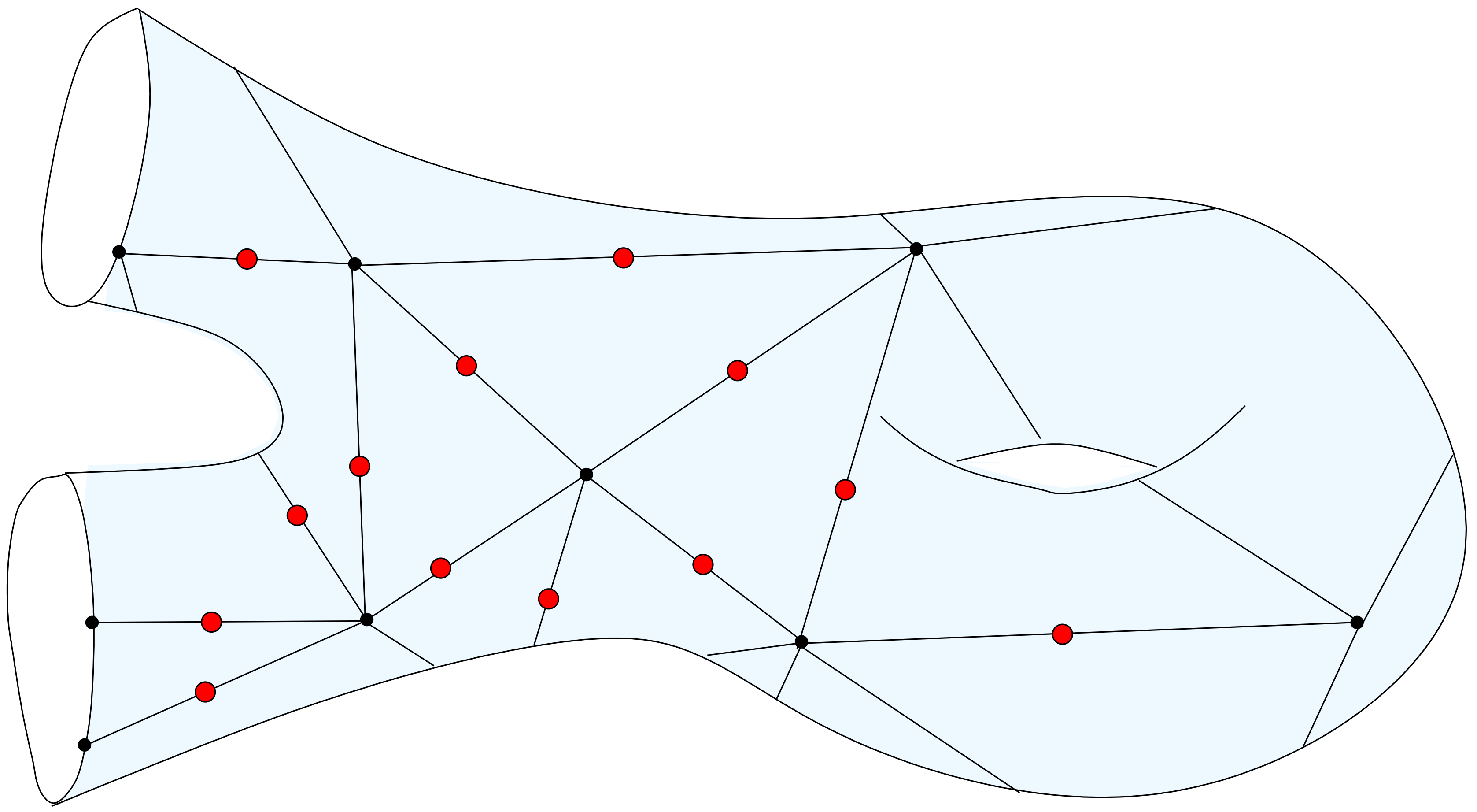

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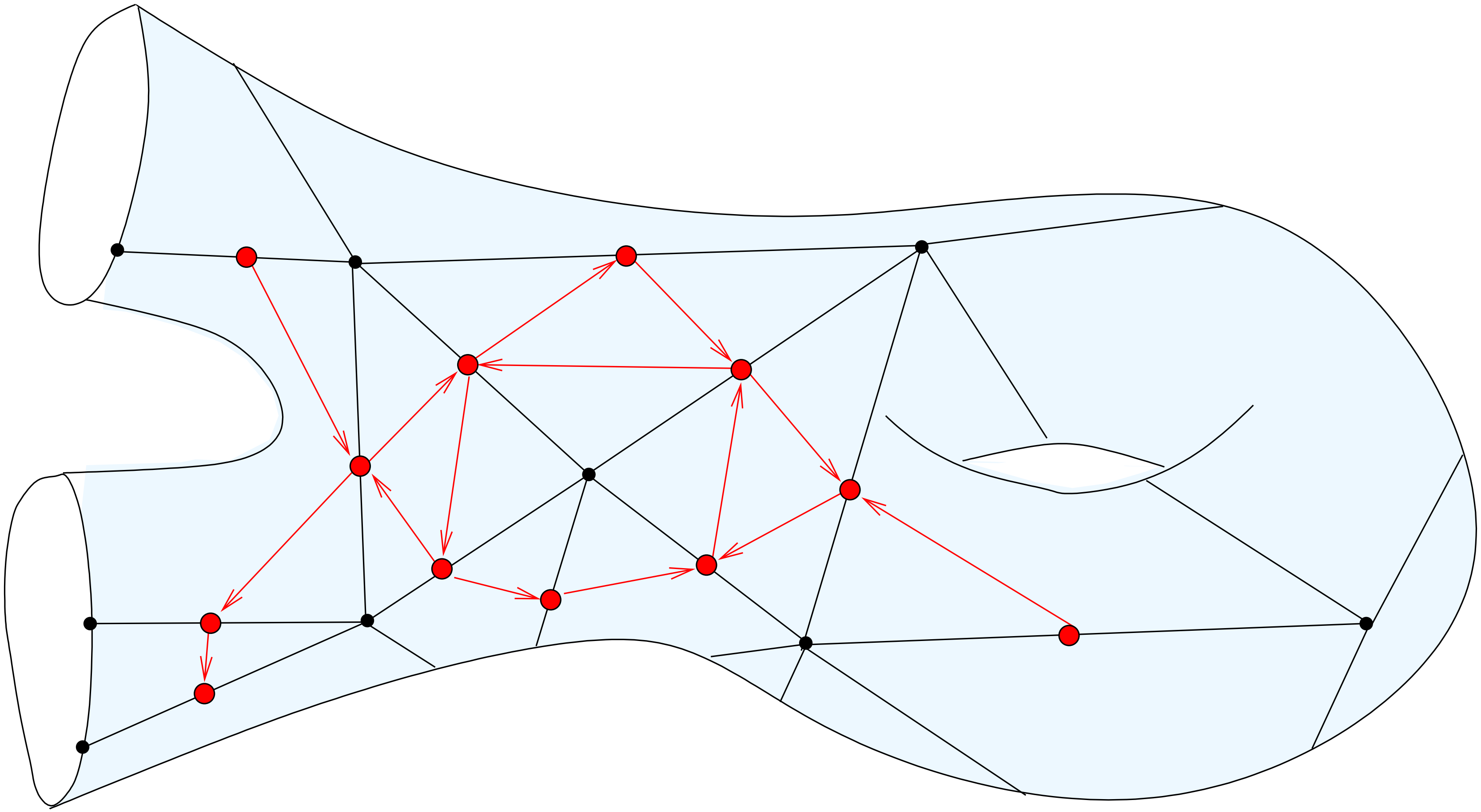

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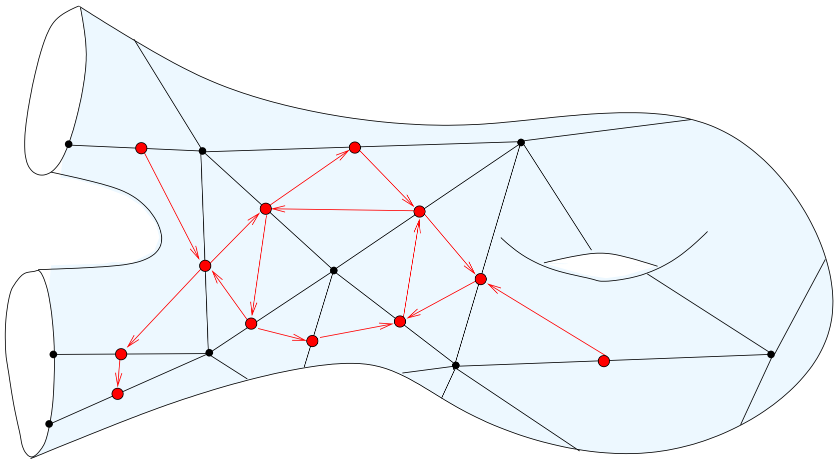
cluster variables



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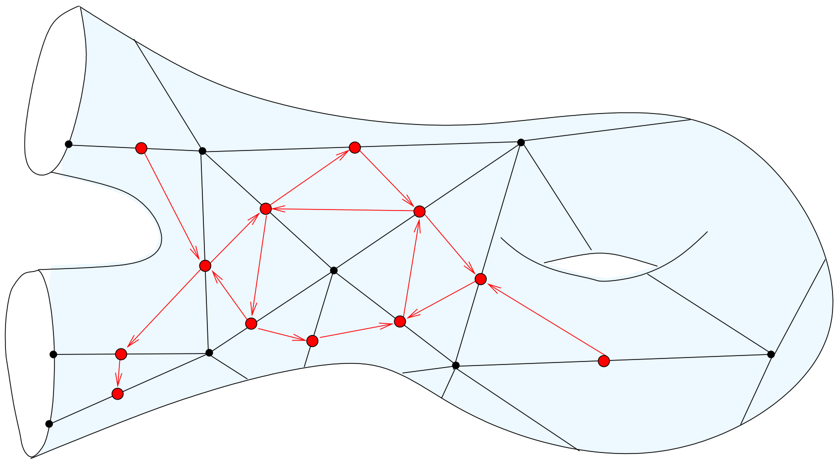

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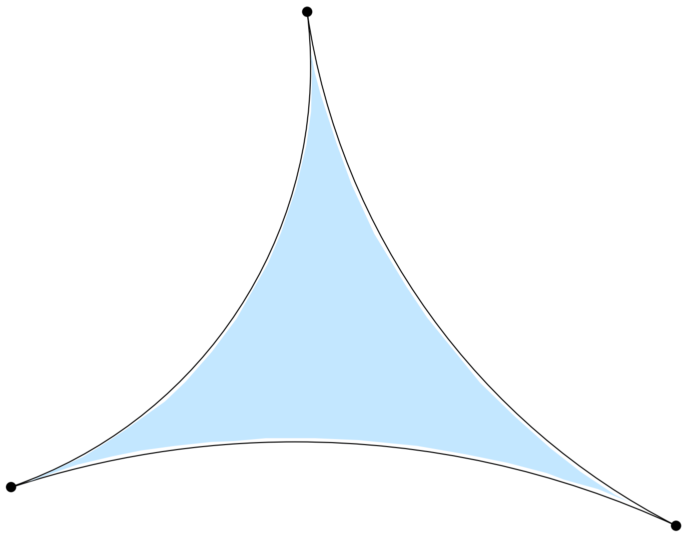

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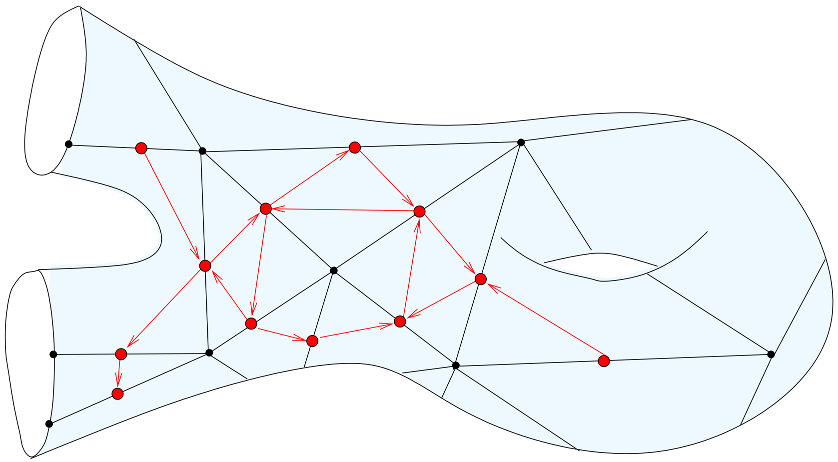

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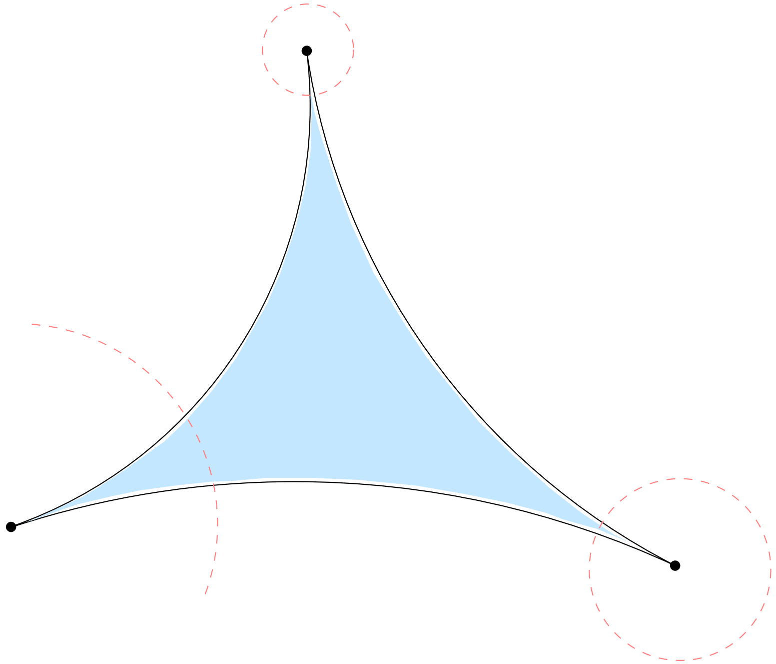

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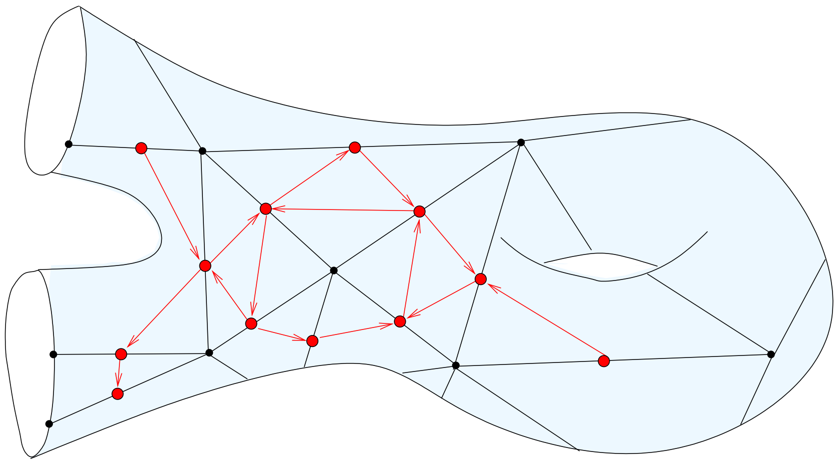

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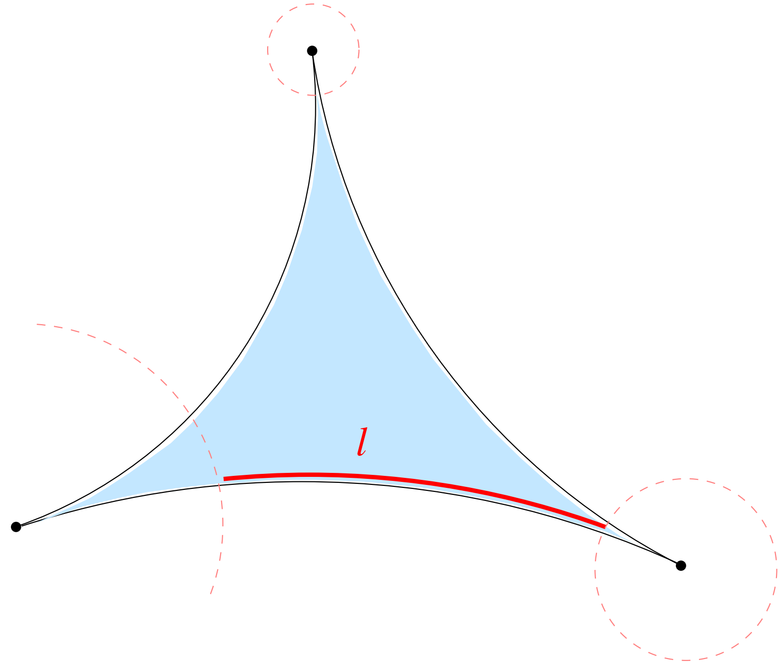
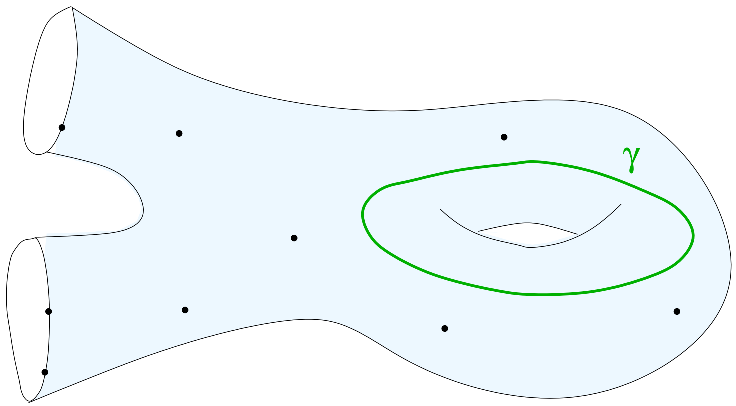
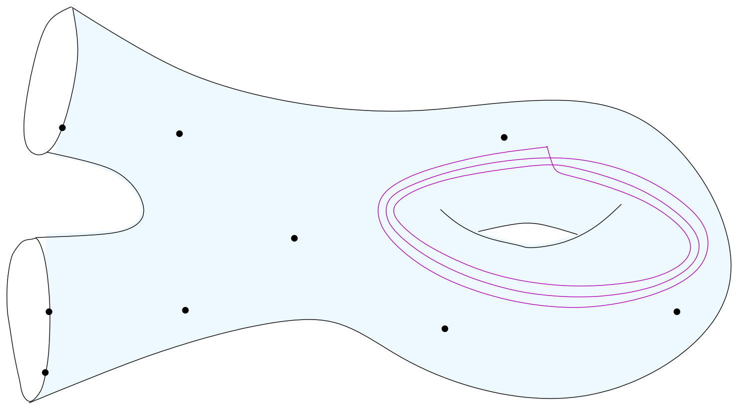


$x_\gamma =$ "λ-length of the arc γ" $= e^{l/2}$

1.2. Bases for algebras from surfaces
[Musiker, Schiffler, Williams'2011]

built 2 bases (Bangles and Bracelets). Bracelets:





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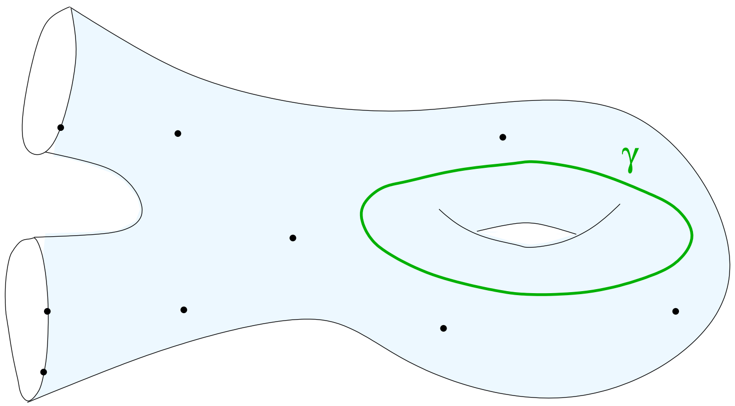

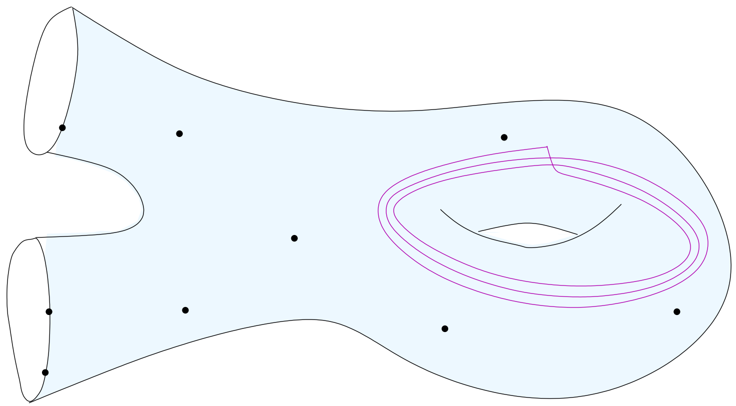

$Brac_3\gamma$

$$x_\gamma = e^{l/2} \qquad x_{\gamma,k} = T_k(x_\gamma)$$

$T_k(x)$ Chebyshev polynomial:

$$T_k(x) = xT_{k-1}(x) - T_{k-2}(x), \qquad T_0(x) = 2, \quad T_1(x) = x.$$

1.2. Bases for algebras from surfaces [Musiker, Schiffler, Williams'2011]

$x_\gamma = x_{\gamma,1}$

$x_{\gamma,k} \in \mathcal{A}$ (skein relations).

- Set of curves $\Sigma :=$ { arcs, bracelets }.
- Compatible subset $C \subset \Sigma$:
  - no two elements cross each other;
  - at most one $Brac_k\gamma$ for a given γ; at most one copy of it.

Bracelet basis: $\mathcal{B} = \{ \prod_{\gamma \in C} x_\gamma \mid C$ compatible$\}$.

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Thm.[MSW] If S is unpunctured, with at least two marked points (on boundary) then $\mathcal{B}$ is a basis for $\mathcal{A}_B$.

- All cluster monomials are in $\mathcal{B}$.
- positive: each elt has a positive Laurent expansion in each cluster;
- strictly positive: $q_1, q_2 \in \mathcal{B} \;\Rightarrow\; q_1 q_2 = \sum_{q_i \in \mathcal{B}} a_i q_i$ with $a_i \geq 0.$
  [Thurston'2013]
- conj. atomic: if $a \in \mathcal{A}^+$ then $a = \sum a_i q_i$ with $a_i \geq 0$

where $\mathcal{A}^+$ is the set of all elts which expand positively in each cluster

2.1. Cluster algebras from orbifolds
[F, Shapiro, Tumarkin' 2011]

Aim:

Algebras from surfaces = all but 11 mutationally finite skew-symmetric cluster algebras of rank > 2.

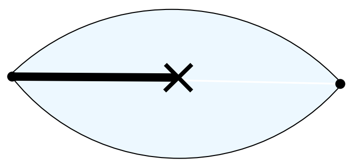
Orbifold construction = all but 22 mutationally finite cluster algebras of rank > 2.

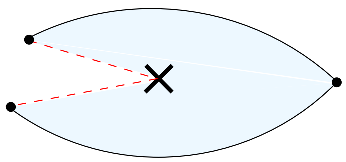
2.1. Cluster algebras from orbifolds

[F, Shapiro, Tumarkin' 2011]

$\mathcal{O}$ = surface with marked points and
orbifold points (i.e. cone points with engle π).

Triangulation: includes orbifold triangles





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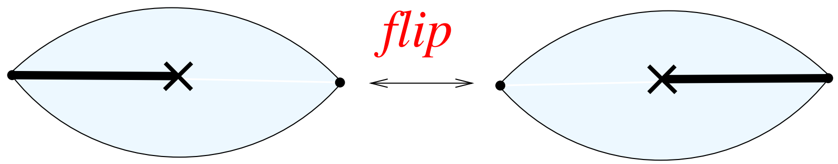
Triangulation: includes orbifold triangles 

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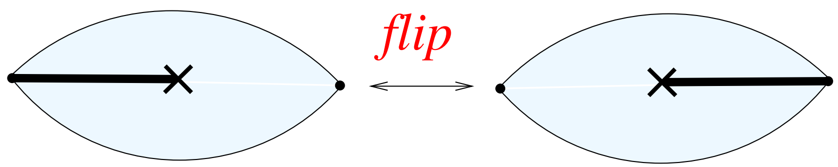
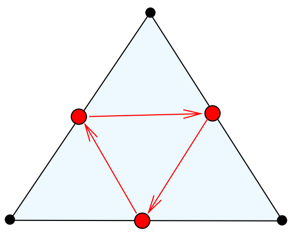
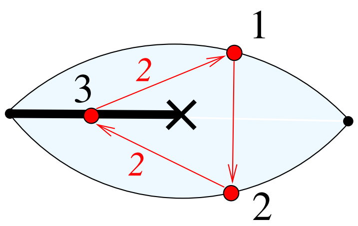


Adjacency matrix:



$$\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$



$$\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 2 & -2 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & 2 \\ 1 & -1 & 0 \end{pmatrix}$$

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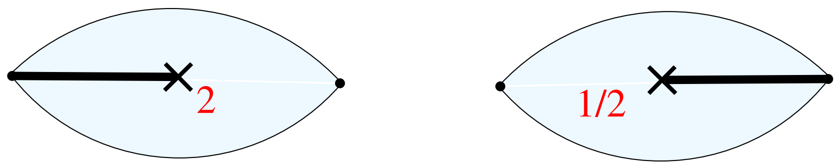

Adjacency matrix: depends on label $\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 2 & -2 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & 2 \\ 1 & -1 & 0 \end{pmatrix}$

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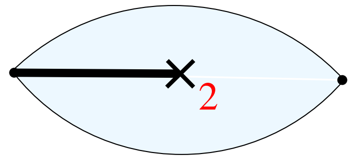

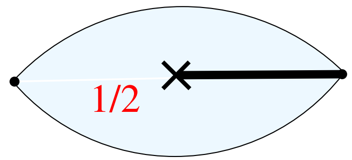

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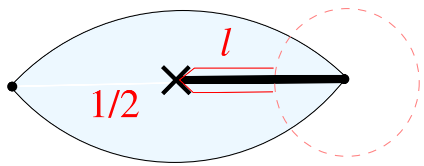
Cluster variables: λ-length of arcs.



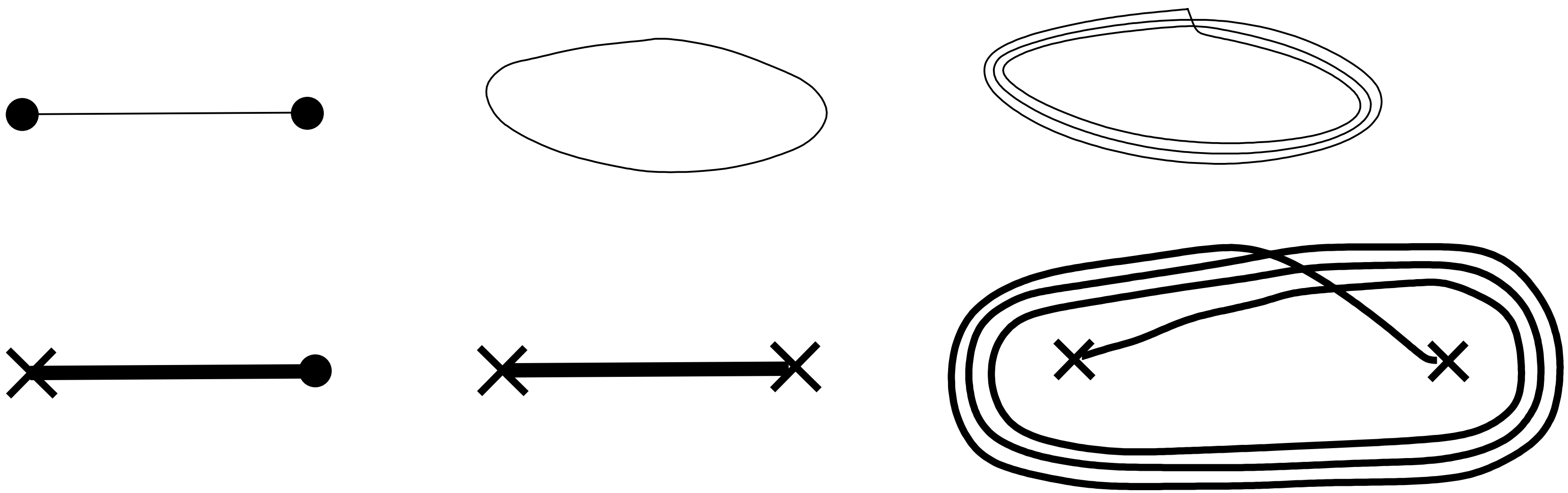
$x_\gamma = e^{l/2}$

For weight 2 orbifold points is more involved, discard for today.

2.2. Bracelet basis for orbifold case

$\mathcal{O}$: no punctures; all orbifold pts of wight $1/2$; ≥ 2 marked points.

Curves: arcs, pending arcs, bracelets, pending bracelets



Compatibility: no intersections; ≤ 1 (pending) bracelet for each γ

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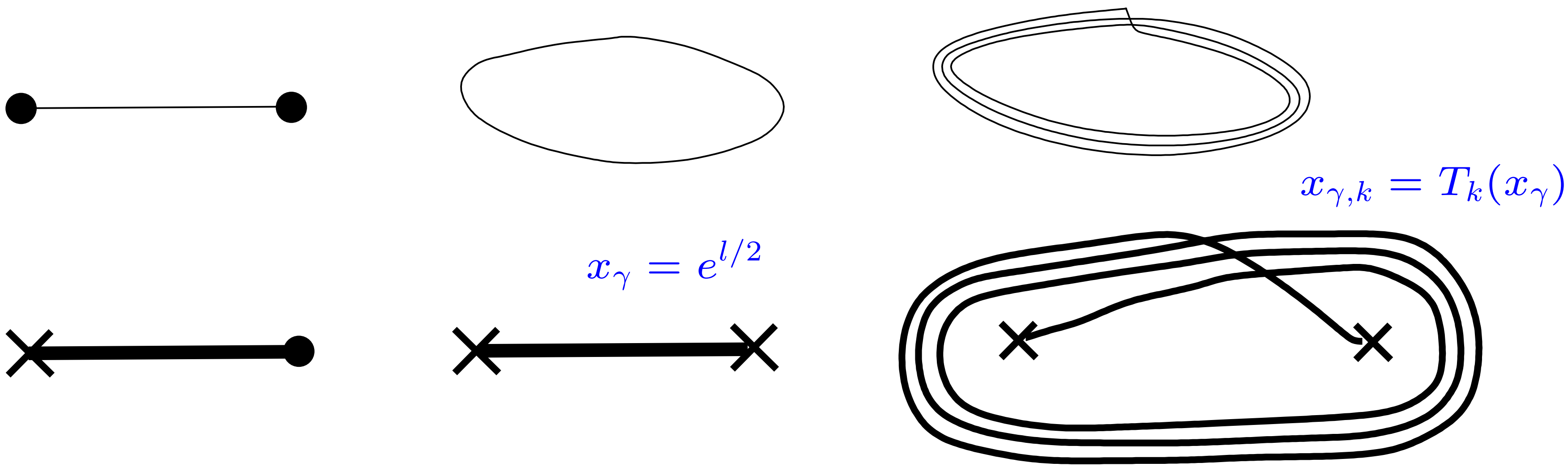

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3. Why does it work?

Need to prove:

1. $\mathcal{B} \in \mathcal{A}_B$
2. $\mathcal{B}$ spans $\mathcal{A}_B$
3. $\mathcal{B}$ is linearly independent

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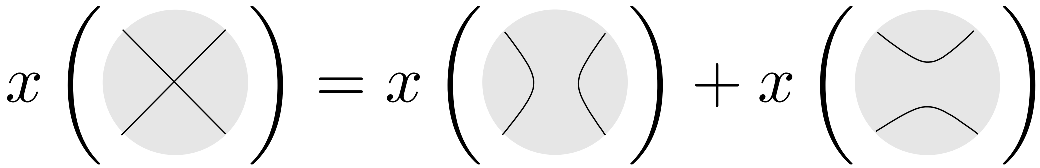
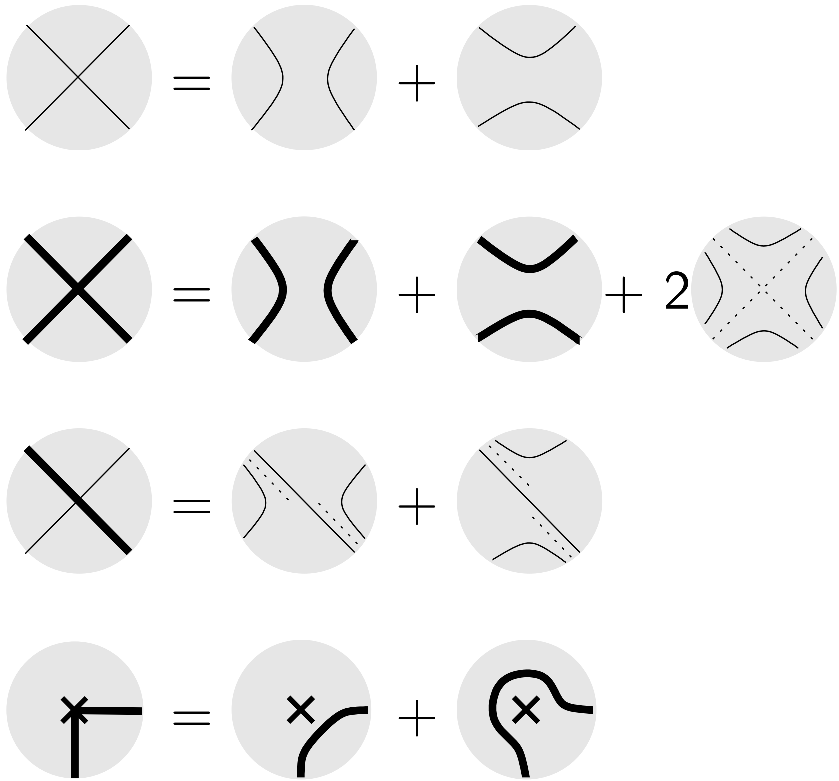
Skein relations: for a multicurve $C = \cup \gamma_i$ denote $x(C) = \prod x_{\gamma_i}$, then

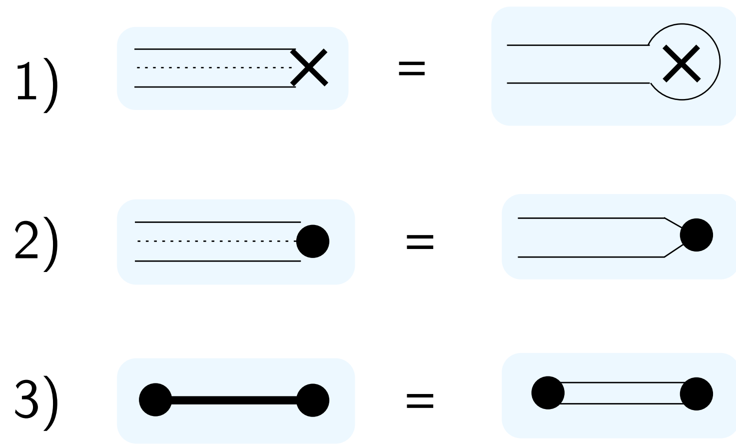


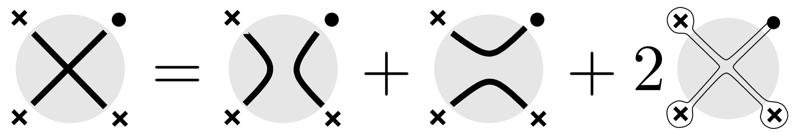
Pf in [MSW]: involves technique of snake graphs.

3.1. Skein relations for orbifolds



where



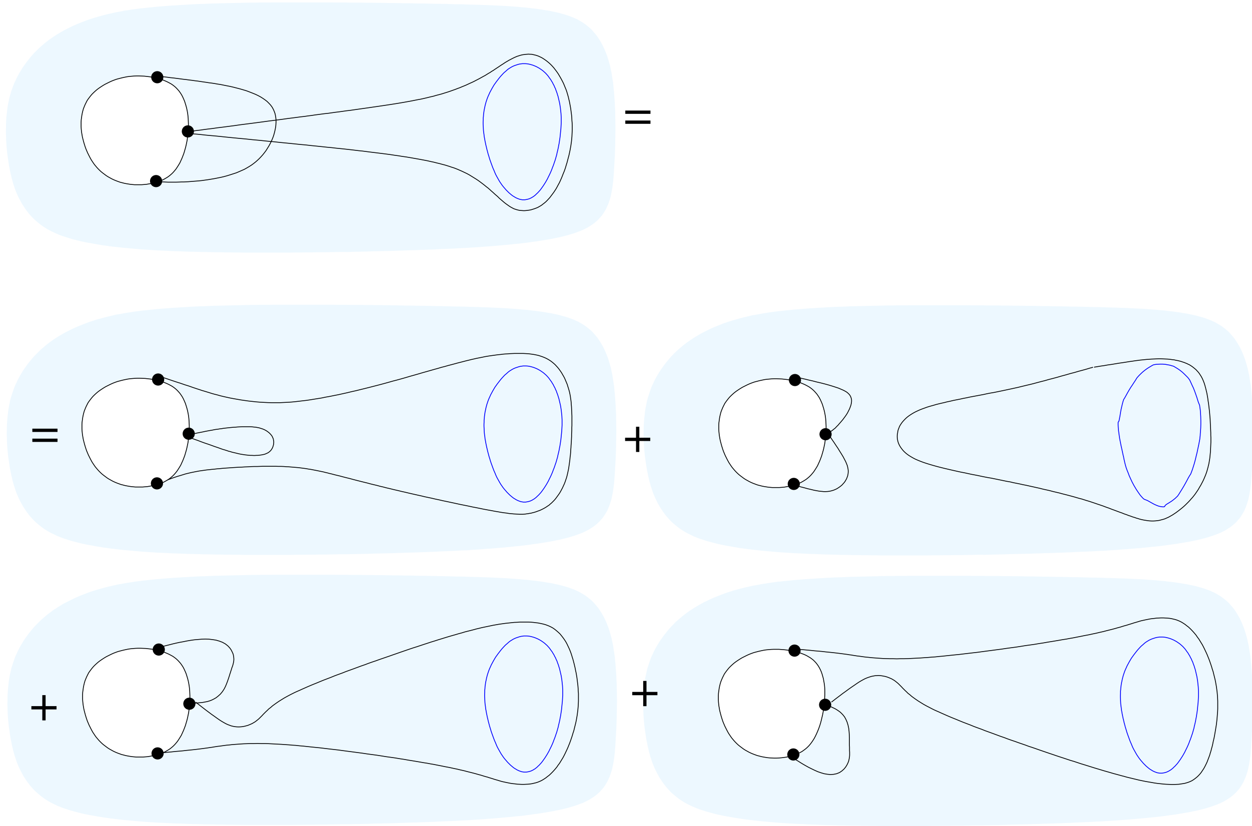
Example: 

Pf: from the double cover of the orbifold by a surface.

3.1. Skein relations for orbifolds

How does it help:

1. $\underline{\mathcal{B} \subset \mathcal{A}_B}$: $\qquad x_\gamma \overset{?}{\subset} \mathcal{A}_B$



2. $\underline{\mathcal{B} \text{ spans } \mathcal{A}_B}$: $\qquad x_1 \dots x_k \overset{?}{=} \sum q_i, \;\; q_i \in \mathcal{B}$

(Yes: <u>resolve crossings</u>, get the sum!)

3.2. Linear independance: $\mathbf{g}$-vectors

[MSW]:

- elements with distict $\mathbf{g}$-vectors are linear independent;
  Pf: modification of arguments from [FZ4].
- elements of $\mathcal{B}$ have distinct $\mathbf{g}$-vectors
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Geometric description of $\mathbf{g}$-vectors: tropical duality of Nakanishi-Zelevinsky:

$$(G_t^{B;t_0})^T = C_{t_0}^{B_t^T;t}$$

$$\mathbf{g}\text{-vector} \xleftrightarrow{NZ} \mathbf{c}\text{-vector} \xleftrightarrow{FG} \text{laminations}$$

Elementary laminations for distinct multicurves	$\longrightarrow$	distinct shear coordinates	$\longrightarrow$	lin. independence

3.2. Linear independance: $\mathbf{g}$-vectors

More precisely, $\mathbf{g}(x_\gamma) = -b_{T^*}(L^*_\gamma)$, where

1. to obtain T^* we take an initial triangulation T on $\mathcal{O}$ and turn the triangulated orbifold inside-out; denote $\mathcal{O}^*$ the inside-out orbifold.

2. take an elementary lamination L^*_γ for $\gamma^* \in \mathcal{O}^*$ (L^*_γ is an image of $-L_\gamma$, negative of the elementary lamination on $\mathcal{O}$).

3. $\mathbf{g}(x_\gamma) =$ shear coordinates of L^*_γ in T^*.

4. Thm. $L \to b_\gamma(T, L)$ is a bijection to $\mathbf{Z}^n$.
([Fomin, Thurston] surface case; [F, Shapiro, Tumarkin] orbifold case)

Thanks!