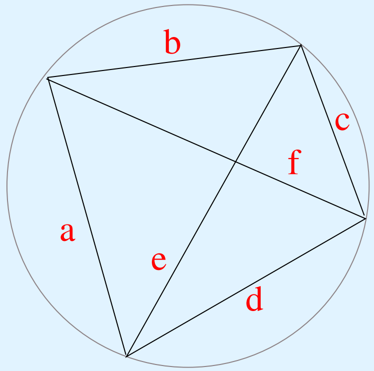
Introduction to cluster algebras

Anna Felikson

Durham University

$$ef = ac + bd$$



LMS Graduate Student Meeting, London
June 29, 2018

Cluster algebras

(Fomin, Zelevinsky, 2001)

Andrei Zelevinsky

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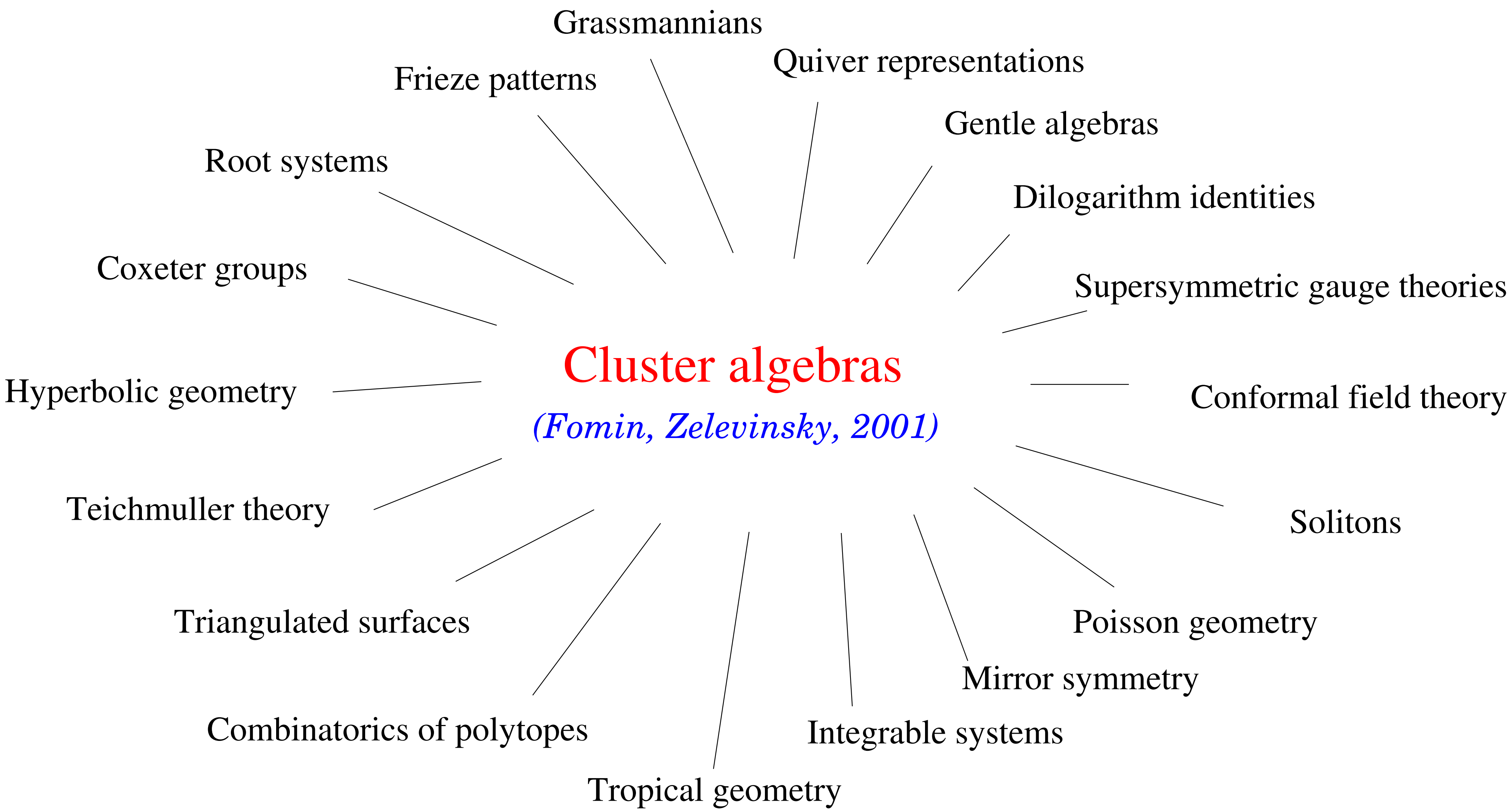
Sergey Fomin


Cluster algebras
(Fomin, Zelevinsky, 2001)
Grassmannians
Quiver representations
Frieze patterns
Gentle algebras
Root systems
Dilogarithm identities
Coxeter groups
Supersymmetric gauge theories
Hyperbolic geometry
Conformal field theory
Teichmuller theory
Solitons
Triangulated surfaces
Poisson geometry
Mirror symmetry
Combinatorics of polytopes
Integrable systems
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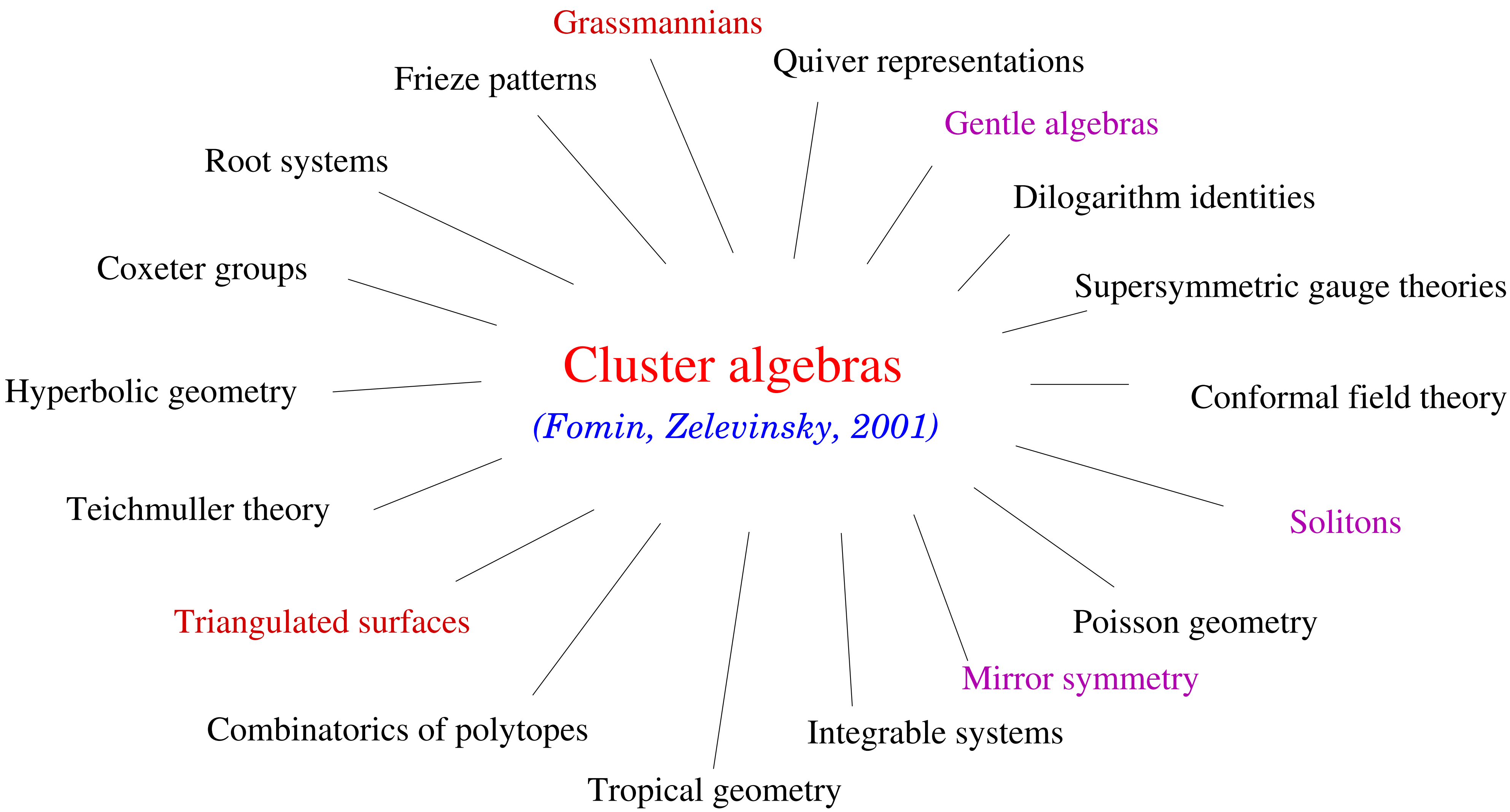
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$d = \frac{(ad-bc)+bc}{a}$

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3×3 matrix: out of 19 only need 9.

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In general [Fomin, Zelevinsky' 1999]: need n^2.
Can go from one test to another by local moves:
each time substituting one of the functions by a new one.

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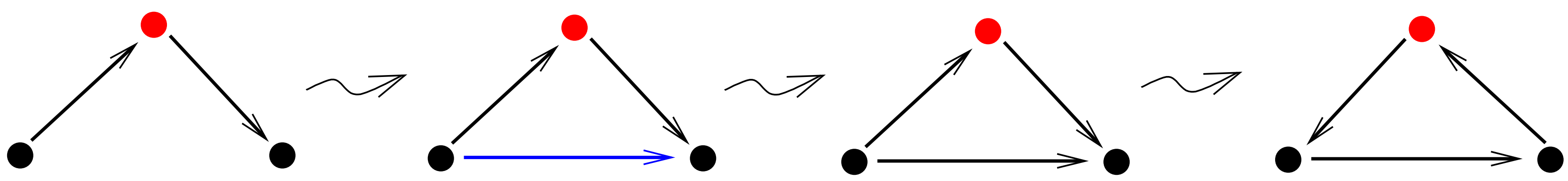

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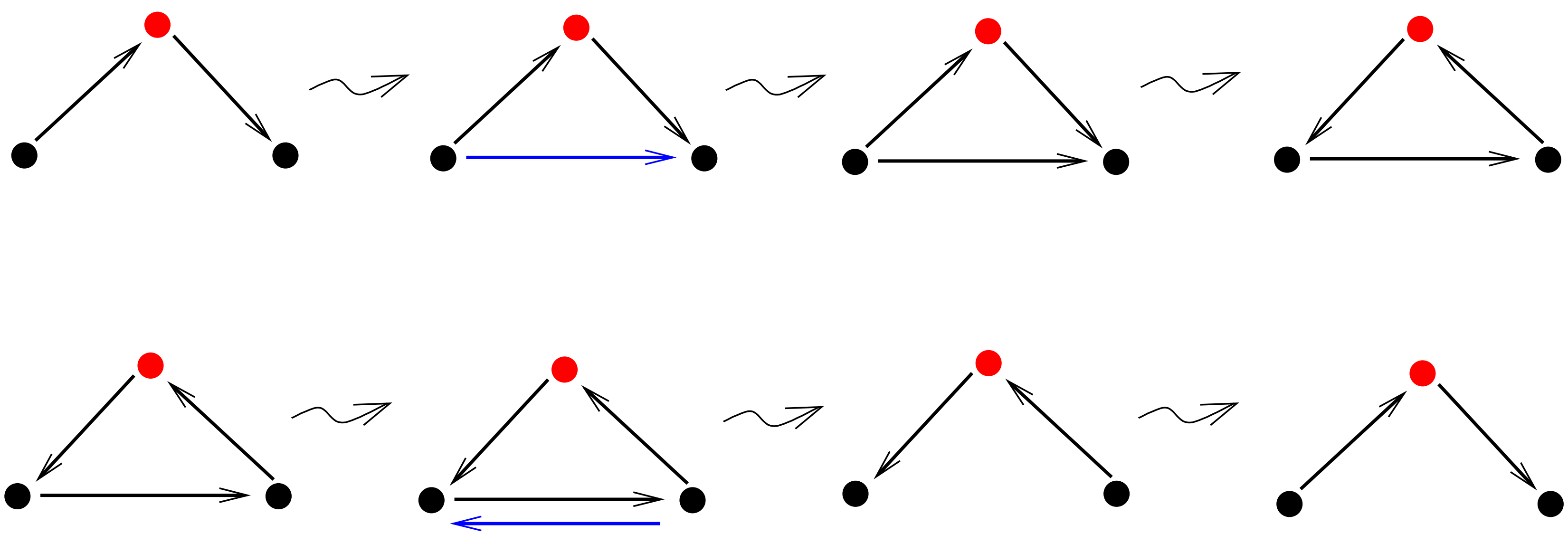

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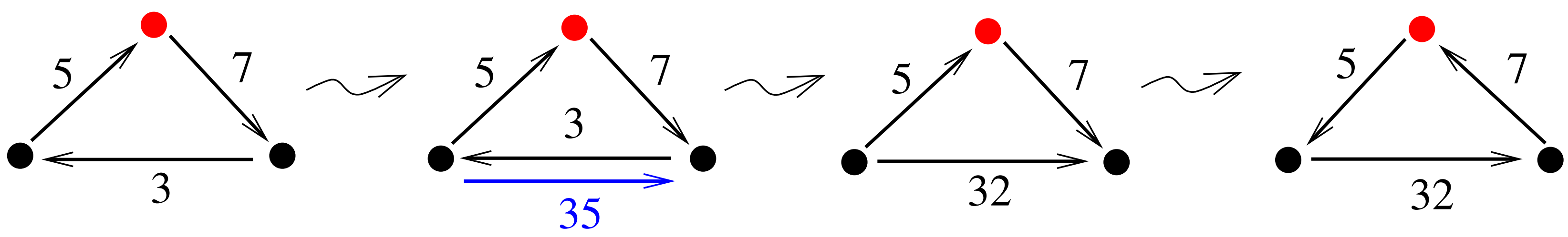

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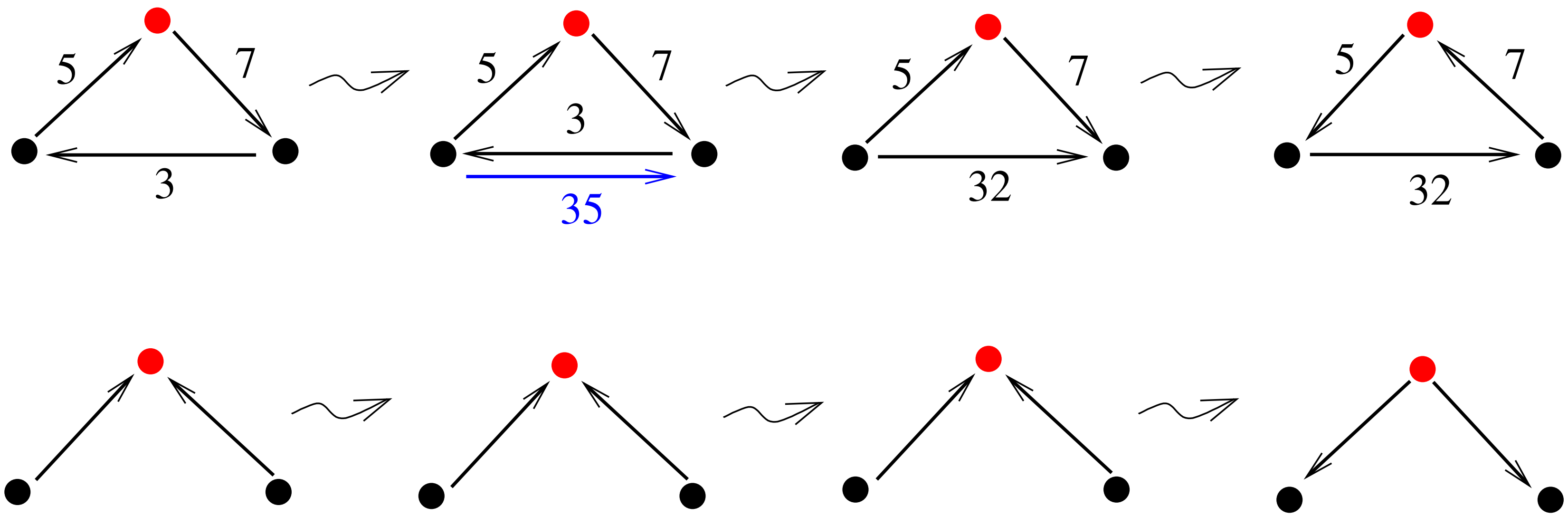

1. Cluster algebras: Seed mutation

A seed s is a pair $(Q, \{u_1, \dots, u_n\})$ where $u_i = u_i(x_1, \dots, x_n)$,

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Mutation: $\mu_k(s) = (\mu_k(Q), \{u_1, \ldots, u'_k, \ldots, u_n\})$

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Functions u_i are called cluster variables.

Definition. Let $\mathcal{F}$ be the set of all rational functions in $x_1, \ldots, x_n$.

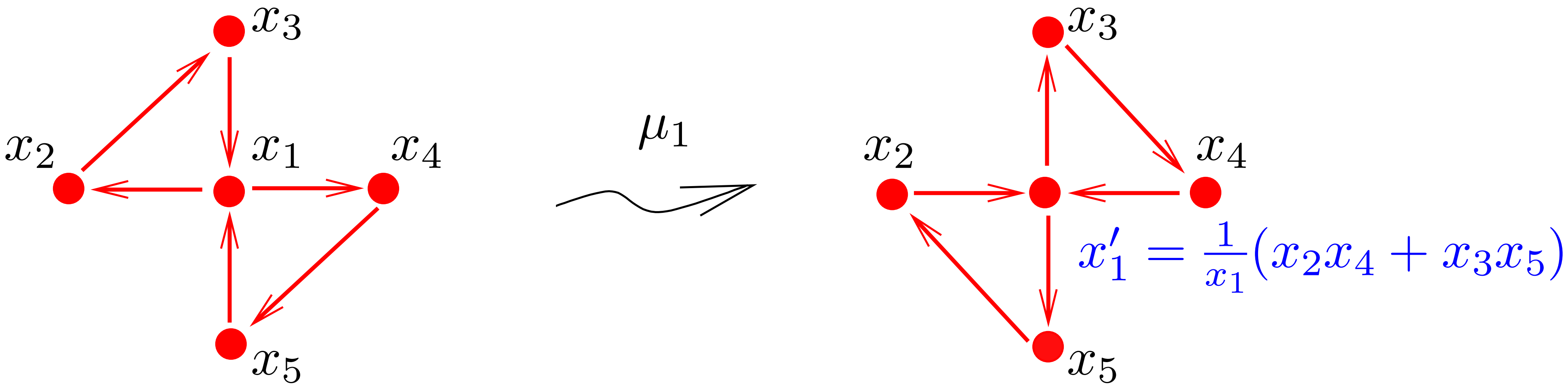
The cluster algebra $\mathcal{A}(Q)$ is the subring of $\mathcal{F}$

generated by all cluster variables.

1. Cluster algebras: Seed mutation

How to think about this?

Example:



Cluster variables: $x_1, x_2, x_3, x_4, x_5, x_1', \ldots$

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By definition $u_i = \frac{P(x_1,\ldots,x_n)}{R(x_1,\ldots,x_n)}$, where P and R are polynomials.

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- Laurent phenomenon [Fomin, Zelevinsky' 2001]:
  $R(x_1,\ldots,x_n)$ is a monomial, $R = x_1^{d_1}\ldots x_n^{d_n}$.

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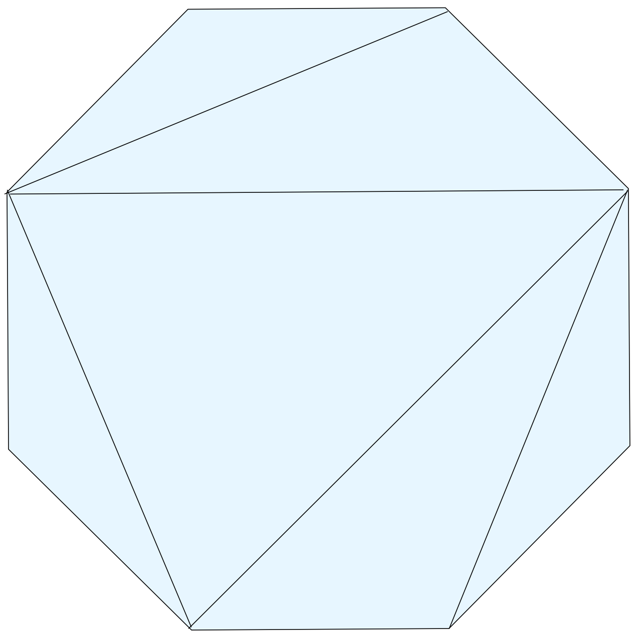
- Positivity [Conj.: Fomin, Zelevinsky' 2001; proved: Lee, Schiffler' 2013]:
  $P(x_1,\ldots,x_n)$ has positive coefficients.

  It is a miracle as we divide: $\frac{a^3+b^3}{a+b} = a^2 - ab + b^2$.

2. Triangulated polygons

Triangulated polygon $\longrightarrow$ Quiver Q



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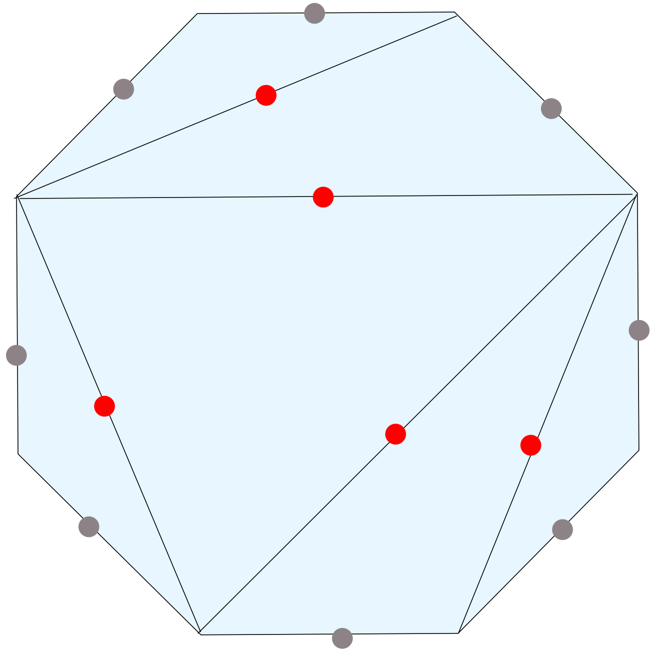
Triangulated polygon	$\longrightarrow$	Quiver Q
diagonal in triangulation		vertex of quiver



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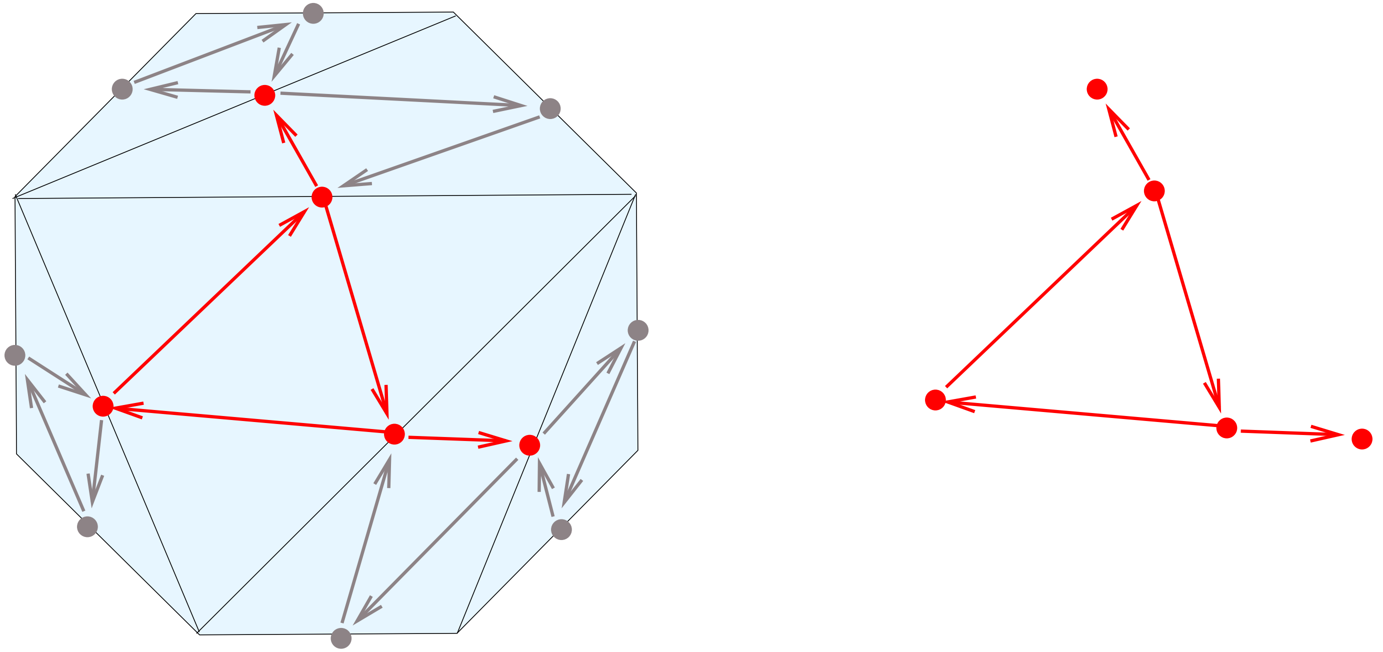

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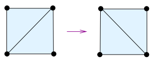
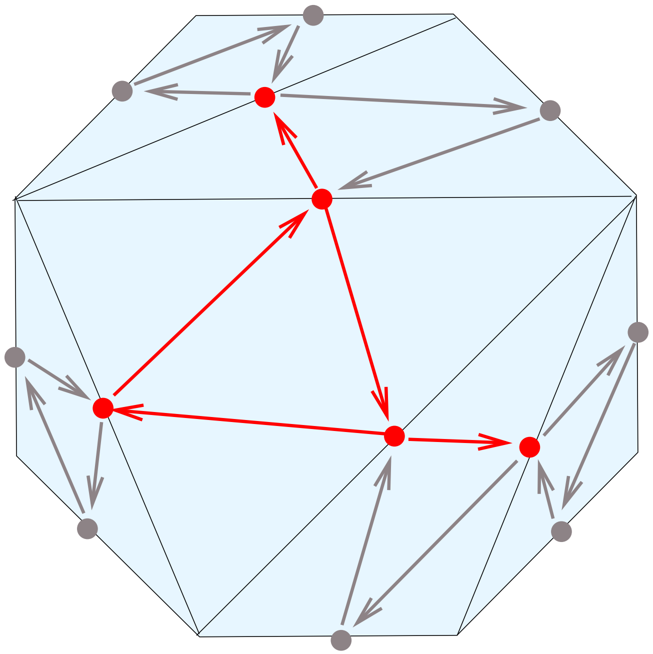
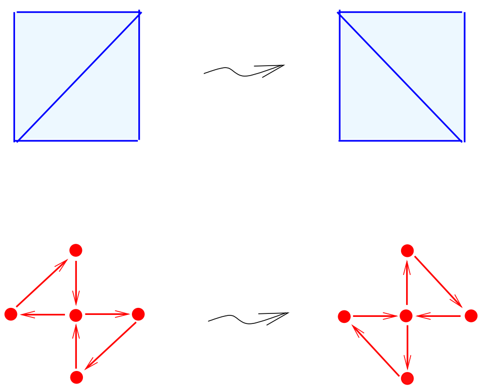
flip of triangulation 		mutation of quiver





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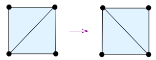
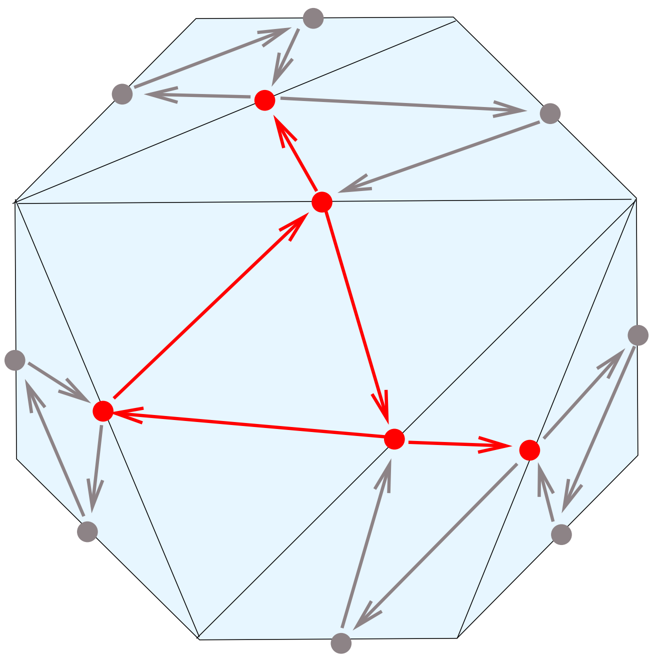
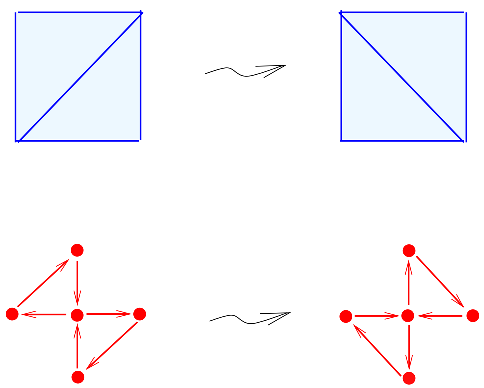
flip of triangulation 		mutation of quiver

$\longrightarrow$ Cluster algebra $\mathcal{A}(Q)$



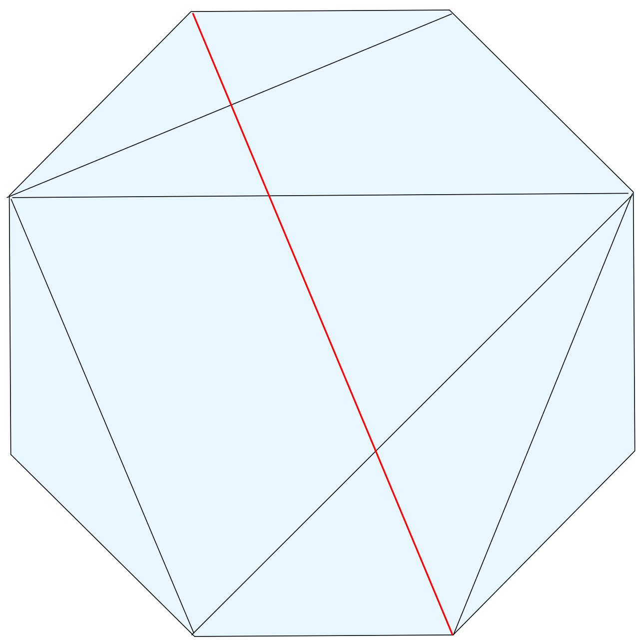


Label vertices of Q by x_i, transform as in $x_1' = \frac{x_2 x_4 + x_3 x_5}{x_1}$ under mutations.

2. Triangulated polygons

Triangulated polygon $\longrightarrow$ Quiver Q $\longrightarrow$ Cluster algebra $\mathcal{A}(Q)$

By sequence of flips can take any triangulation to any other.



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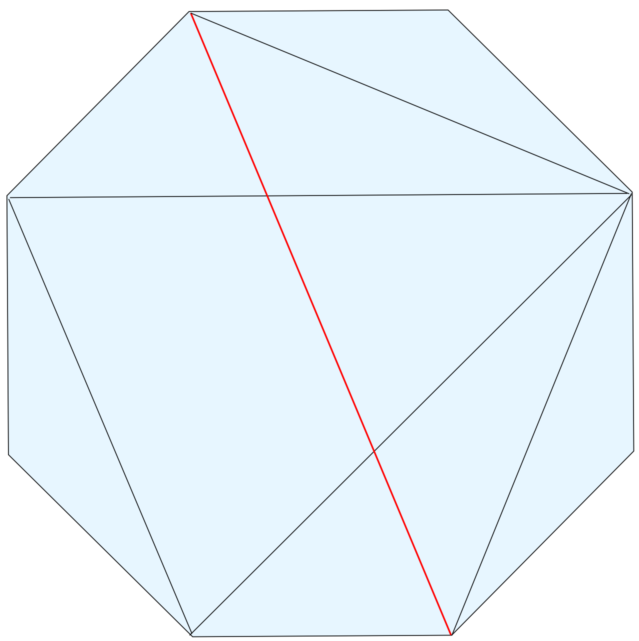

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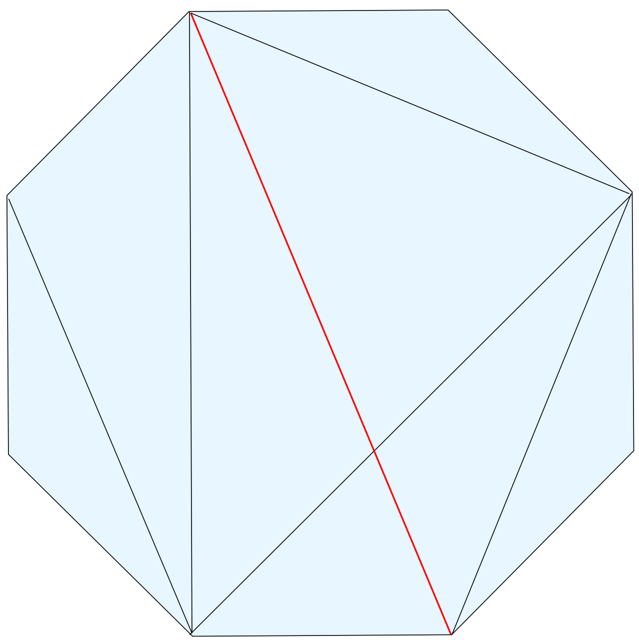

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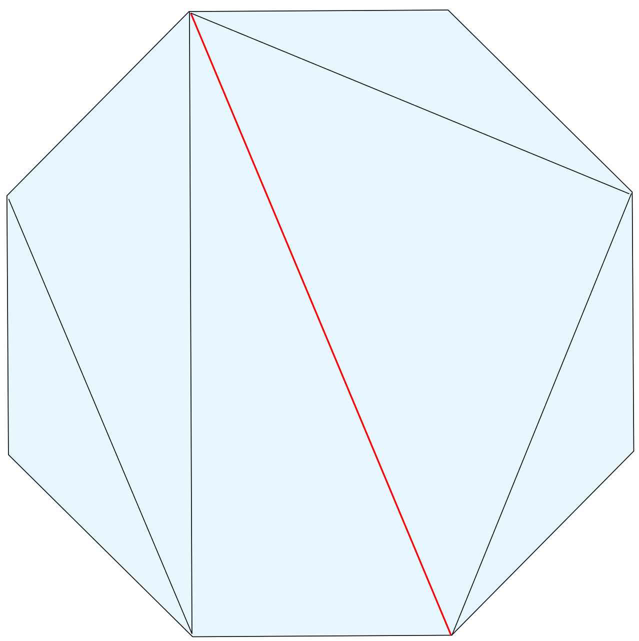

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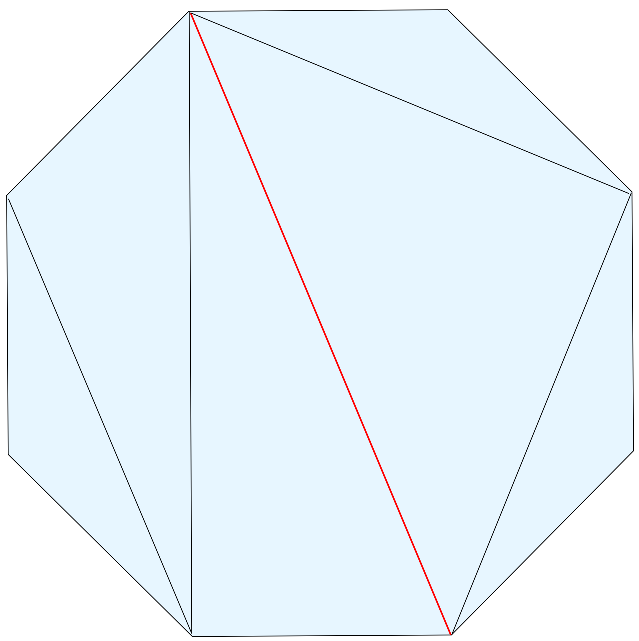

Triangualtions $\leftrightarrow$ Seeds of $\mathcal{A}(Q)$

diagonals $\leftrightarrow$ cluster variables

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Remark: Transformation $x_1' = \frac{x_2x_4+x_3x_5}{x_1}$ makes sense for lengths of diagonals in $\mathbb{E}^2$:

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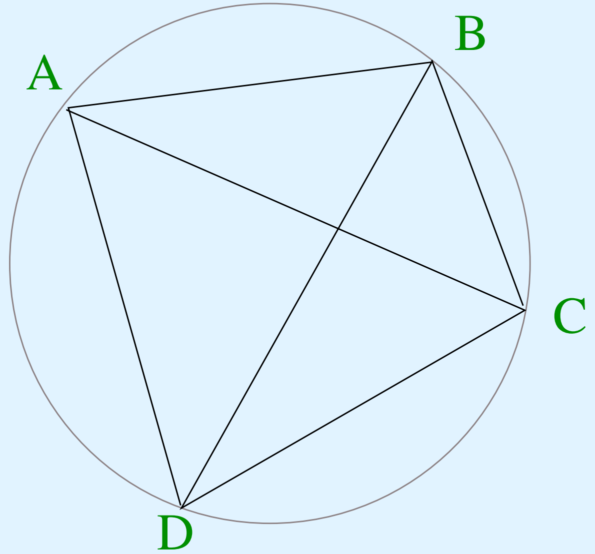
Given an inscribed quadrilateral $ABCD \subset \mathbb{E}^2$,

$$AC \cdot BD = AB \cdot CD + BC \cdot AD$$



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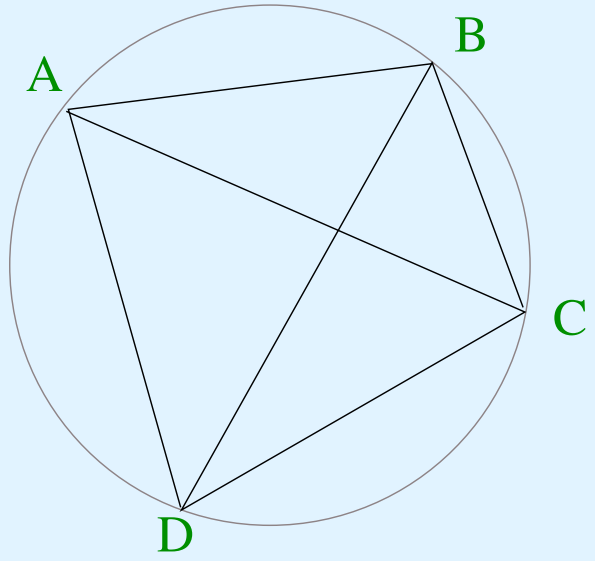

The lengths of diagonals in an inscribed polygon are not independent variables.
This can be fixed by using hyperbolic geometry instead of Euclidean.

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Grassmannian $\quad Gr_{k,n}(\mathbb{R}) = \{V \mid V \in \mathbb{R}^n,\ dim\ V = k\}$
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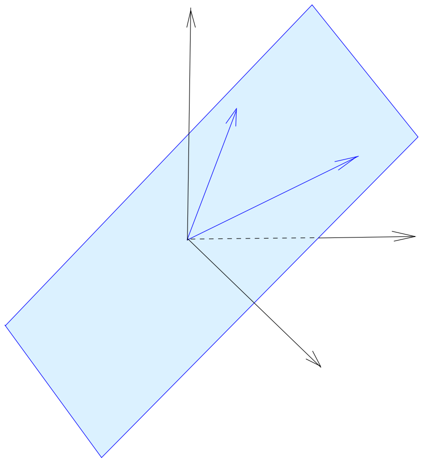
$k = 2$: $\quad Gr_{2,n} = \{\ A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \end{pmatrix} \mid rk\ A = 2\}$.



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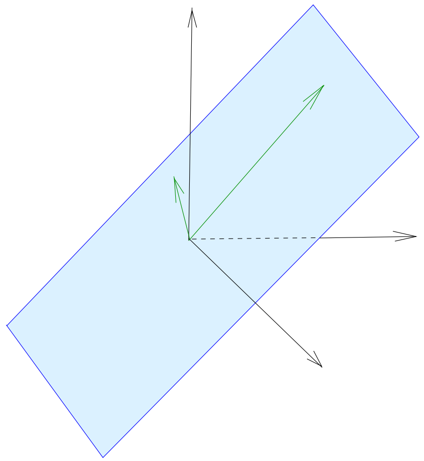

This representation is not unique: can take another basis of the 2-plane.

So, $Gr_{2,n} = Mat_{2,n}/\sim$, $\quad$ where $\sim$ stays for changes of basis in $\mathbb{R}^2$.

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Determinants $\Delta_{ij} = \begin{vmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{vmatrix}$ are called Plücker coordinates.

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This represents a point with $\Delta_{12} \neq 0$ uniquely.

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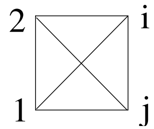

3. Grassmannians $Gr_{2,n} = \{\text{2-planes in } \mathbb{R}^n \}$

Determinants $\Delta_{ij} = \begin{vmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{vmatrix}$ are called Plücker coordinates.

If $\Delta_{12} \neq 0$ then $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} \begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \end{pmatrix} = \begin{pmatrix} 1 & 0 & c_3 & \dots & c_n \\ 0 & 1 & d_3 & \dots & d_n \end{pmatrix}$

This represents a point with $\Delta_{12} \neq 0$ uniquely.

Moreover: $c_i = -\frac{\Delta_{2i}}{\Delta_{12}}, \quad d_i = \frac{\Delta_{1i}}{\Delta_{12}}.$

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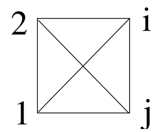
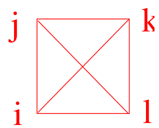


Similarly, $\Delta_{ik}\Delta_{jl} = \Delta_{ij}\Delta_{kl} + \Delta_{il}\Delta_{jk}$ for $i < j < k < l$.



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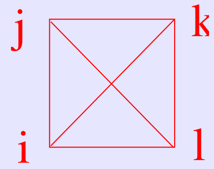

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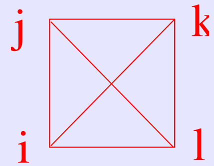

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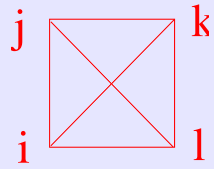

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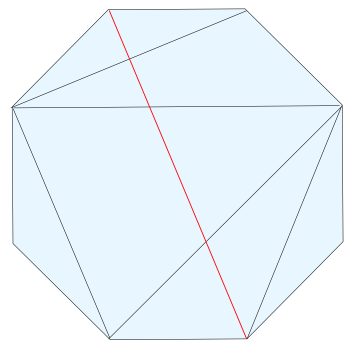
Take any triangulation T of an n-gon.

Then $\{\Delta_{ij} \mid ij = \text{ side or diagonal in } T\}$ is sufficient to find all Δ_{lk}:

apply Plücker (=Ptolemy) relations to resolve crossings!



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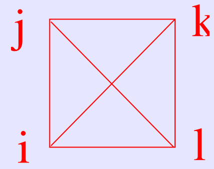

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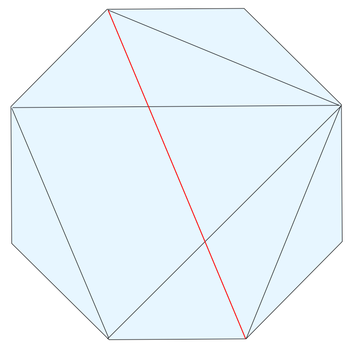

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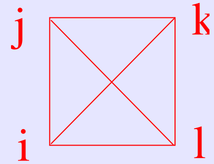

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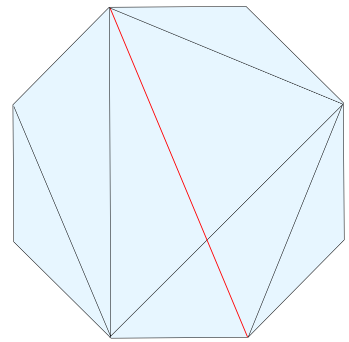

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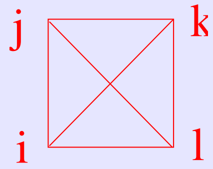

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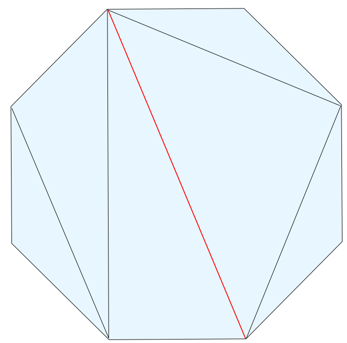

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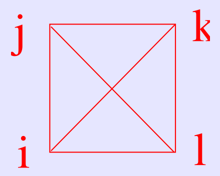

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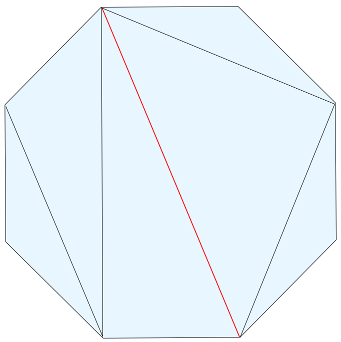
Get a cluster algebra structure on $Gr_{2,n}$ with

cluster variables $\leftrightarrow$ diagonals of n-gon

seeds $\leftrightarrow$ triangulations of n-gon



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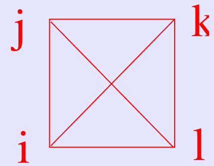

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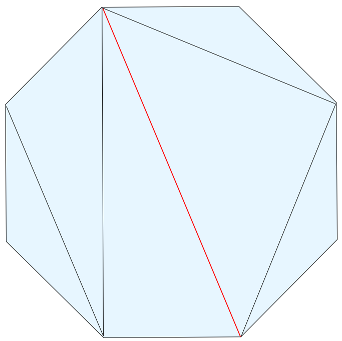

A totally positive Grassmannian $Gr^{tp}_{2,n}$ is a subset of $Gr_{2,n}$
where $\Delta_{ij} > 0$ for all i, j.

By positivity of cluster variables,
only need to check initial variables.

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A cluster algebra is of finite type if it contains finitely many cluster variables.

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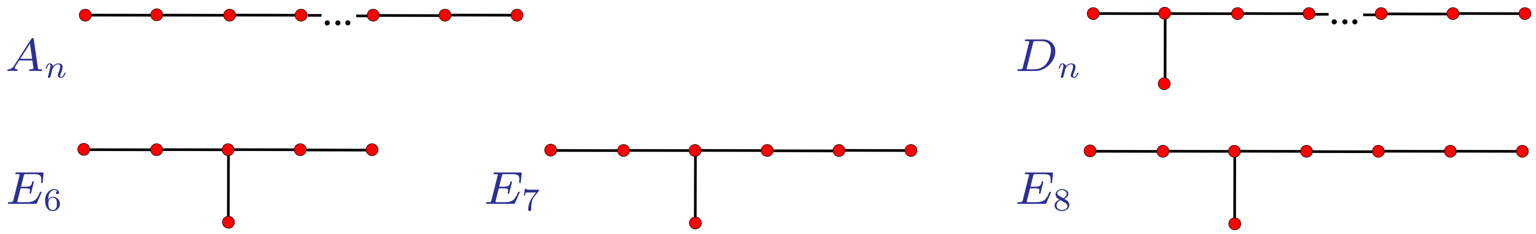

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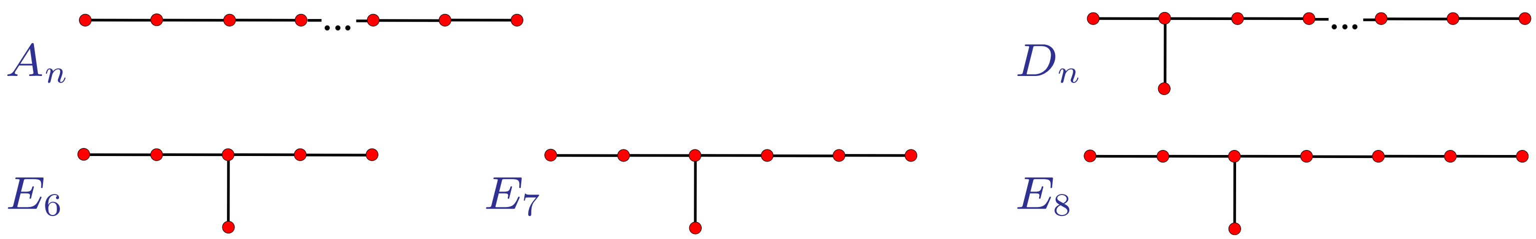

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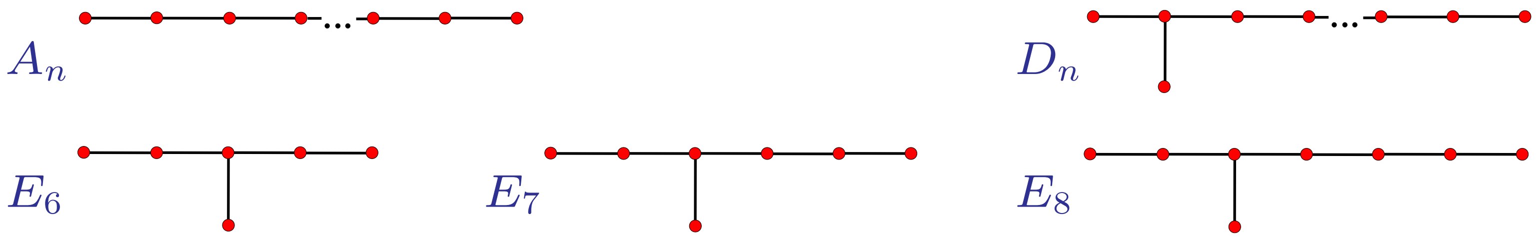

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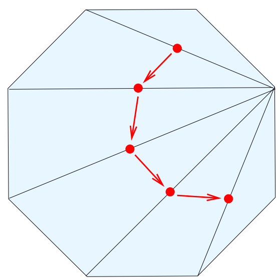
Cluster algebra related to
an n-gon and Grassmannian $G_{2,n}$
is of type A_n:



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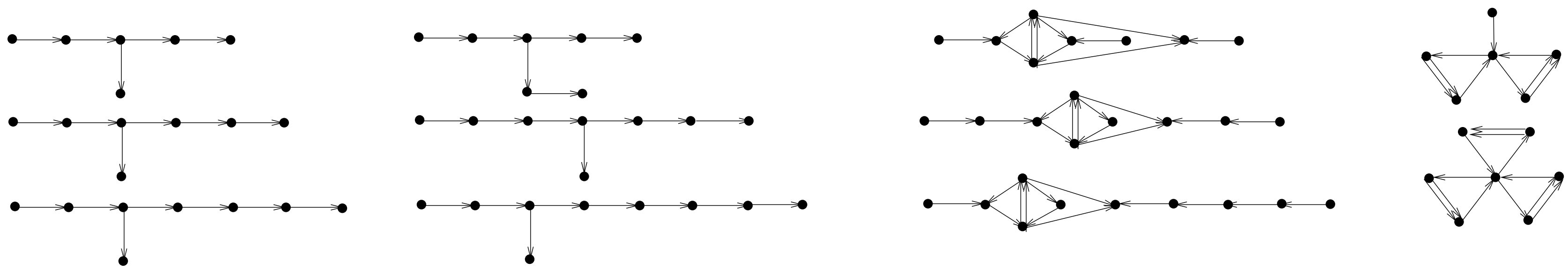
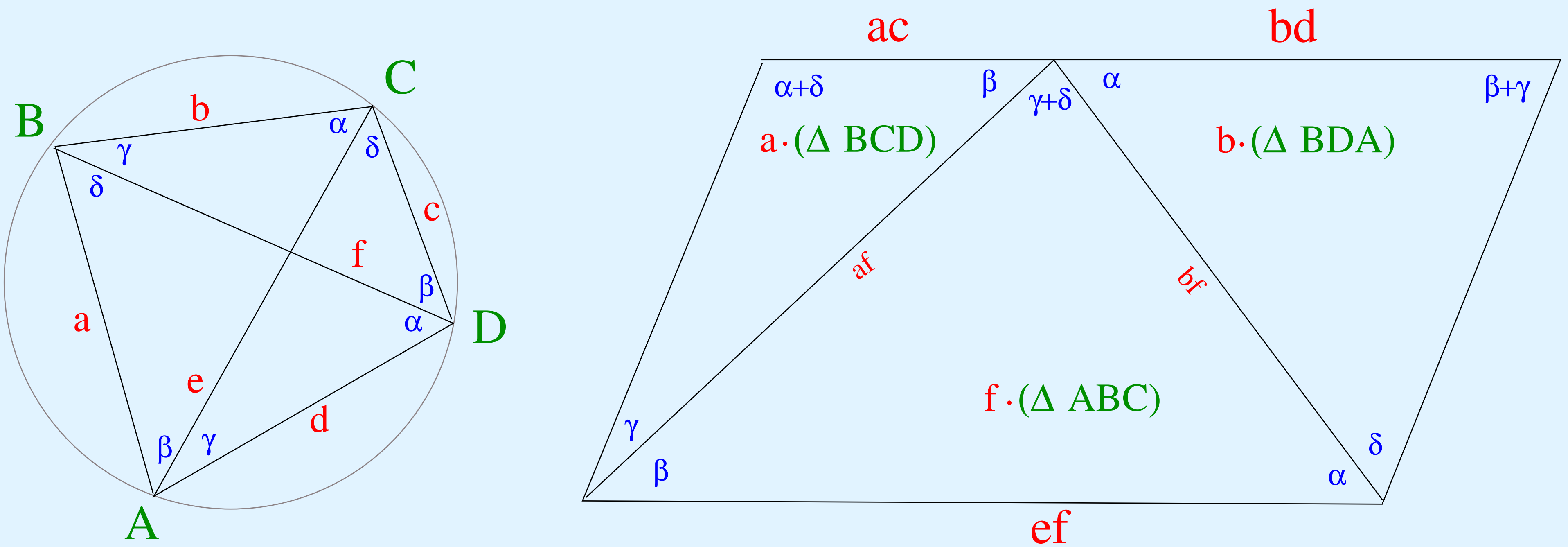
Theorem [A.F, M.Shapiro, P.Tumarkin' 2008]
A connected quiver Q s.t. $|Q| > 2$ is of finite mutation type iff
- either Q is obtained from a triangulated surface;
- or Q is mut.-equivalent to one of the following 11 quivers:



5. Bonus: proof of Ptolemy theorem

ef=ac+bd



(Proof borrowed from cut-the-knot portal)

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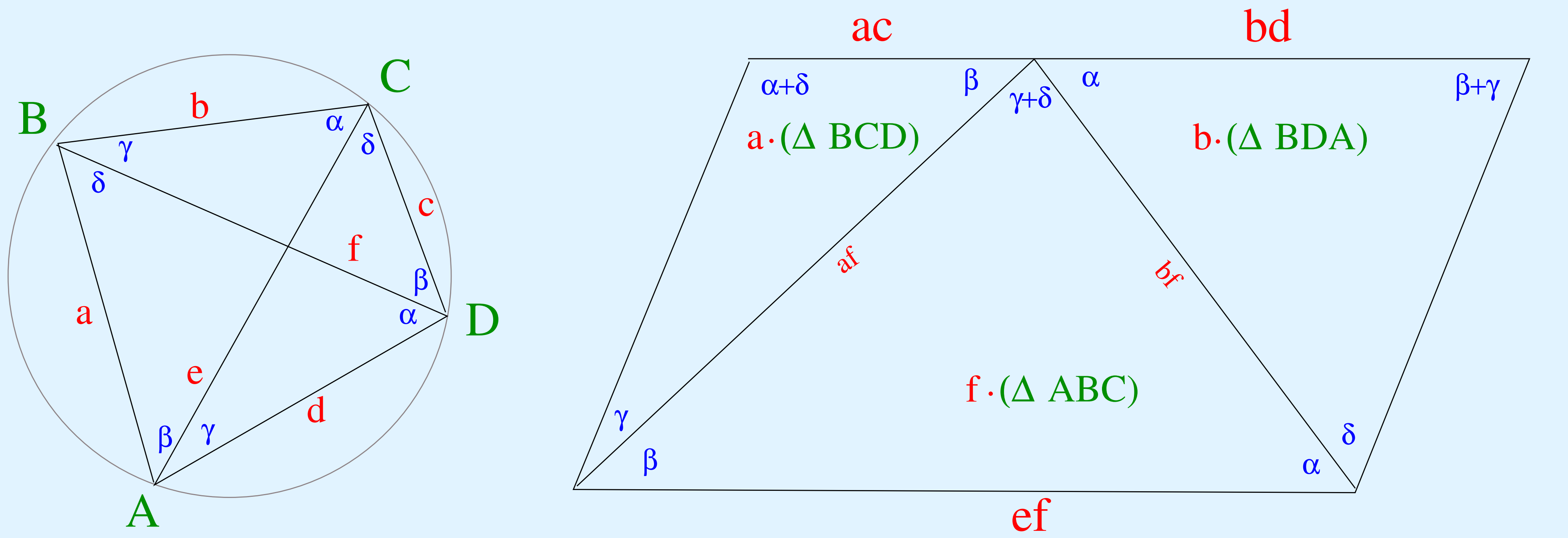

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Thanks!