Combinatorics of Coxeter polytopes

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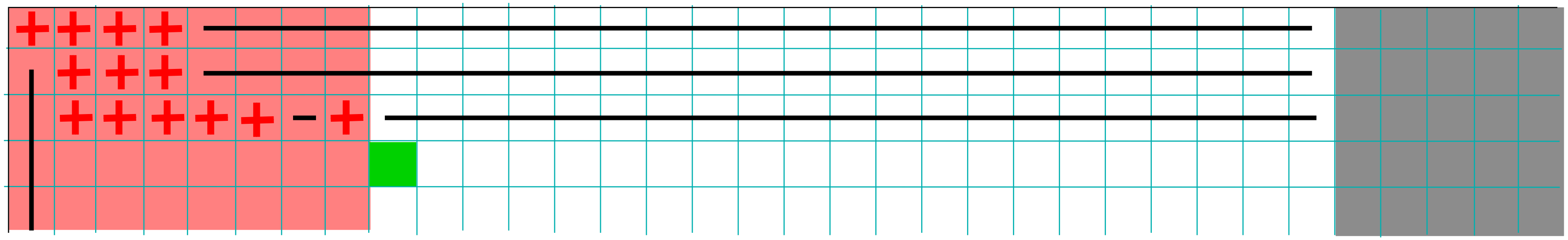
$P \in \mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$ is a Coxeter polytope
if all its dihedral angles are submultiples of π.

Example: Euclidean Coxeter triangles.

- $P \in \mathbb{S}^n$. Classified (Coxeter, 1934).
- $P \in \mathbb{E}^n$. Classified (Coxeter, 1934).
- $P \in \mathbb{H}^n$. No classification.

Compact hyperbolic Coxeter polytopes

- Simple polytopes.
- Do not exist for $dim > 29$ (Vinberg '84).
- Examples known only for $dim \leq 8$.
- Cases $n = dim + 1, dim + 2, dim + 3$ are classified (where n is the number of facets of P).



9-polytopes with 13 facets

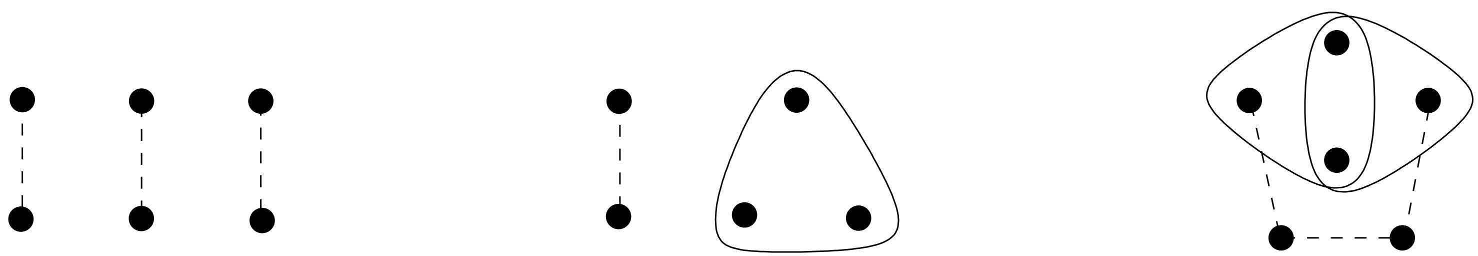
- Finitely many combinatorial types.

- Given a combinatorial type, may try to "reconstruct" the polytope (i.e. to find its dihedral angles).

Combinatorics:	Dihedral angles:
Diagram of missing faces	Coxeter diagram

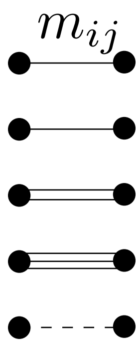
Diagram of missing faces

- Nodes $\longleftrightarrow$ facets of P
- Missing face is a minimal set of facets $f_1, \ldots, f_k$, such that $\bigcap_{i=1}^{k} f_i = \emptyset$.
- Missing faces are encircled.
- Ex:



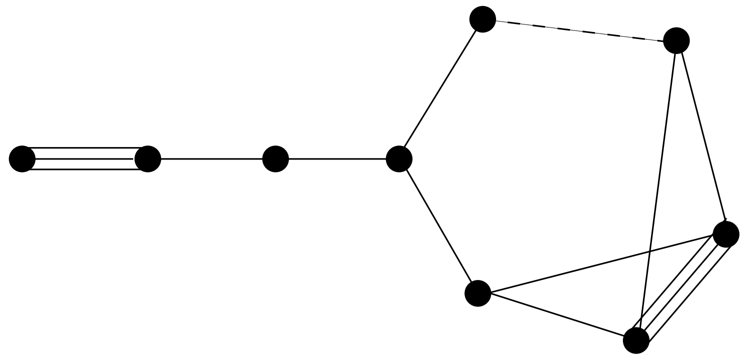
Coxeter diagrams

- Nodes $\longleftrightarrow$ facets f_i of P
- Edges:



if $\angle(f_i f_j) = \pi/m_{ij}$
if $\angle(f_i f_j) = \pi/3$
if $\angle(f_i f_j) = \pi/4$
if $\angle(f_i f_j) = \pi/5$
if $f_i \cap f_j = \emptyset$

Ex:



Lanner subdiagrams are minimal subdiagrams which are neither elliptic nor parabolic.

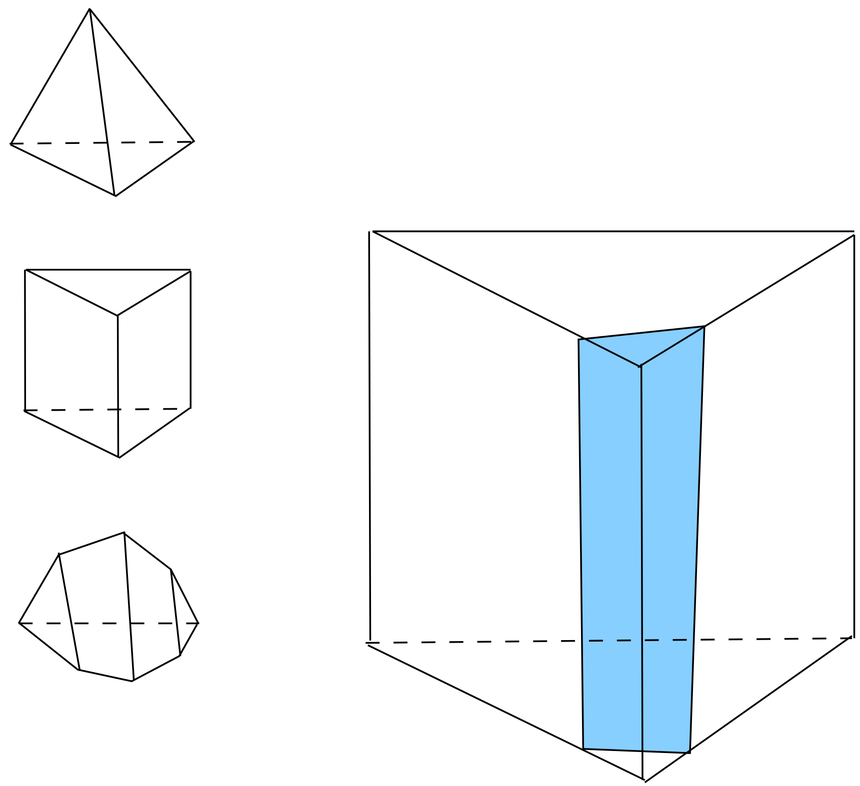
Lanner subdiagrams $\longleftrightarrow$ Missing faces

- If L is a Lanner diagram then $|L| \le 5$.
- $\#$ of Lanner diagrams of order $4, 5$ is finite.
- For any two Lanner subdiagrams s.t. $L_1 \cap L_2 = \emptyset$, $\exists$ an edge joining these subdiagrams.

Given a combinatorial type may try to check

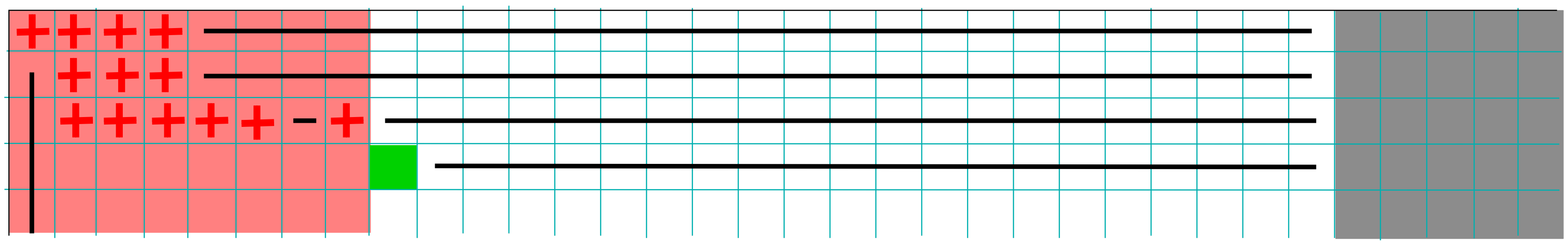
if there is a Coxeter polytope of this type.

To list all simple d-polytopes with $d+k$ facets:



While C++ program ran (for 9-polytopes with 13 facets):

- Checked combinatorial types obtained by the program.
- Proved that there are no Coxeter 9-polytopes with 13 facets.



- Tried to improve the program.

Problem: How to determine if two combinatorial polytopes are of the same combinatorial types?

- Polytopes represented by lists of vertices

	triangular 3-prism		1,3,4	2,3,4
Example:	with bases 1 and 2	$\longleftrightarrow$	1,3,5	2,3,5
	and sides 3,4,5		1,4,5	2,4,5

- Same combinatorial type $\longleftrightarrow$ Same representaion up to permutation

Idea: compare the numbers of missing faces of each size.

$(t_2, t_3, t_4, \ldots, t_n)$, $\quad t_i = \#$(missing faces of order i).

Simple polytope	$\longleftrightarrow$	Dual simplicial complex Σ	$\longleftrightarrow$	Complex Σ_1 of missing faces of Σ

Missing face is a minimal set of vertices defining no simplex in the complex.

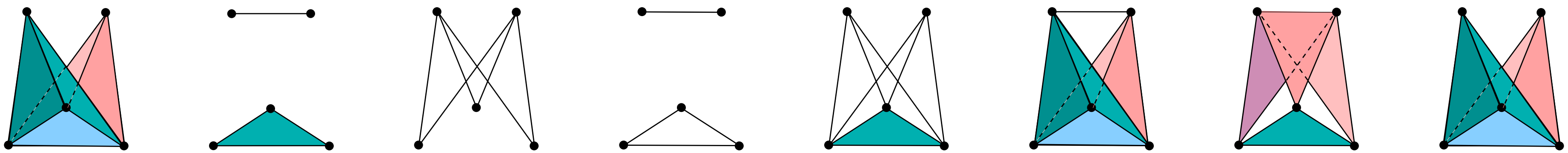
- Complex of missing faces for any simplicial complex:

$$\Sigma \longrightarrow \Sigma_1 \longrightarrow \Sigma_2 \longrightarrow \Sigma_3 \longrightarrow \ldots$$

$(t_2^j, t_3^j, t_4^j, \ldots, t_n^j)$, $\quad t_i^j = \#$(missing faces of order i in Σ_j).

Example: triangular prism.

Σ	Σ_1	Σ_2	Σ_3	Σ_4	Σ_5	Σ_6	Σ_7
1,3,4	1,2	1,3	1,2	1,3	1,2	3,4,5	1,3,4
1,3,5	3,4,5	1,4	3,4	1,4	1,3,4	1,2,3	1,3,5
1,4,5		1,5	3,5	1,5	1,3,5	1,2,4	1,4,5
2,3,4		2,3	4,5	2,3	1,4,5	1,2,5	2,3,4
2,3,5		2,4		2,4	2,3,4		2,3,5
2,4,5		2,5		2,5	2,3,5		2,4,5
				3,4,5	2,4,5		

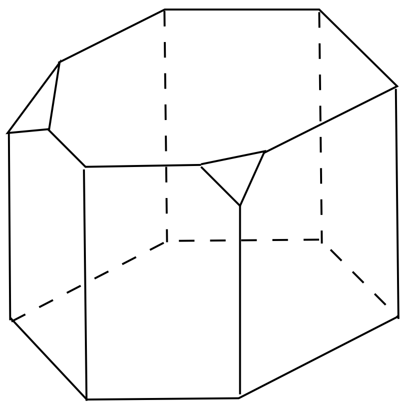


$$\Sigma \longrightarrow \Sigma_1 \longrightarrow \Sigma_2 \longrightarrow \ldots \longrightarrow \Sigma_k$$

$$(t_2^0, t_3^0, t_4^0, \ldots, t_n^0)$$
$$(t_2^1, t_3^1, t_4^1, \ldots, t_n^1)$$
$$(t_2^2, t_3^2, t_4^2, \ldots, t_n^2)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$(t_2^k, t_3^k, t_4^k, \ldots, t_n^k)$$

Conjecture: Do this numbers determine a polytope?

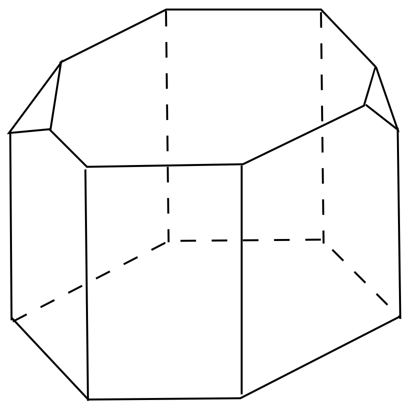
Example:



$$(0, 16, 0, 0, 0, 0, 0, 0, 0)$$
$$(21, 2, 0, 0, 0, 0, 0, 0, 0)$$
$$(18, 20, 0, 0, 0, 0, 0, 0, 0)$$
$$(1, 92, 2, 0, 0, 0, 0, 0, 0)$$

....

k=158381



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$$(18, 21, 0, 0, 0, 0, 0, 0, 0)$$
$$(1, 91, 2, 0, 0, 0, 0, 0, 0)$$

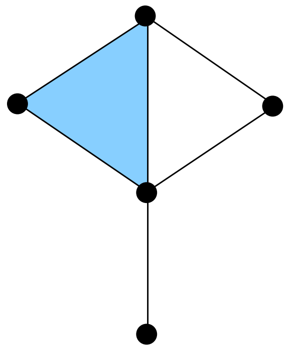
....

k=666517

Question: Do this numbers determine

a simplicial complex?

No: They do not differ



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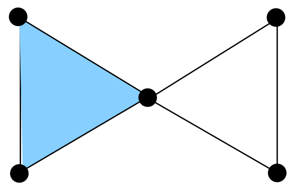
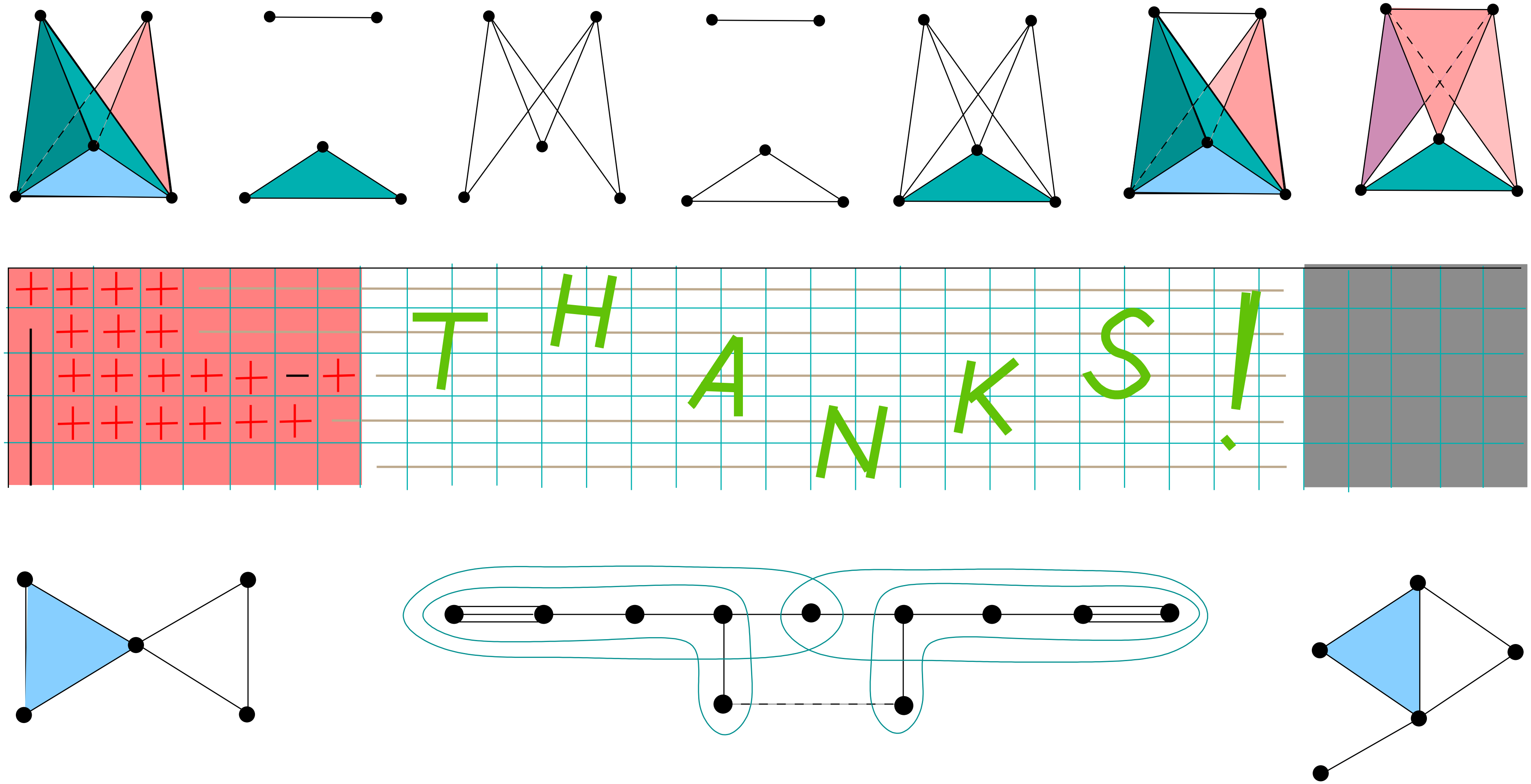


Def: Σ^* is **Alexander dual** to Σ if

$$\Delta \in \Sigma^* \quad \Leftrightarrow \quad \Sigma \setminus \Delta \;\notin\; \Sigma.$$

$$\begin{array}{ccccccccc}
\Sigma_0 & \longrightarrow & \Sigma_1 & \longrightarrow & \Sigma_2 & \longrightarrow & \Sigma_3 & \longrightarrow & \ldots \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \\
\Sigma_0^* & \longleftarrow & \Sigma_1^* & \longleftarrow & \Sigma_2^* & \longleftarrow & \Sigma_3^* & \longleftarrow & \ldots
\end{array}$$


THANKS!