Non-integer quivers: geometry and mutation-finiteness

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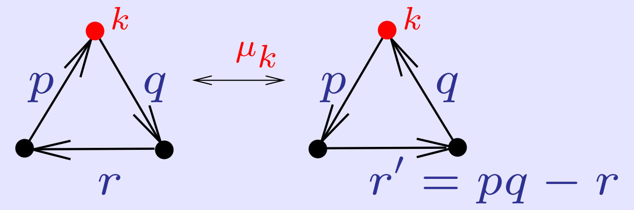
Introduction

- $B = \{b_{ij}\}$ a skew-symmetric matrix with $b_{ij} \in \mathbb{R}$.
- Mutate B by usual mutation rule:

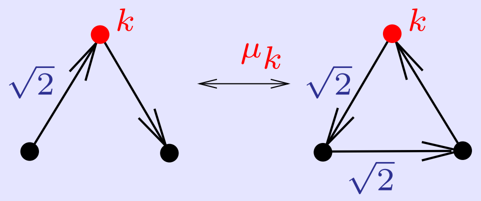
$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } i = k \text{ or } j = k \\ b_{ij} + \frac{1}{2}(|b_{ik}|b_{kj} + b_{ik}|b_{kj}|), & \text{otherwise} \end{cases}$$

- B defines a **non-integer quiver** (with arrows of real weights $b_{ij} = -b_{ji}$).

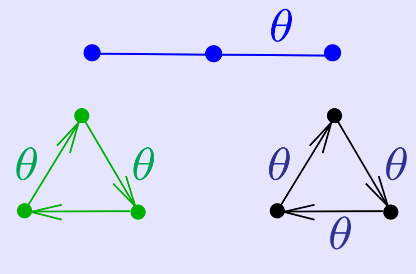
- **Mutation** μ_k of a non-integer quiver:
  1) reverse all arrows incident to k;
  2) for every path $i \xrightarrow{p} k \xrightarrow{q} j$ with $p, q > 0$ apply:



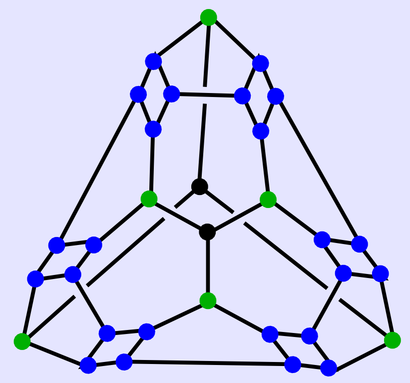
- Example: B_3



- **Question:** when a real quiver Q is mutation-finite?
- Example: H_3 (mutation class and "exchange graph")



where $\theta = 2\cos\frac{\pi}{5}$

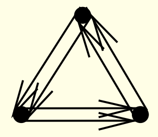


(cf. generalised associahedron in [FR])

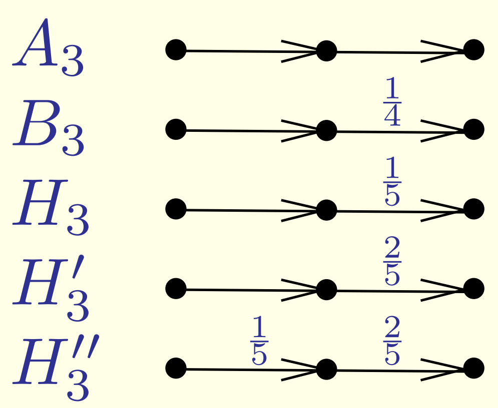
Quivers of rank 3

Theorem [FT1]. Any mutation-finite rank 3 quiver is mutation-equivalent to one of

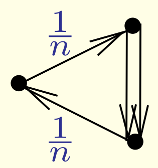
- Markov quiver:



- Finite type quivers:



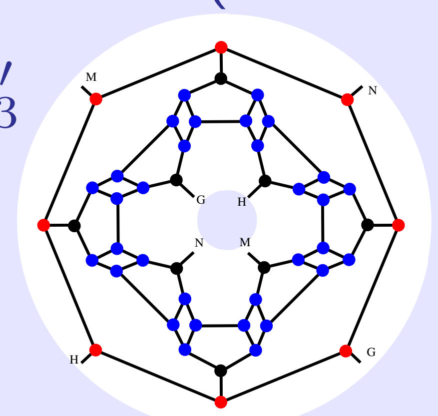
- Affine quivers:



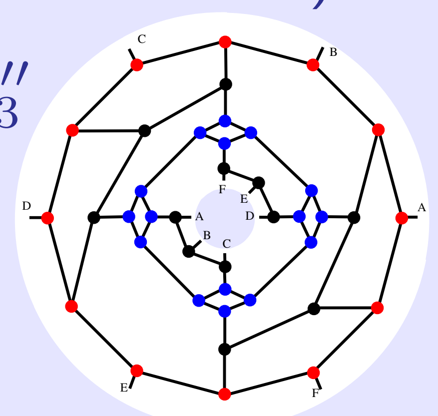
Here, a label $\frac{k}{m}$ stays for the weight $|b_{ij}| = 2\cos\frac{k\pi}{m}$.

- All of these (but Markov) are mutation-acyclic.
- "Exchange graphs" for H'_3 and H''_3 are graphs on a torus (with two acyclic belts each):

H'_3

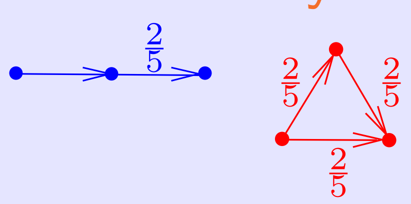


H''_3

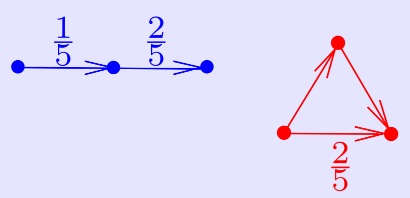


- Each of the mutation classes H'_3 and H''_3 has two different acyclic representatives:

H'_3:



H''_3:



Geometric realisation by reflections (GR)

- GR of a quiver of rank n: vectors $v_1,\dots,v_n$ in a quadratic vector space V s.t. $(v_i, v_i) = 2$ and $(v_i, v_j) = -|b_{ij}|$.
- Mutation = partial reflection:

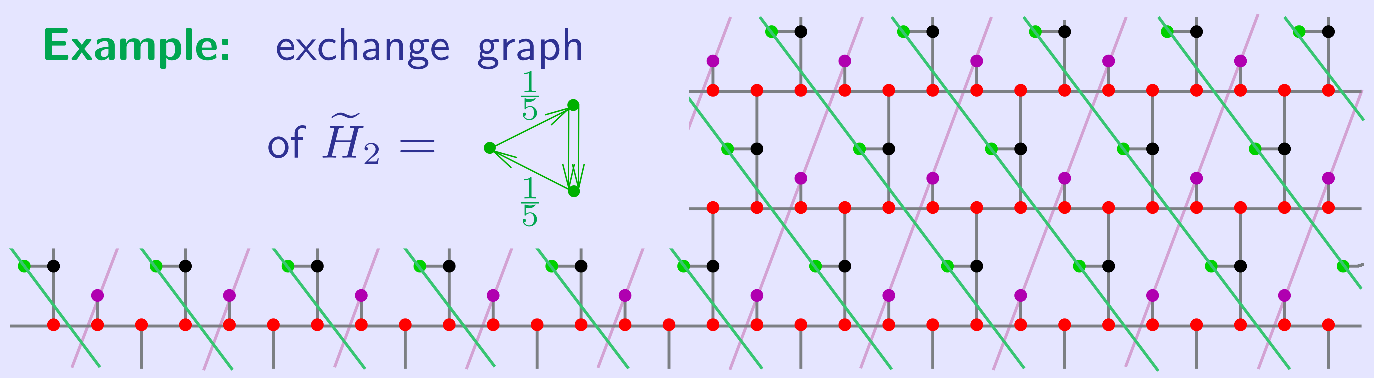
$$\mu_k(v_i) = \begin{cases} v_i, & \text{if } b_{ki} \geq 0, i \neq k \\ -v_i, & \text{if } i = k \\ v_i - (v_i, v_k)v_k, & \text{if } b_{ki} < 0 \end{cases}$$

- Mutation class has a GR if GR of quivers commute with mutations.

Theorem [FT1,FT2]. Mutation class of any real acyclic quiver with $|b_{ij}| \geq 2\ \forall i,j$ admits a GR.

- When GR exists, we define (geometric) Y-seeds (n-tuples of vectors in V) and "exchange graphs".

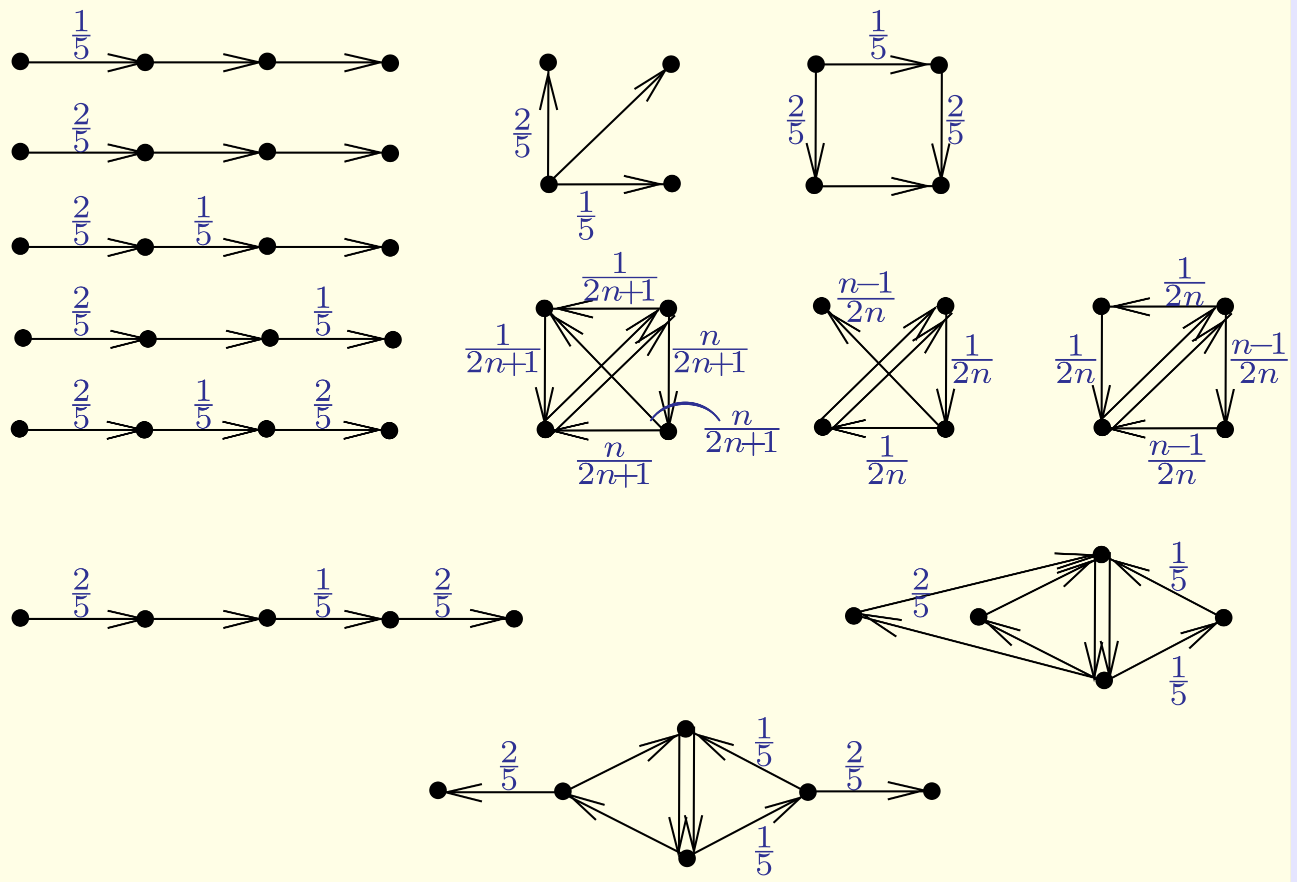
Example: exchange graph of $\widetilde{H}_2 =$



Theorem [FL]. Let Q be an affine type rank 3 mut.-fin. quiver. Then "exchange graph" of Q grows polynomially and is quasi-isometric to a lattice of some dimension.

Finite mutation type: classification

Theorem [FT3]. Any mut.-fin. non-integer quiver of rank $n > 3$ is mutation-equivalent to either one of B_n, F_4, $\widetilde{B}_n$, $\widetilde{C}_n$, $\widetilde{F}_4$ or one of



Here, a label $\frac{k}{m}$ stays for the weight $|b_{ij}| = 2\cos\frac{k\pi}{m}$.

References

[FL] A. Felikson, Ph. Lampe, *Exchange graphs for non-integer affine quivers with 3 vertices*, in preparation.
[FT1] A. Felikson, P. Tumarkin, *Geometry of mutation classes of rank 3 quivers*, arXiv:1609.08828,
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[FR] S. Fomin, N. Reading, *Root systems and generalized associahedra*, Geometric combinatorics, 63-131, IAS/Park City Math. Ser., 13, Amer. Math. Soc., Providence, RI, 2007.
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