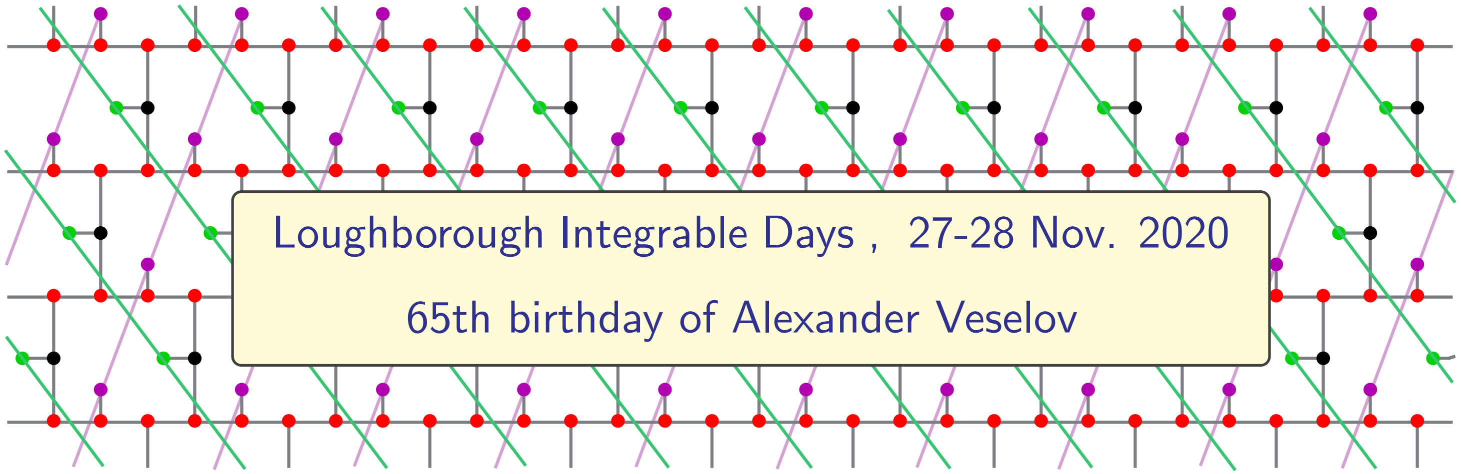
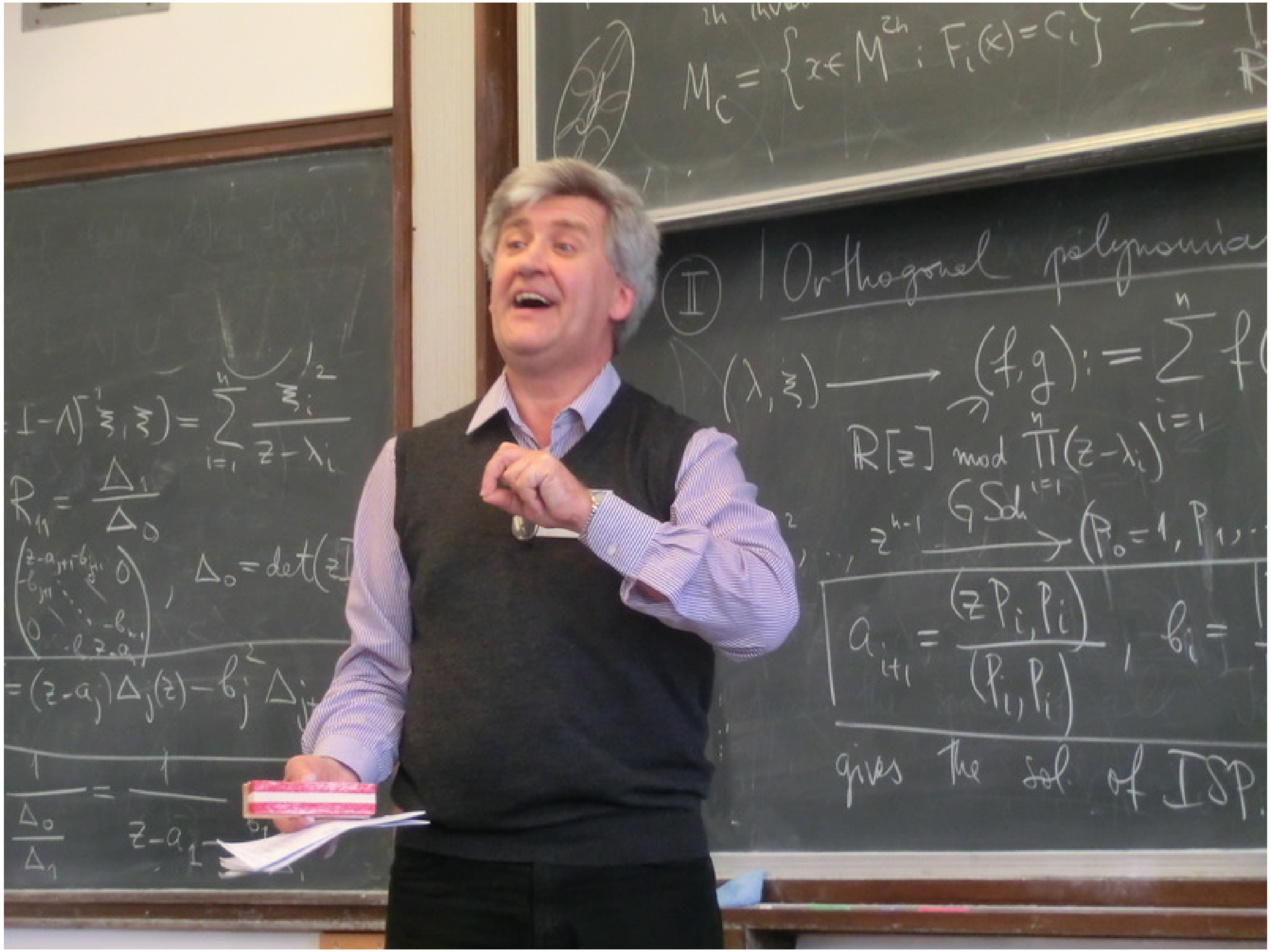
Mutations of non-integer quivers: finite mutation type

Anna Felikson

Durham University

(joint works with Pavel Tumarkin and Philipp Lampe)





Durham, 2015

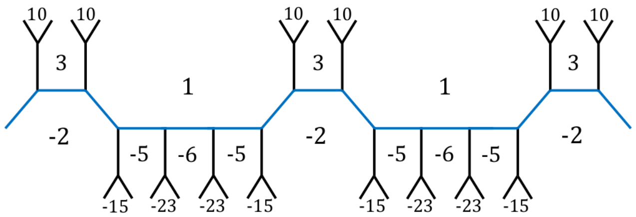
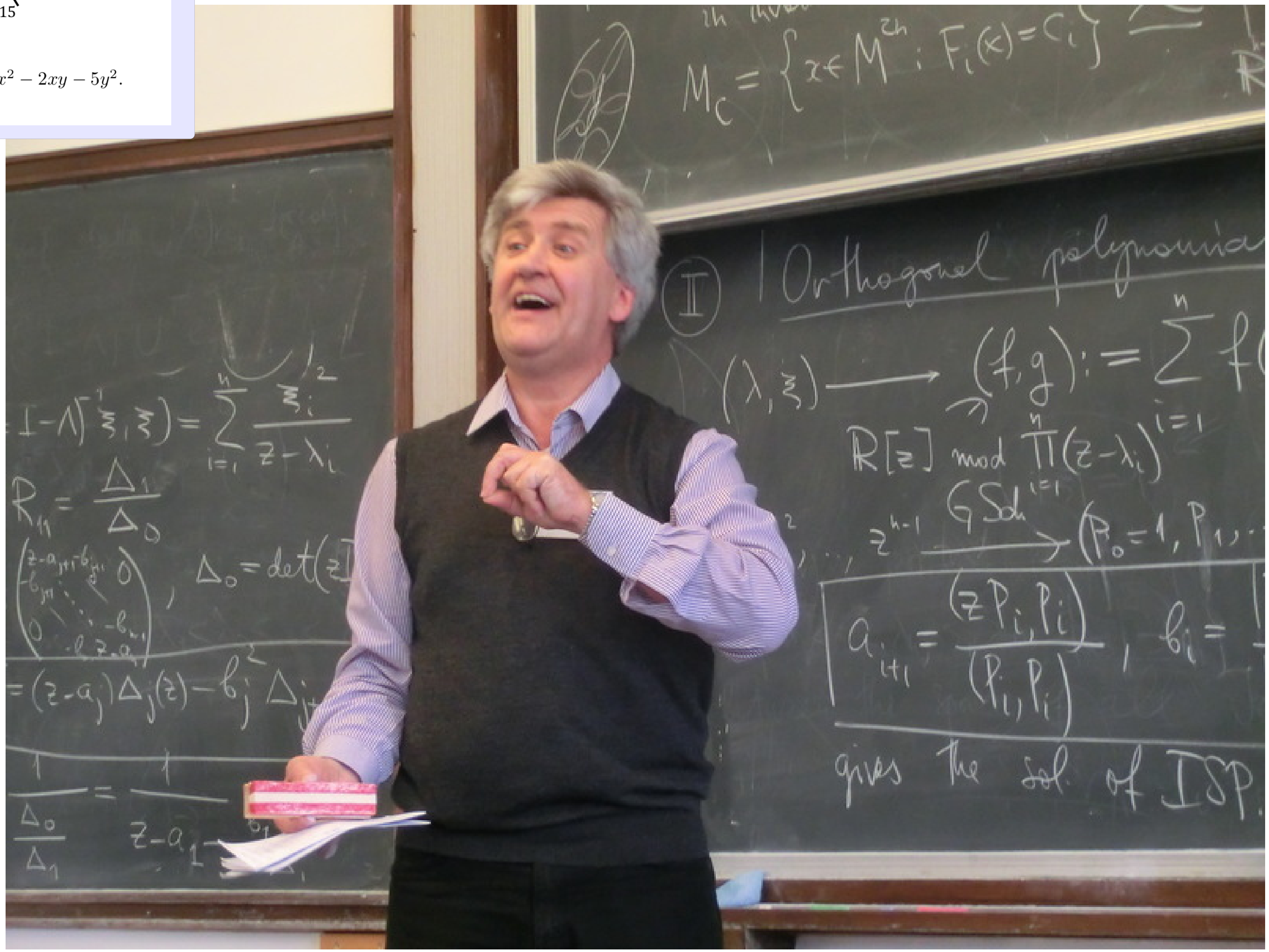


FIGURE 3. Conway river for the quadratic form $Q = x^2 - 2xy - 5y^2$.

Picture shamelessly stolen from

"Conway river and Arnold Sail"

by K. Spalding, A. P. Veselov



Durham, 2015

1. Mutations of non-integer quivers:

- $B = \{b_{ij}\}$ a skew-symmetric $n \times n$ matrix with $b_{ij} \in \mathbb{R}$.
- Mutate B by usual mutation rule: (introduced by Fomin, Zelevinsky)

$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } i = k \text{ or } j = k \\ b_{ij} + \frac{1}{2}(|b_{ik}|b_{kj} + b_{ik}|b_{kj}|), & \text{otherwise} \end{cases}$$

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Why:

- Philipp Lampe, *On the approximate periodicity of sequences attached to noncrystallographic root systems*, Experimental Mathematics (2016).
- Integer finite type contains types A, B, C, D, E, F .... - but not H_3, H_4!
- Geometric realization of acyclic mutation classes by partial reflections allow non-integer values.

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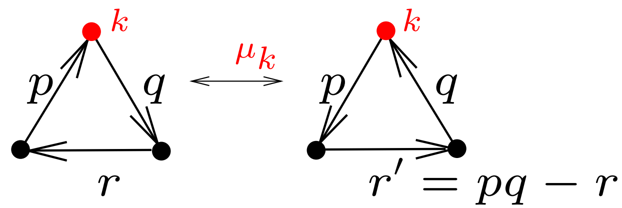

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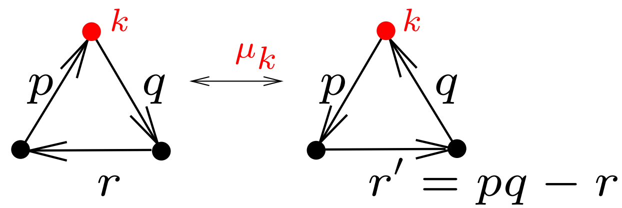
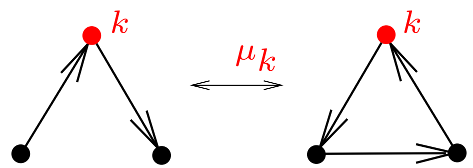


- Example: A_3



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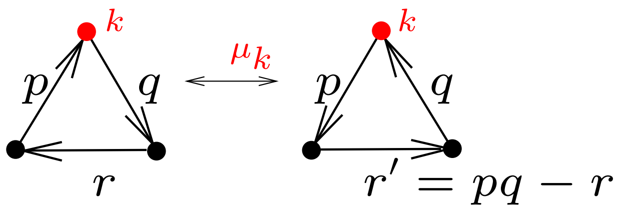
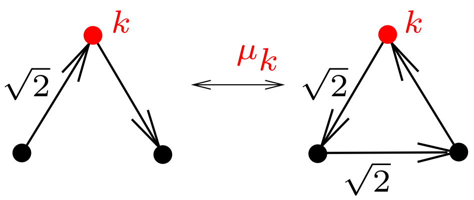


- Example: B_3



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  – rank 2 quivers;
  – "quivers coming from triangulated surfaces" (incl. A_n, D_n, $\widetilde{A}_n$, $\widetilde{D}_n$).
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But no H_2, H_3, H_4 and no I_n!!!

1. Mutations of non-integer quivers:

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- Finite mutation type?

  For which quivers their mutation class is finite?

- Geometry and combinatorics.

2. In Rank 3: $\left\{\begin{array}{l}\text{acyclic}\\\text{mutation class}\end{array}\right\} \longrightarrow \left\{\begin{array}{l}\text{Geometric realisation}\\\text{by partial reflections}\end{array}\right\}$

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Acyclic quiver $=$ quiver containing no oriented cycles.

Acyclic mutation class $=$ mutation class containing an acyclic quiver.

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- $Q = (b_{ij}) \qquad \rightsquigarrow \qquad M = \begin{pmatrix} 2 & & -|b_{ij}| \\ & 2 & \\ -|b_{ij}| & & 2 \end{pmatrix} = \langle v_i, v_j \rangle$

  $(v_1, \ldots, v_n)$ - basis of quadratic space V of same signature as M.

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- Let $G = \langle s_1, \ldots, s_n \rangle$ where $s_i = r_{v_i}$.

  G acts discretely in a cone $C \subset V$ with fundamental domain

$$F = \bigcap_{i=1}^{n} \Pi_i^-, \quad \text{where } \Pi_i^- = \{u \in V \mid \langle u, v_i \rangle < 0\}.$$

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Theorem. (Barot, Geiss, Zelevinsky'06; Seven'15)

The values $\langle v_i, v_j \rangle$ change under mutations
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For integer quivers (but also for real ones in rank 3):
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$Q = (p, q, r)$,
mutation-cyclic $\rightsquigarrow$ $\begin{pmatrix} -2 & p & q \\ p & -2 & r \\ q & r & -2 \end{pmatrix} = (v_i, v_j)$

Mutation: $\rightsquigarrow$ $\mu_k(v_i) = \begin{cases} -v_i - (v_i, v_j)v_j, & \text{if } i \to k \text{ in } Q \\ v_i, & \text{otherwise} \end{cases}$

Proposition. (FT)

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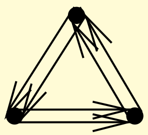
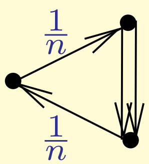
Theorem [FT'16]. Any mutation-finite rank 3 quiver is mutation-equivalent to one of

- Markov quiver:



- Affine quivers:



- Finite type quivers:

A_3 $\quad \bullet \longrightarrow \bullet \longrightarrow \bullet$

B_3 $\quad \bullet \longrightarrow \bullet \xrightarrow{\frac{1}{4}} \bullet$

H_3 $\quad \bullet \longrightarrow \bullet \xrightarrow{\frac{1}{5}} \bullet$

H_3' $\quad \bullet \longrightarrow \bullet \xrightarrow{\frac{2}{5}} \bullet$

H_3'' $\quad \bullet \xrightarrow{\frac{1}{5}} \bullet \xrightarrow{\frac{2}{5}} \bullet$

(Here, a label $\frac{k}{d}$ stays for the weight $|b_{ij}| = 2\cos\frac{k\pi}{d}$.)

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- All of them arise from reflection groups!

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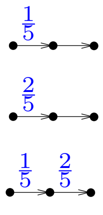
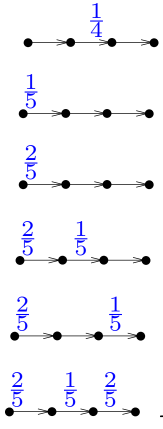
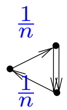
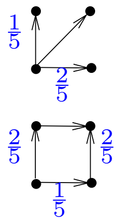
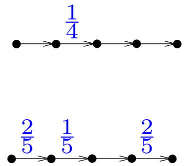
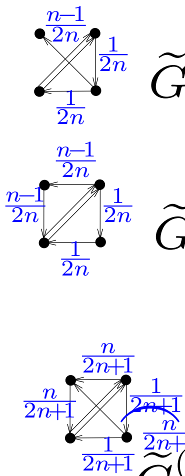
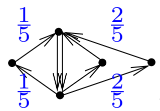
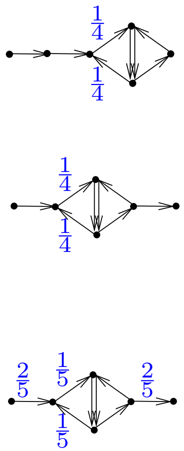
Answer:

Thm. [FT'18]
Non-int.
mut.-fin. $\Leftrightarrow$
- $G_{2,n}$ or
- orbifold or
- as in Table:

	rank 3	rank 4	rank 5	rank 6
Finite type	 H_3, H_3', H_3''	 F_4, H_4, H_4', H_4'', H_4''', H_4''''		
Affine type	 $\widetilde{G}_{2,n}$	 $\widetilde{H}_3$, $\widetilde{H}_3'$	 $\widetilde{F}_4$, $\widetilde{H}_4$	
Extended affine type		 $\widetilde{G}_{2,2n}^{(*,+)}$, $\widetilde{G}_{2,2n}^{(*,*)}$, $\widetilde{G}_{2,2n+1}^{(*,*)}$	 $H_3^{(1,1)}$	 $F_4^{(*,+)}$, $F_4^{(*,*)}$, $H_4^{(1,1)}$

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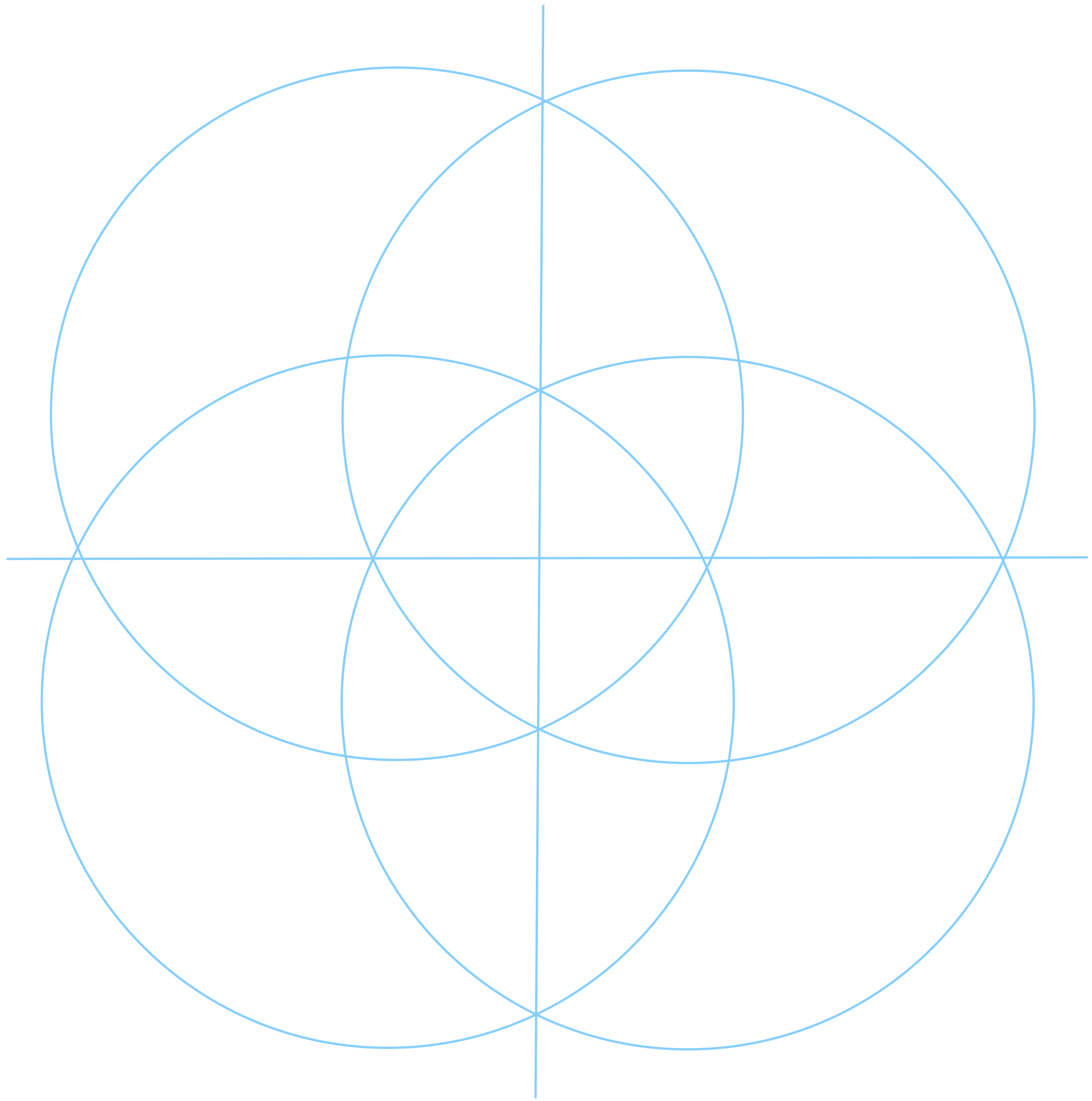

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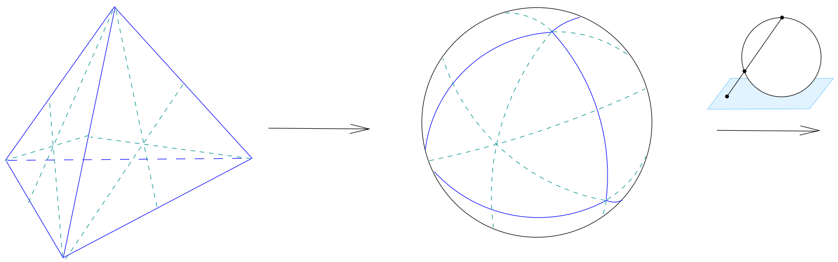

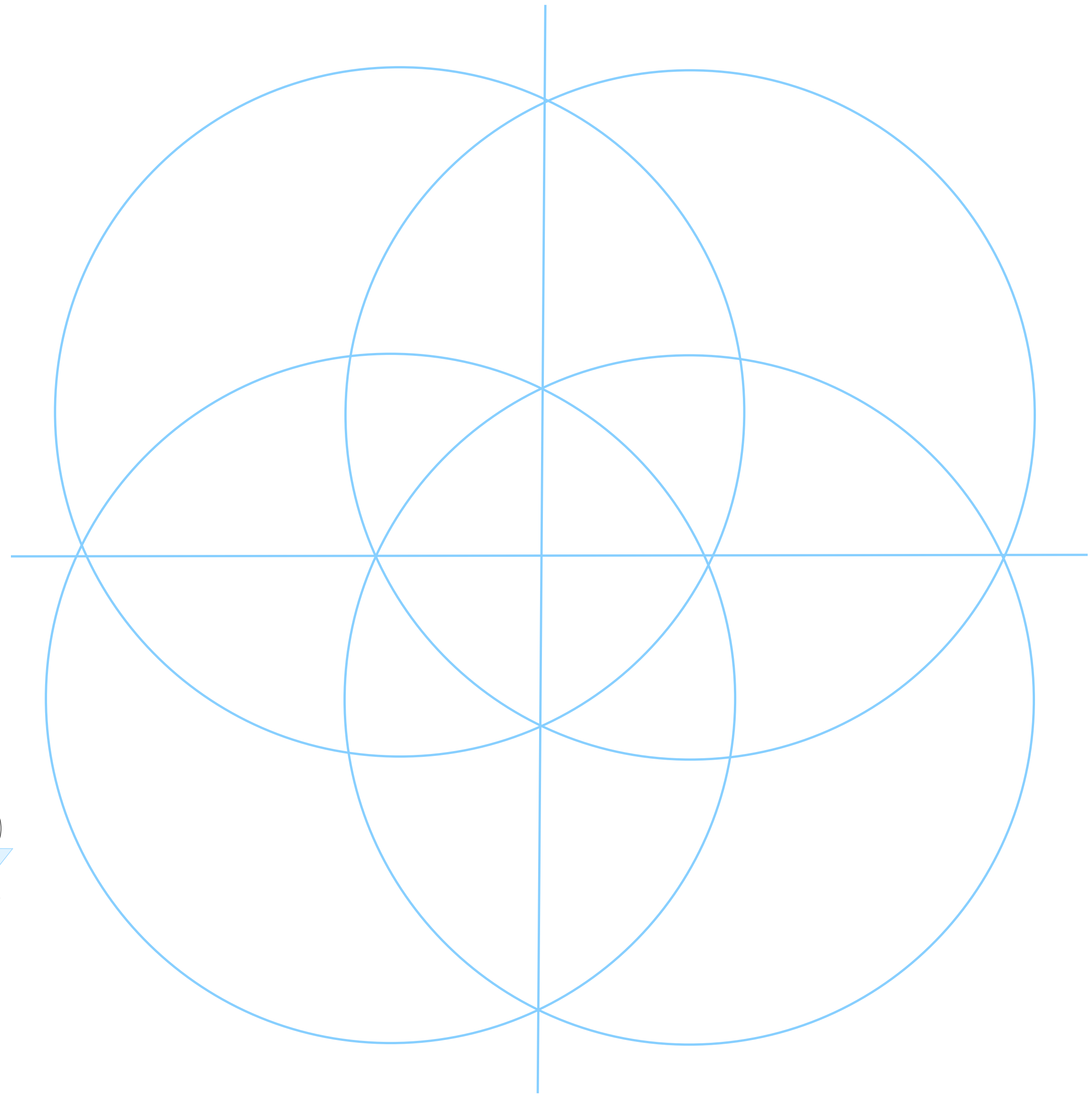

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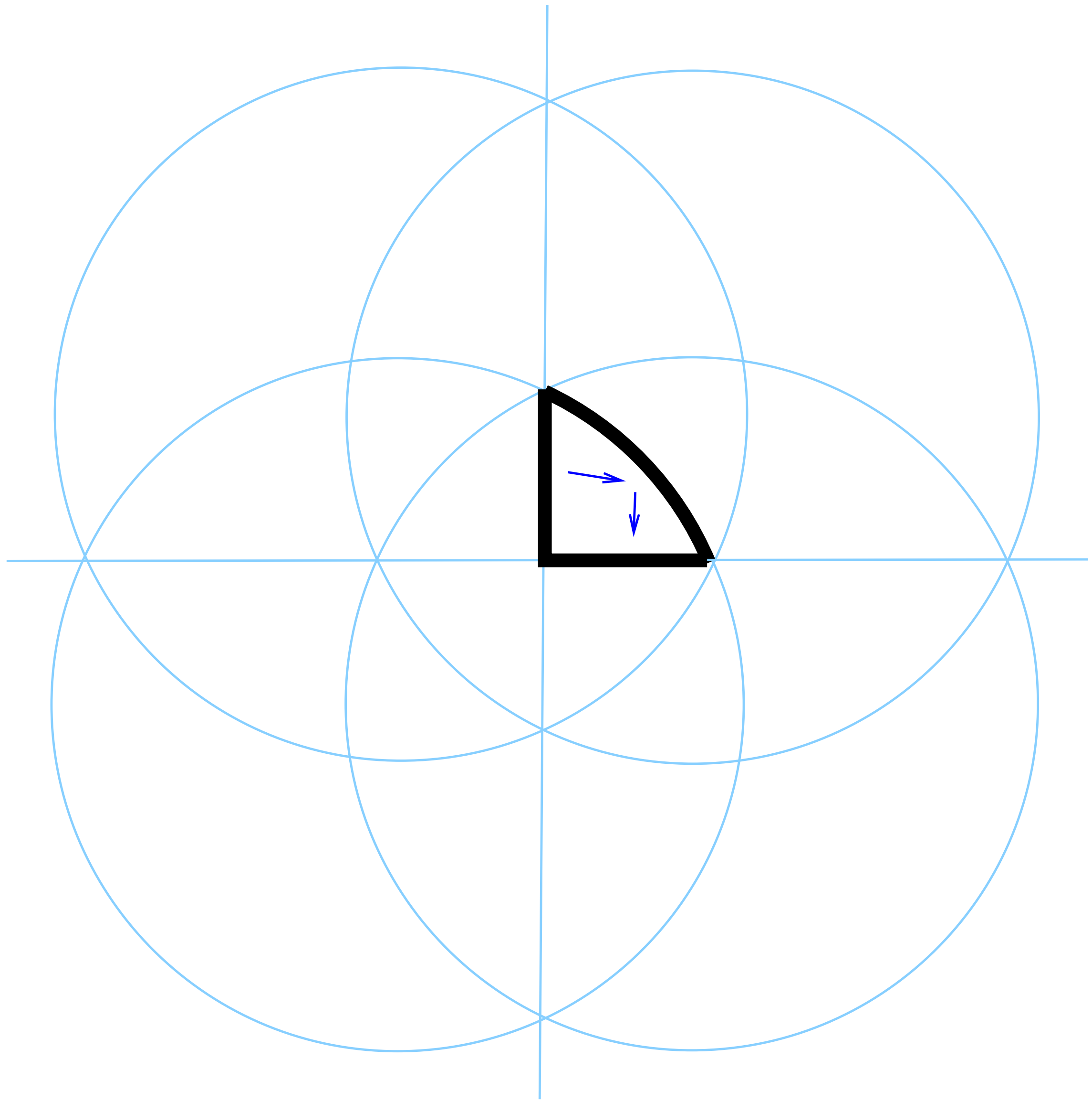

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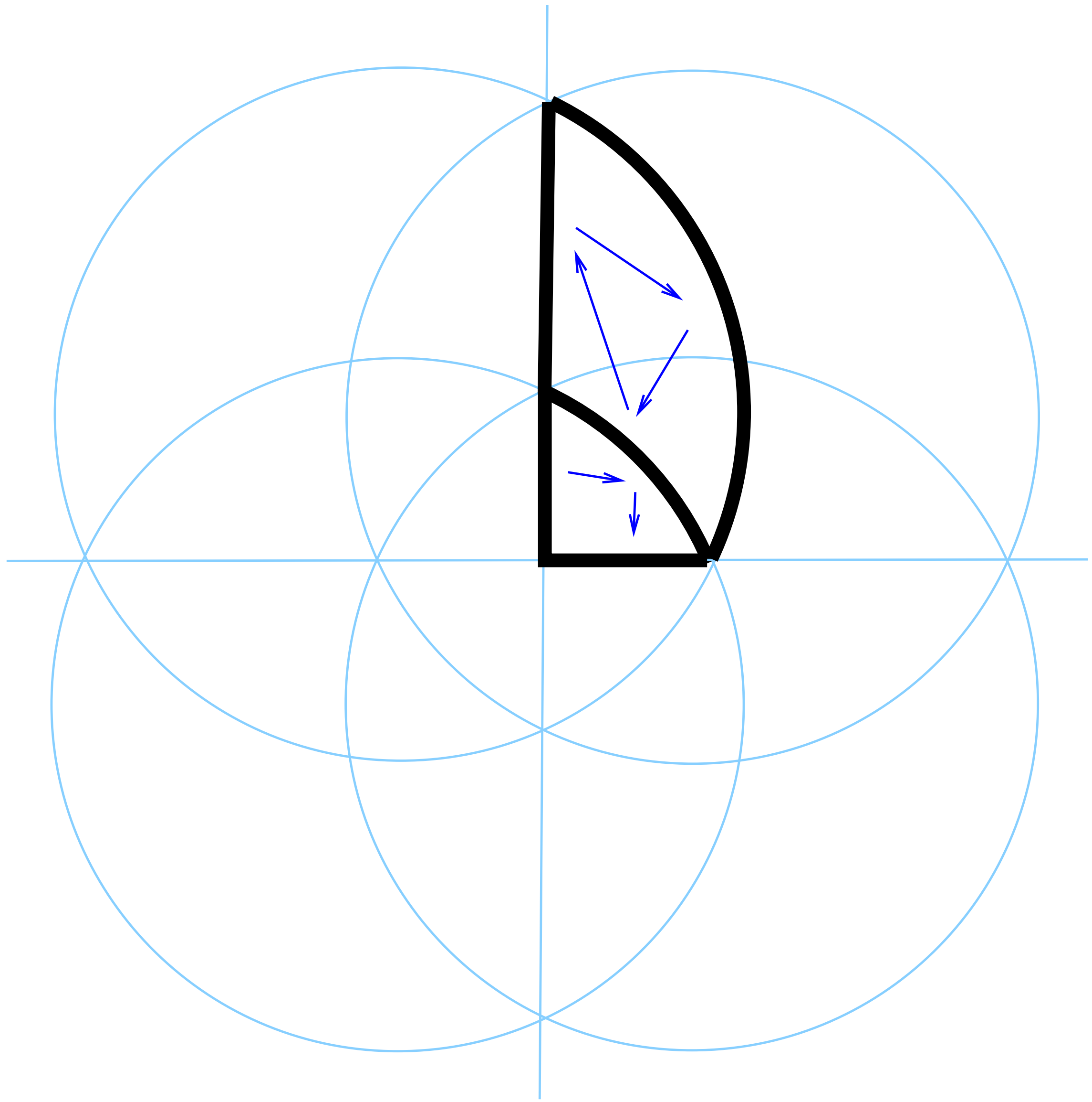

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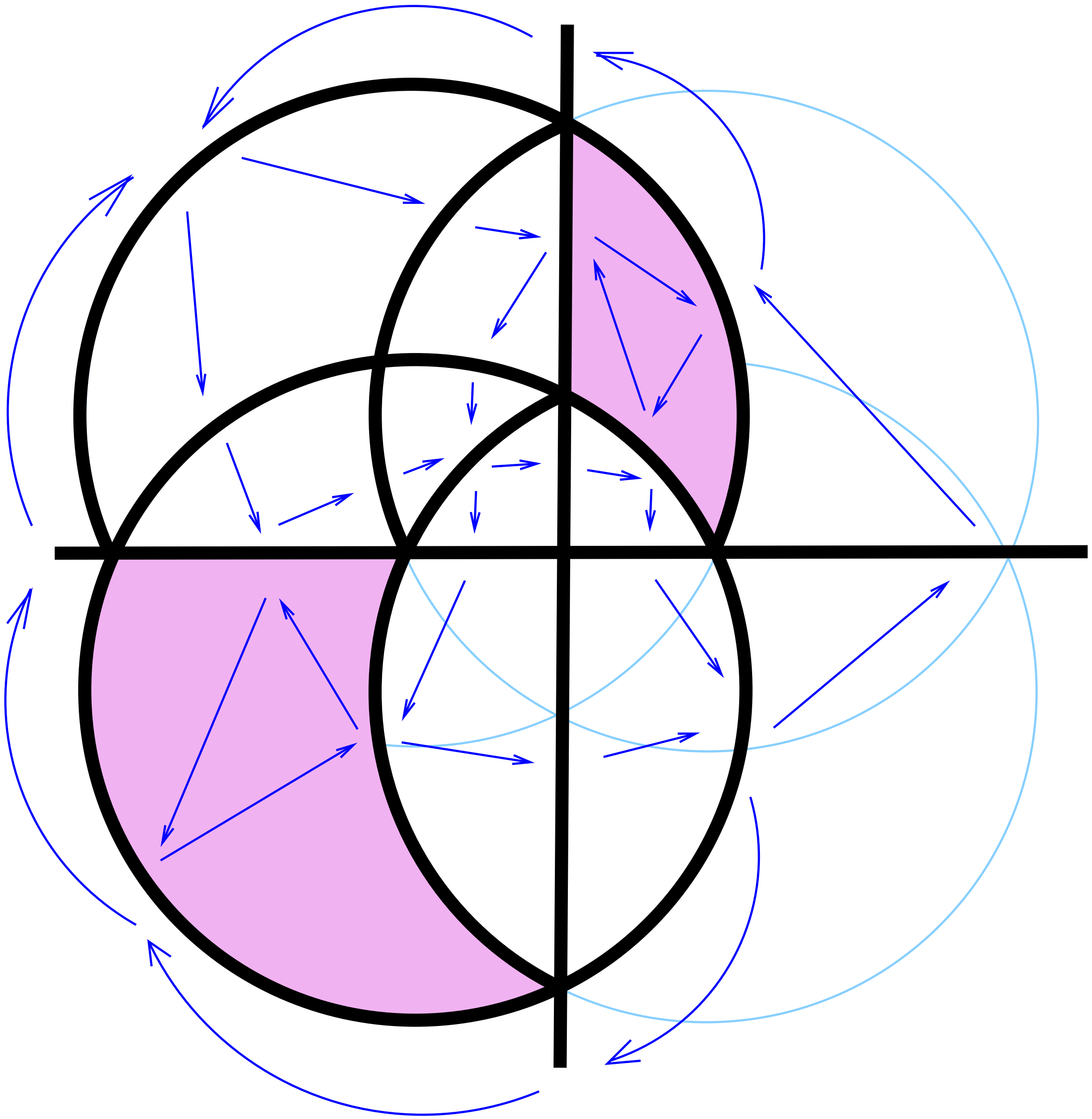

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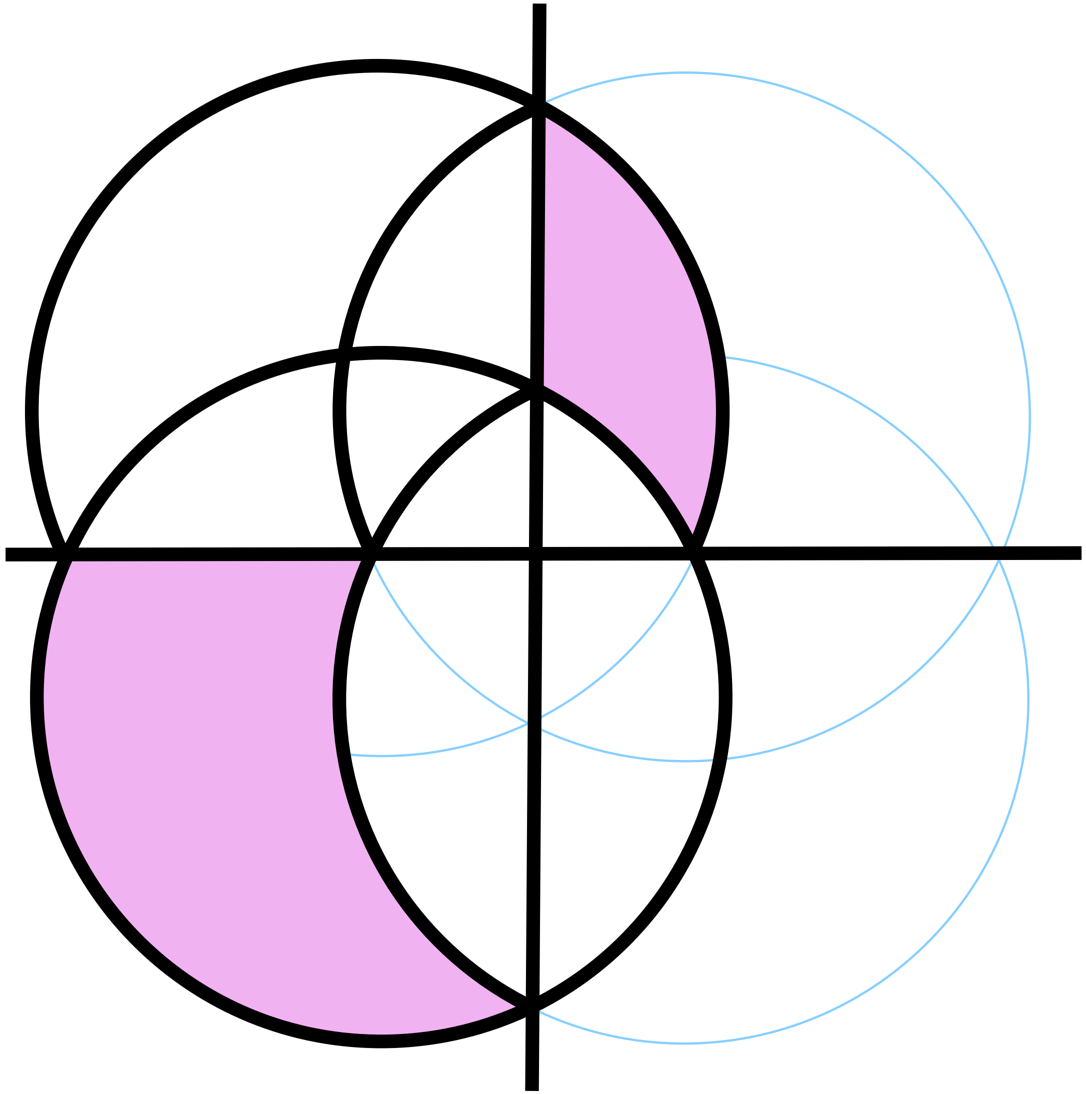

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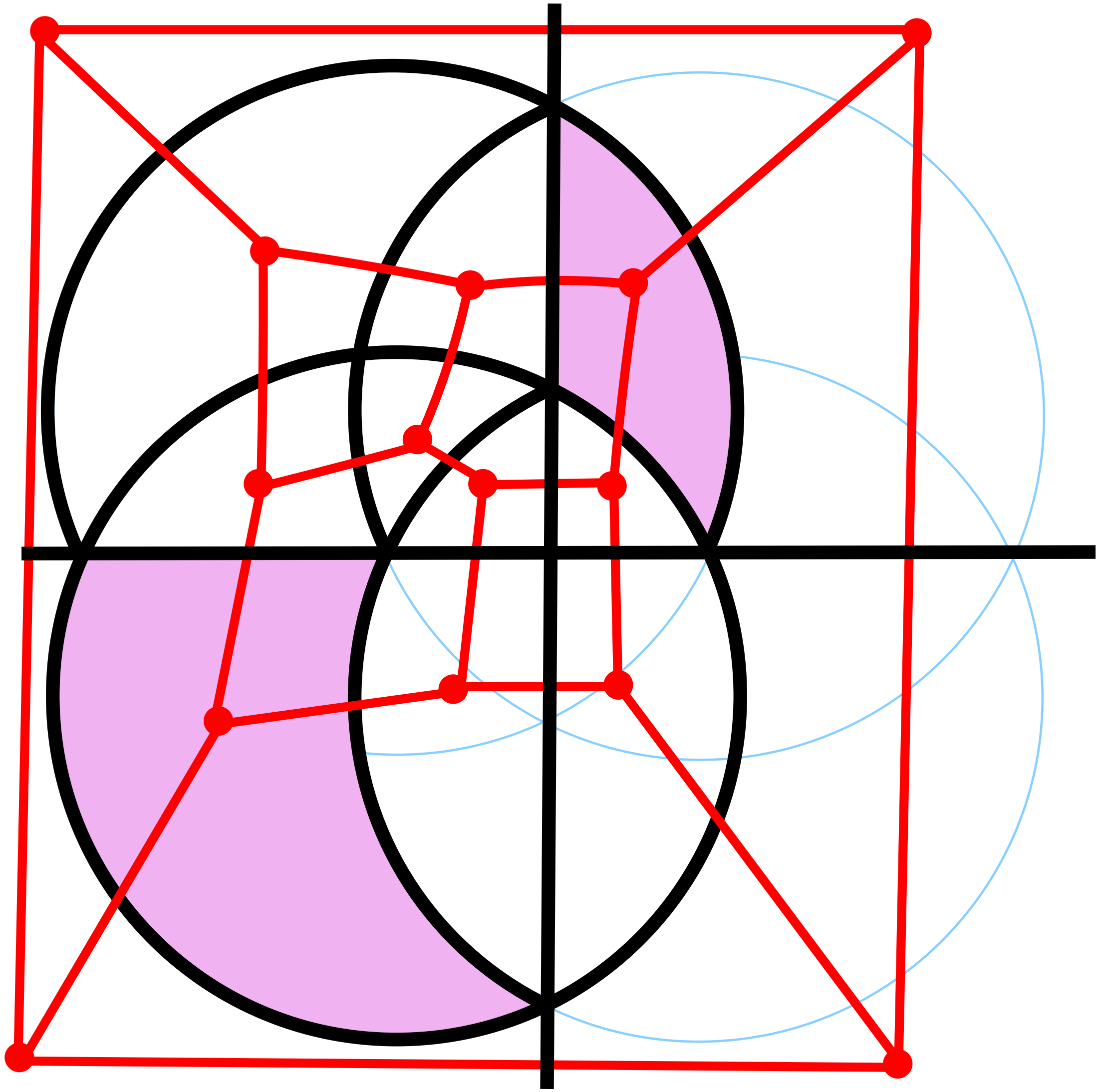

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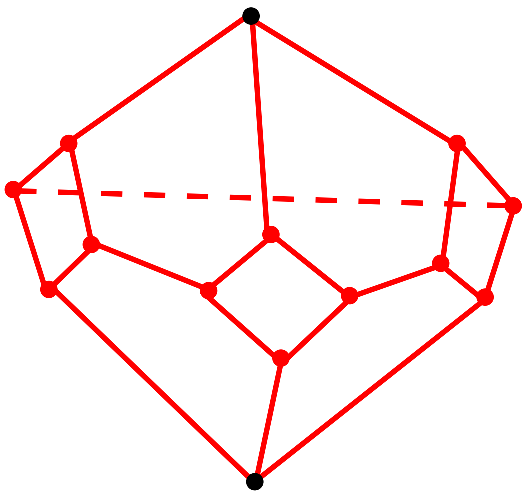
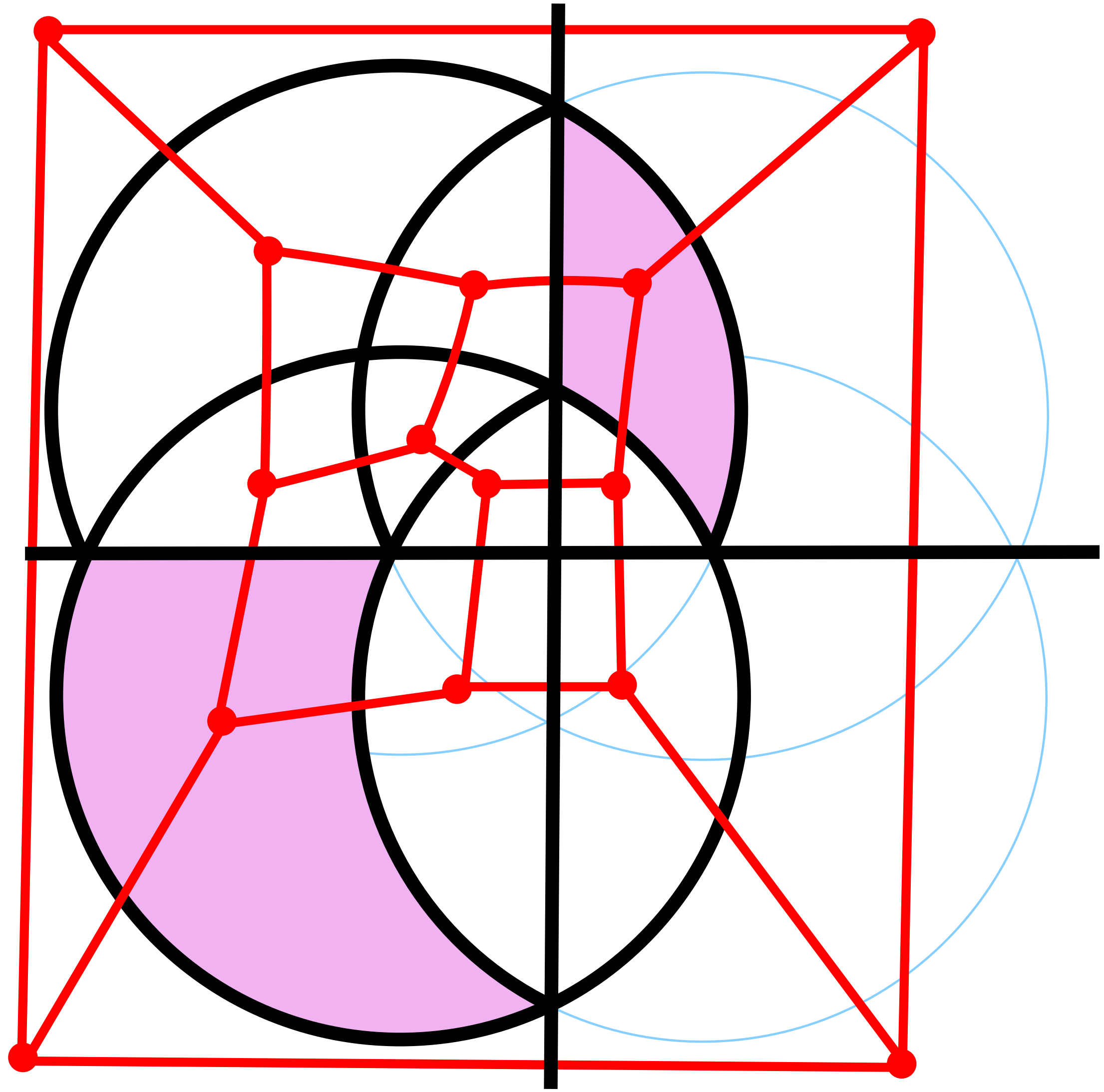


Associahedron



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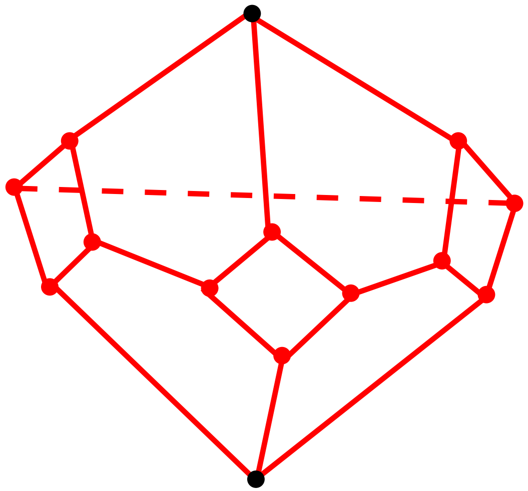
In integer case:

- Exchange graph is a one-skeleton of a polytope.



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Vertices $\leftrightarrow$ seeds (i.e. states), edges $\leftrightarrow$ mutations

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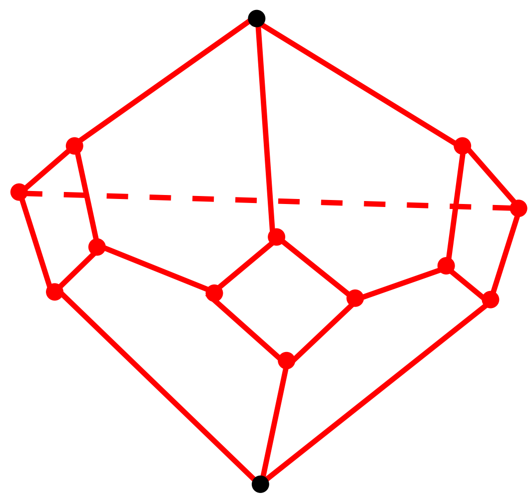
- All acyclic quivers in the mutation class are the same (up to orientations of arrows).



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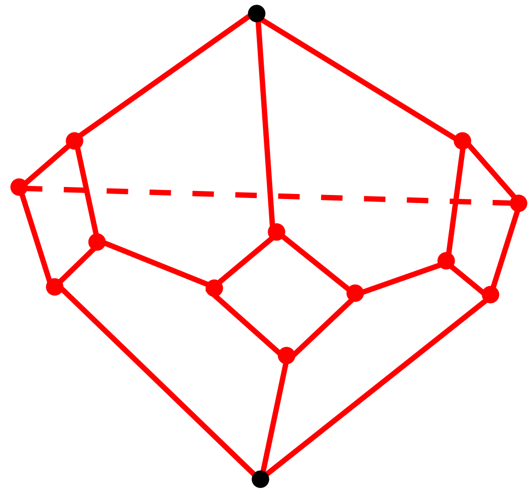

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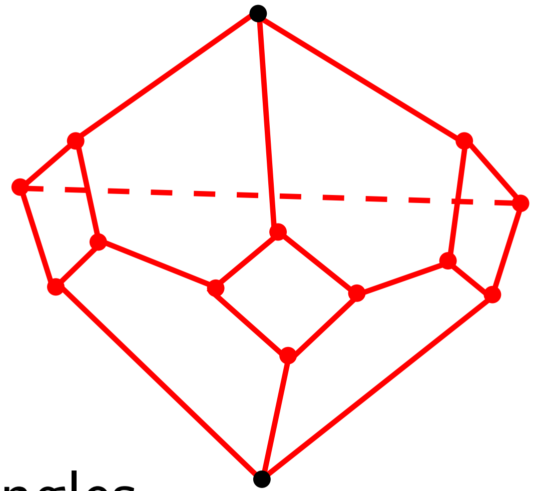
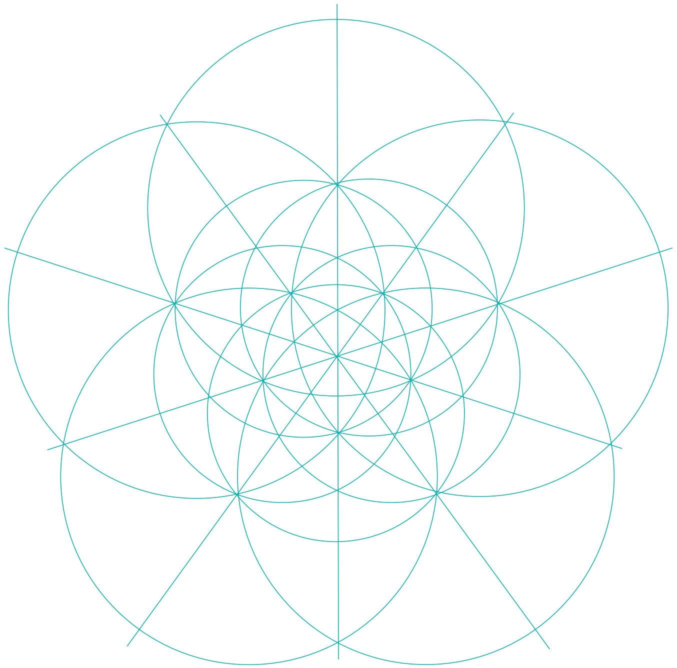


- Acyclic quiver correspond to "acute-angled" triangles.

4. Exchange graph:

Rank 3 - finite type

- Example: H_3



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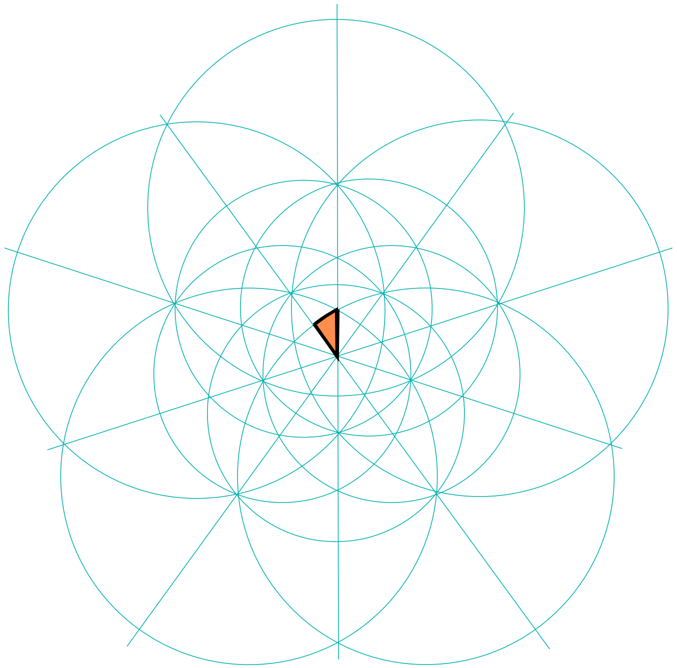

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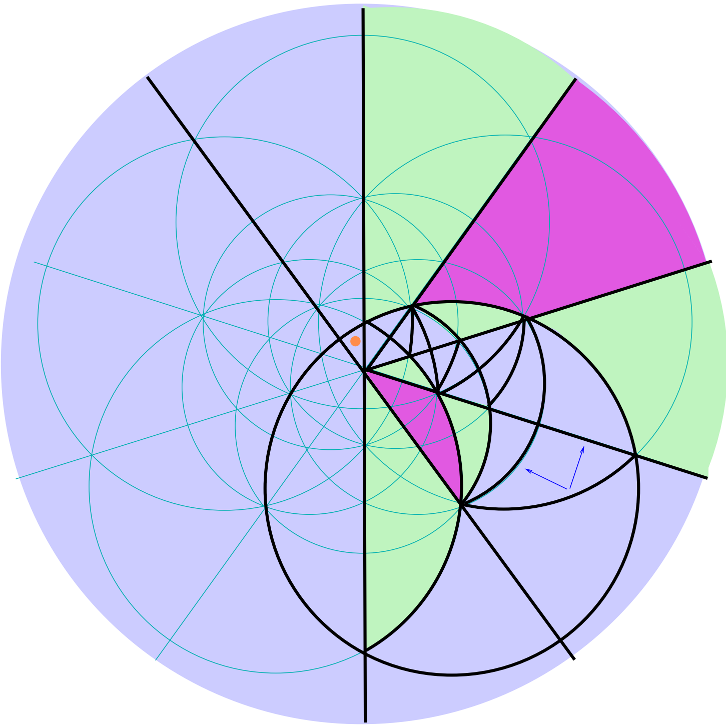
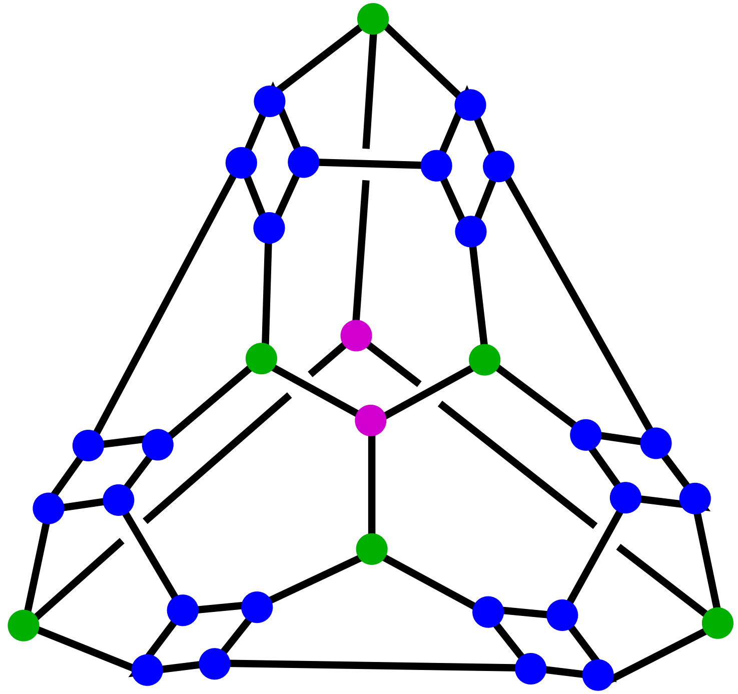
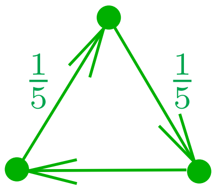
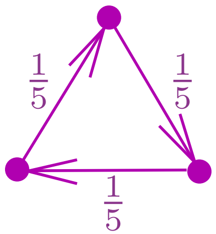
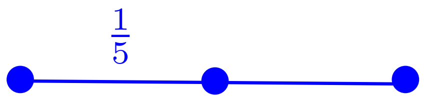




(cf. generalised associahedron in Fomin - Reading)

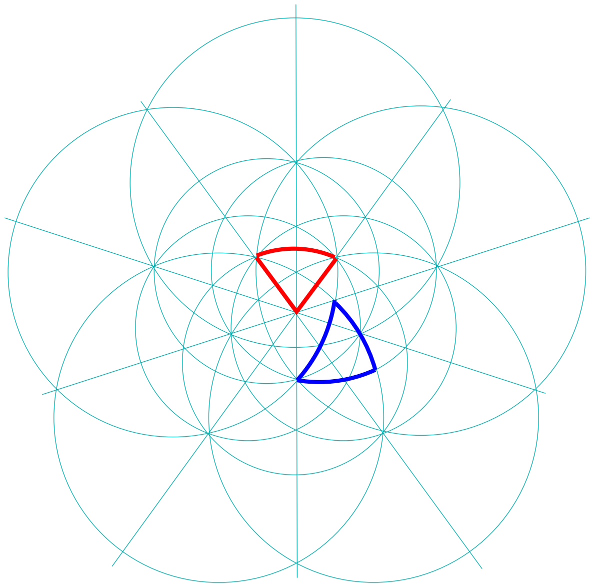


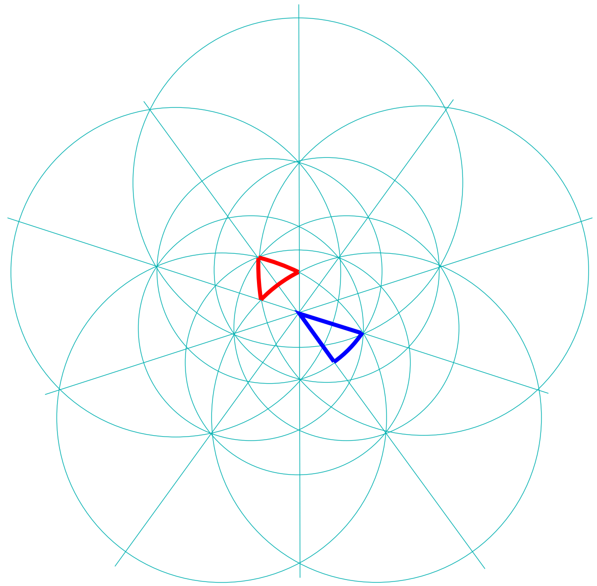


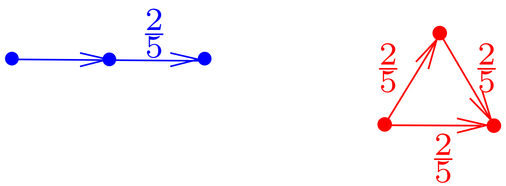


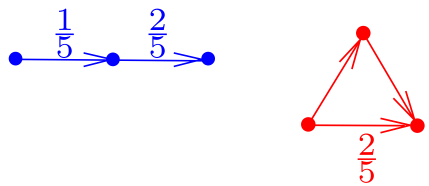
4. Exchange graph:

Rank 3 - finite type



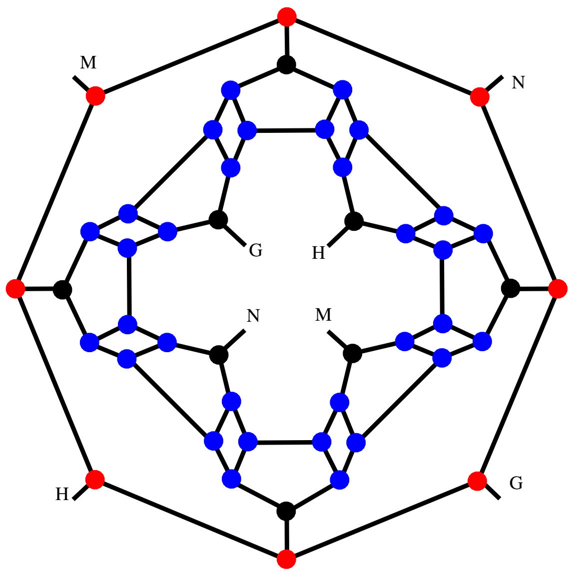




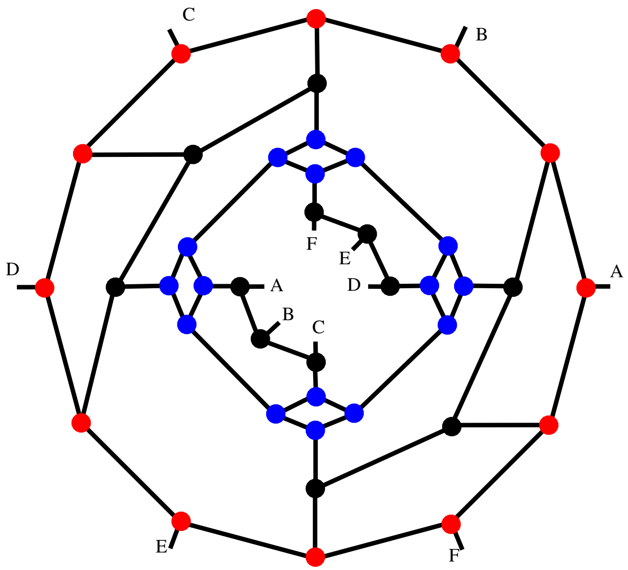


4. Exchange graph: Rank 3 - finite type

- Exchange graphs for H_3' and H_3'' are graphs on a torus (with two acyclic belts each):

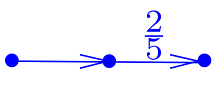


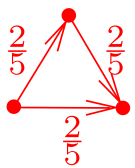
H_3'

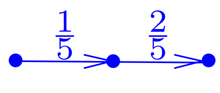


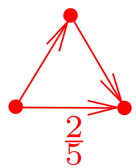
H_3''

- Two different acyclic representatives in each of H_3' and H_3'':









4. Exchange graph: Rank 3 - affine type

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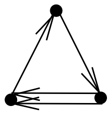
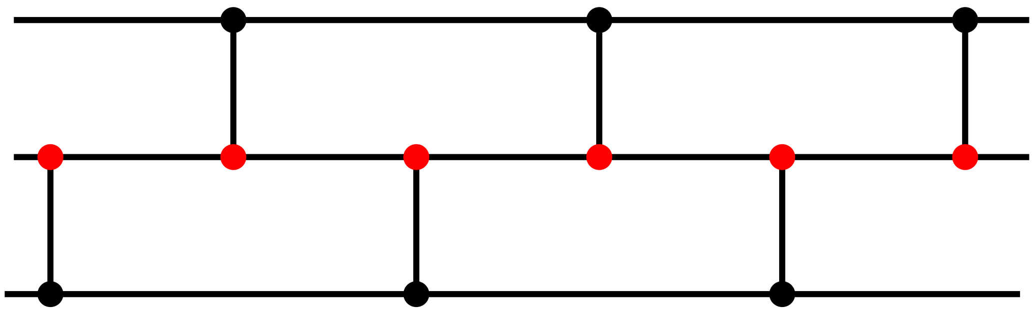
In integer case:

$\widetilde{A}_2$





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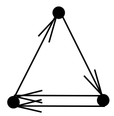
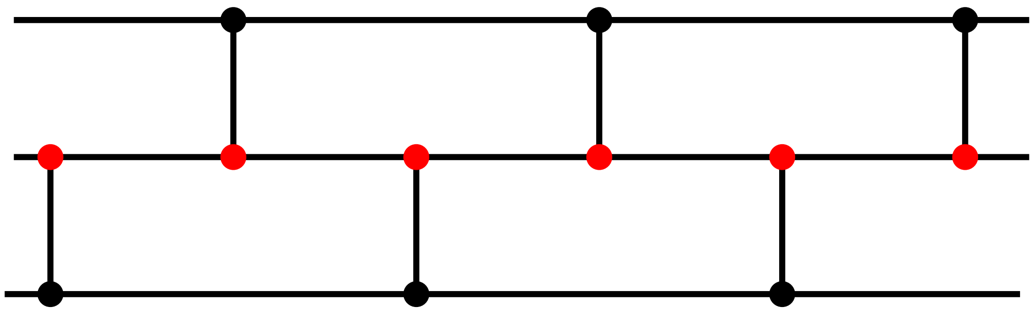
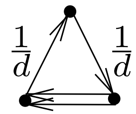




- One infinite acyclic belt;
- Finitely many seeds modulo shift along the belt.

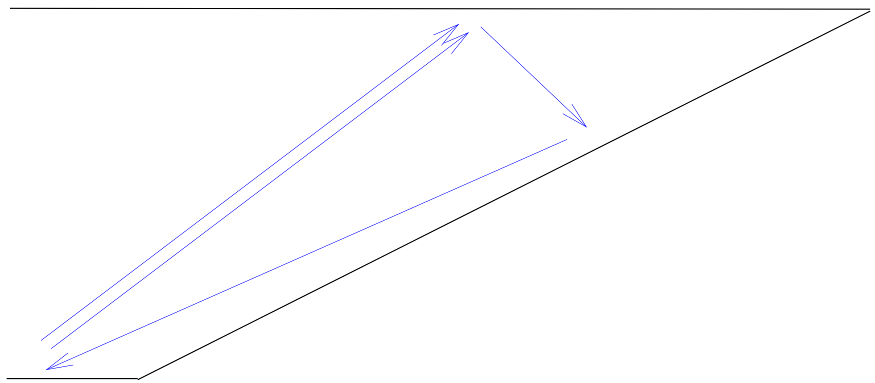
4. Exchange graph:

Rank 3 - affine type



from here: joint with Philipp Lampe

arXiv:1904.03928



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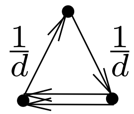
Rank 3 - affine type 

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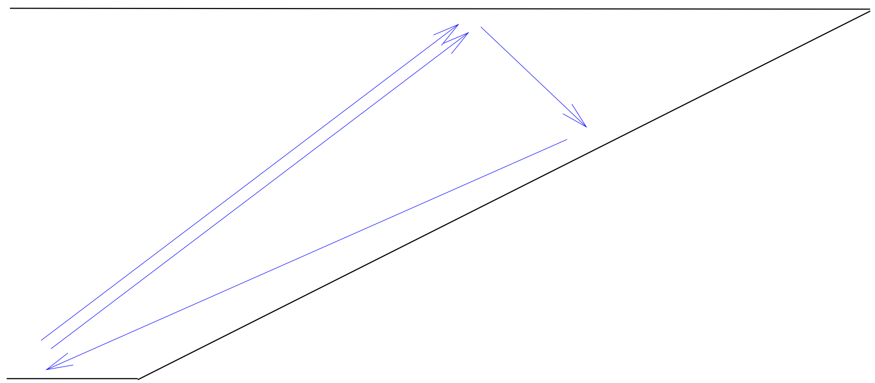

- It is of finite mutation type:

by reflections one can obtain finitely many slopes.

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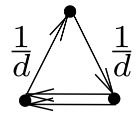

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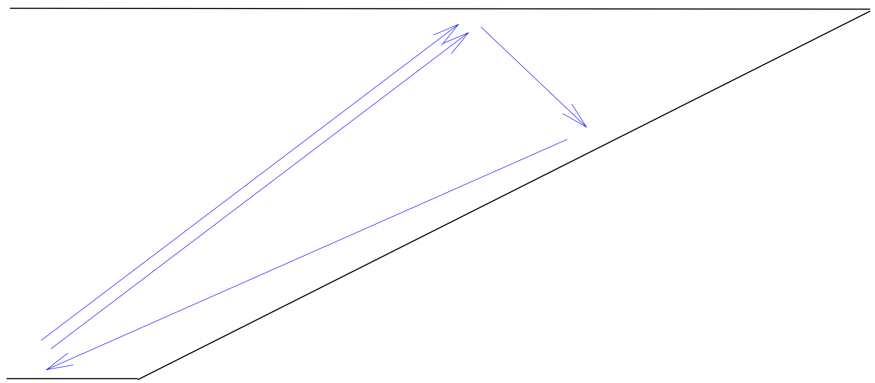

- It is of finite mutation type:

  by reflections one can obtain finitely many slopes.

- All triples of suitable angles $(\frac{p\pi}{d}, \frac{q\pi}{d}, \frac{r\pi}{d})$, with $p+q+r=d$ are in the mutation class.

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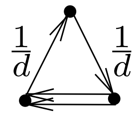

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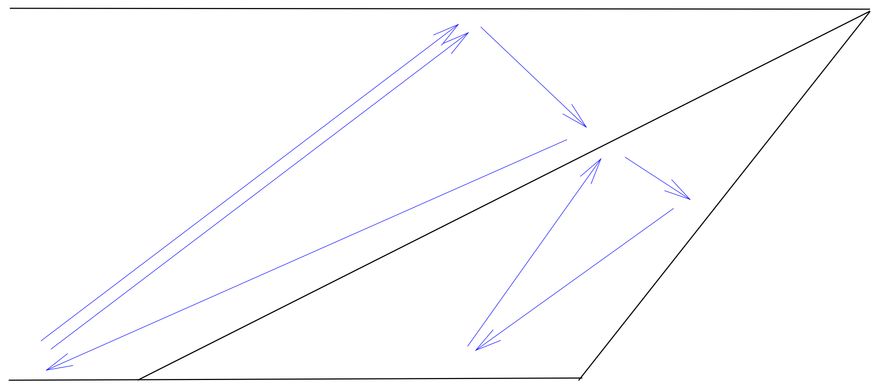

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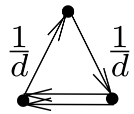

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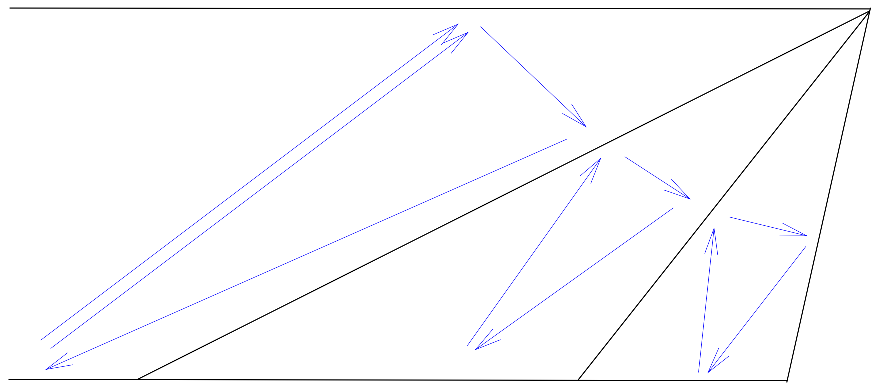

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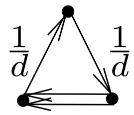
Rank 3 - affine type 

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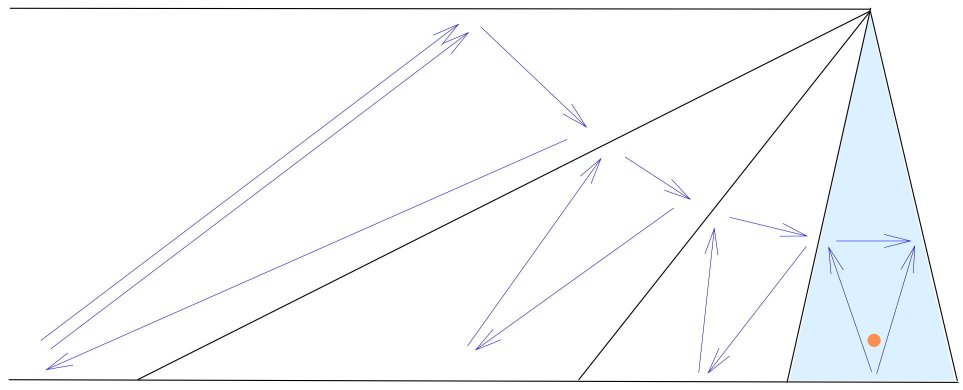

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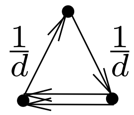
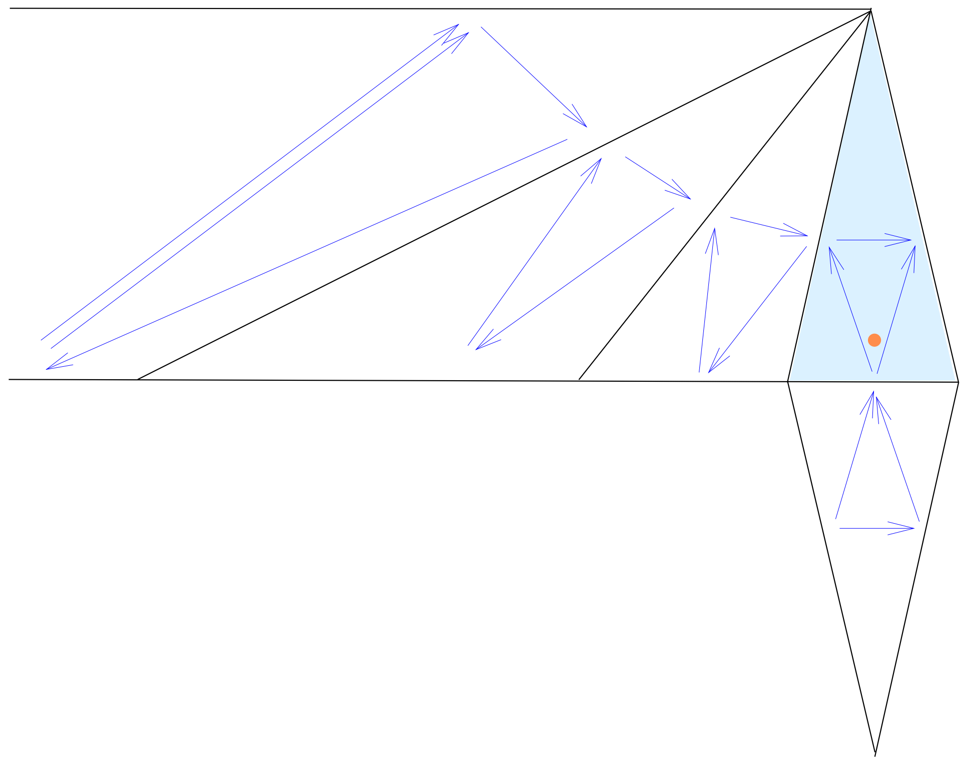


from here: joint with Philipp Lampe

- Initial acyclic seed

- An acyclic belt



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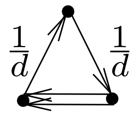
Rank 3 - affine type 

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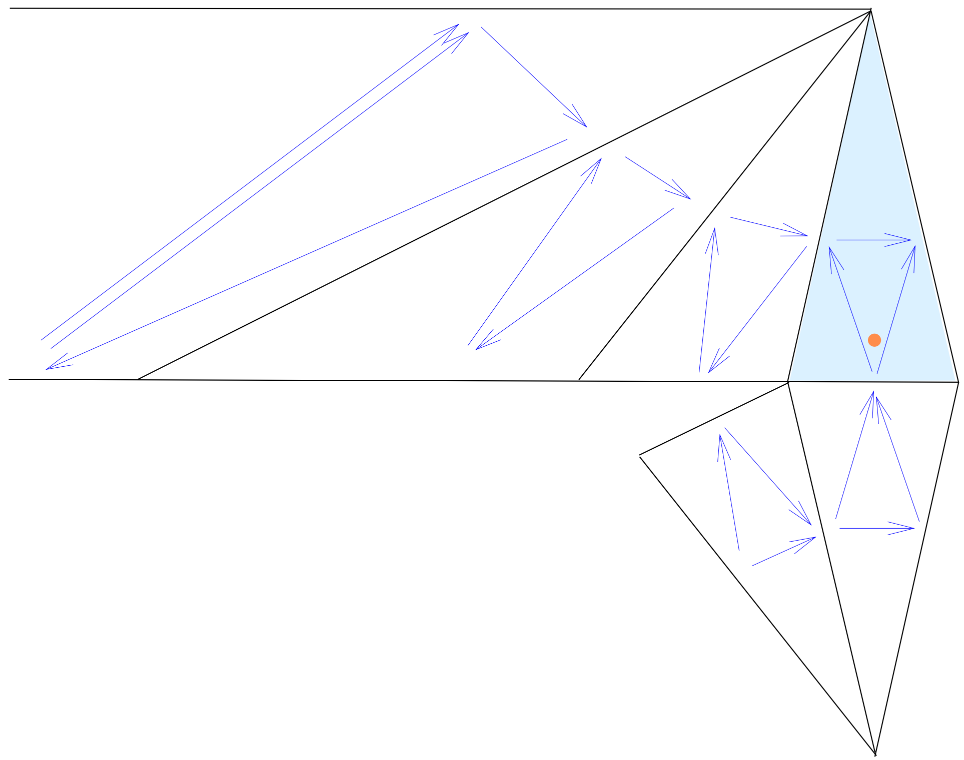

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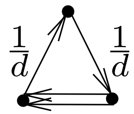
Rank 3 - affine type 

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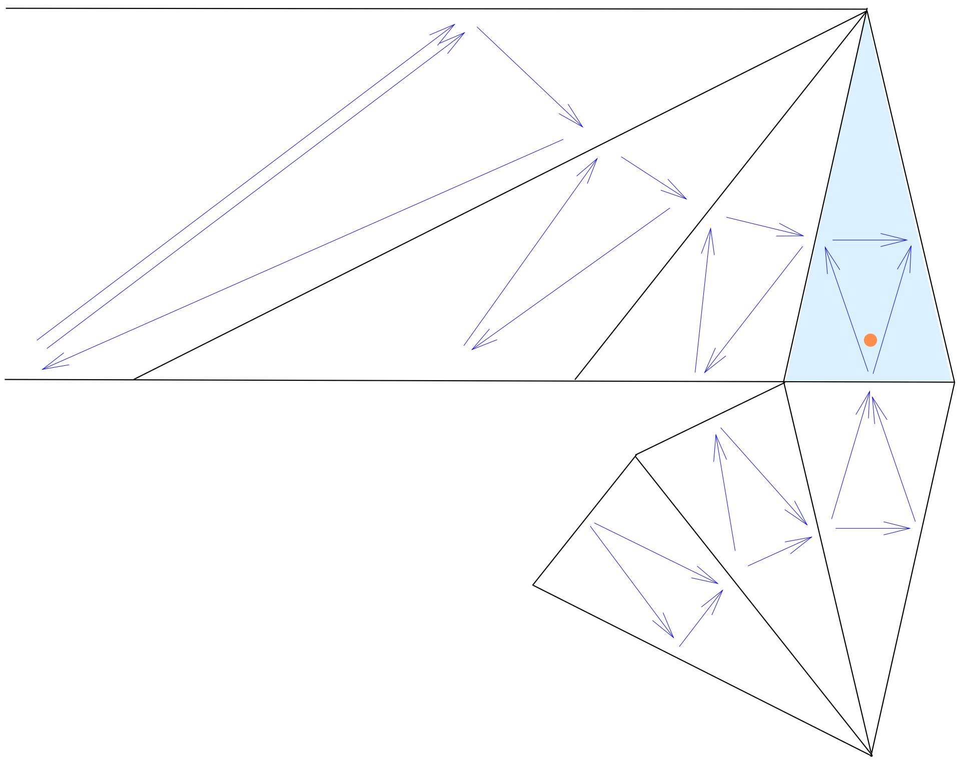

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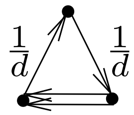

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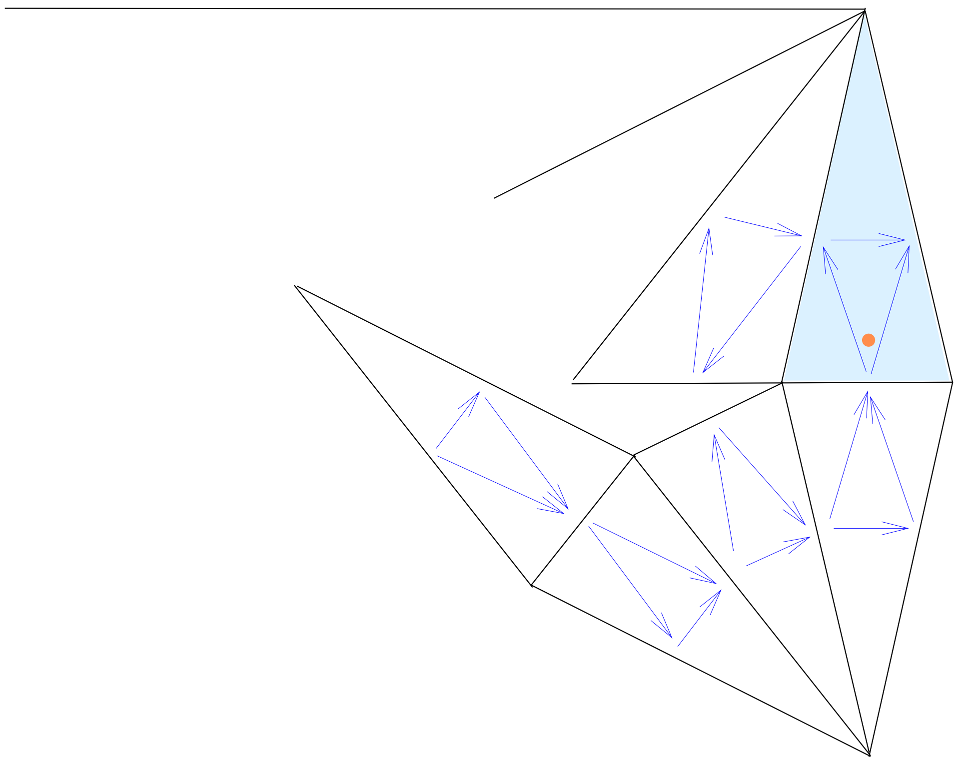

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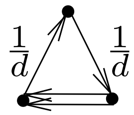
Rank 3 - affine type 

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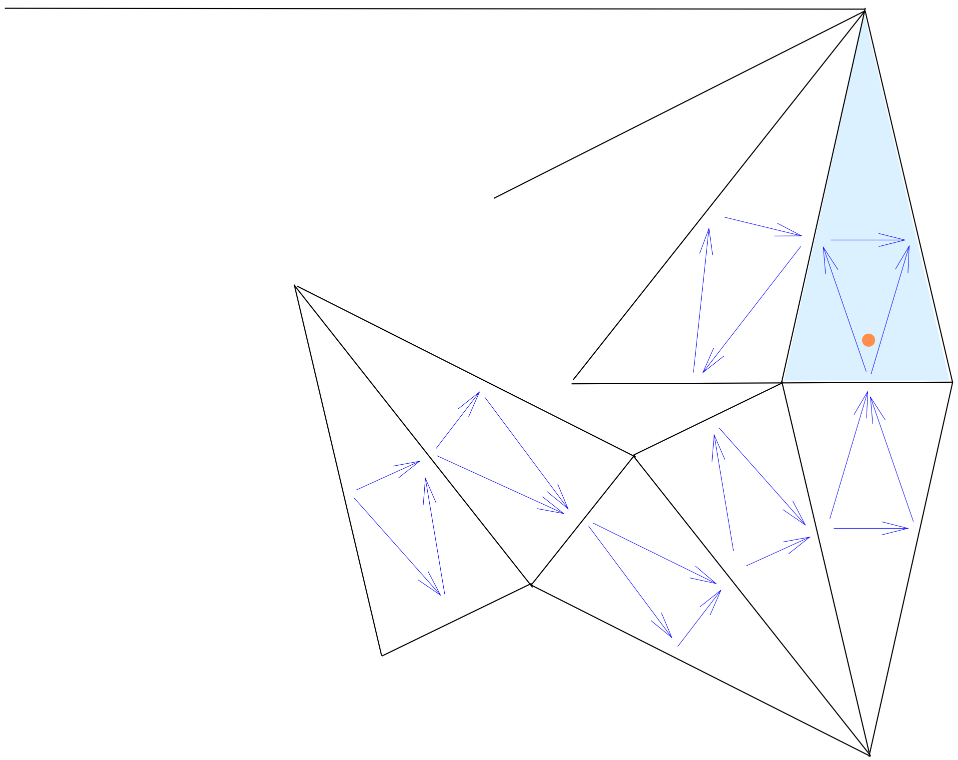

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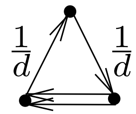
Rank 3 - affine type 

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- Initial acyclic seed
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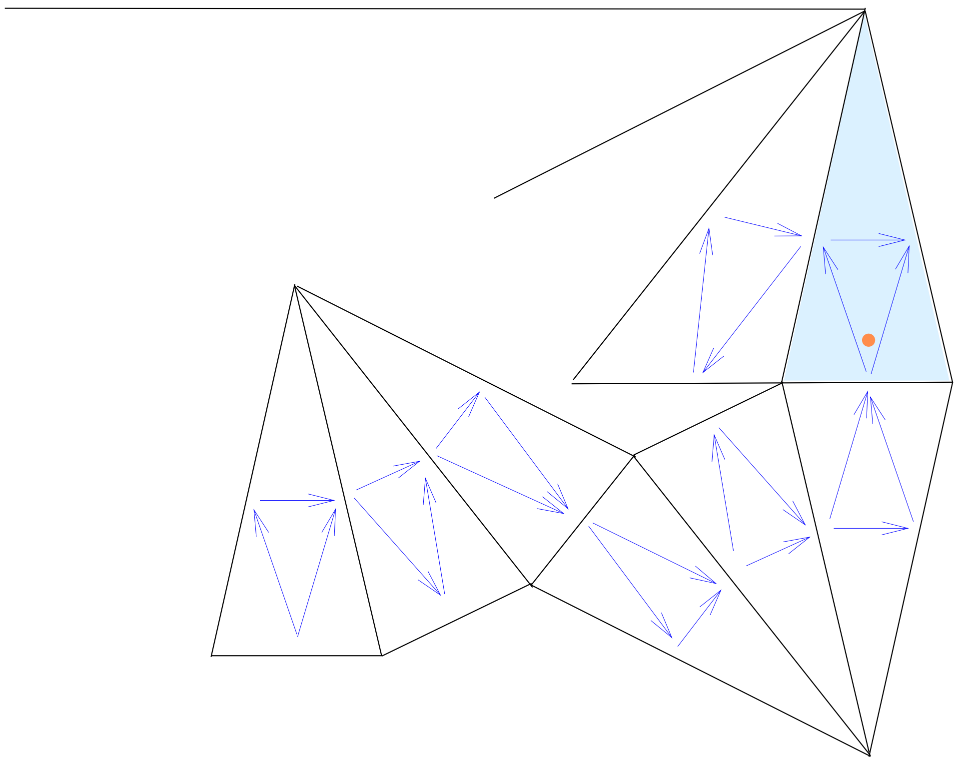

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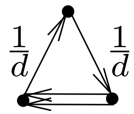


from here: joint with Philipp Lampe

- Initial acyclic seed

- An acyclic belt

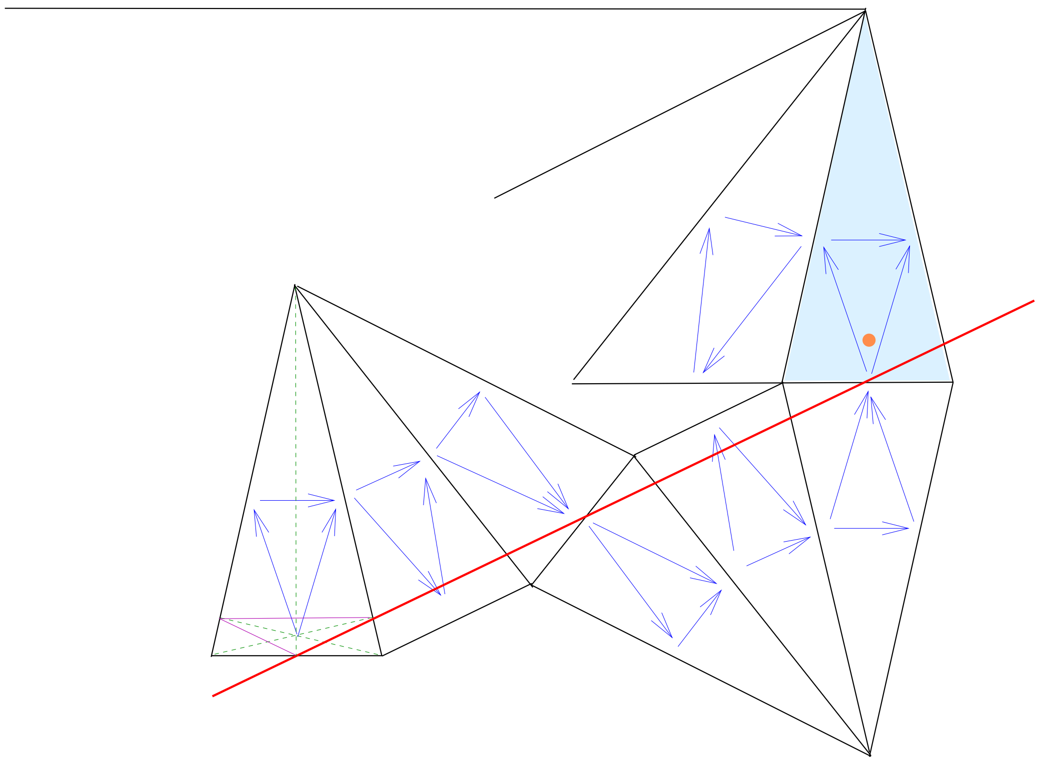
- Belt (or billiard) line

  passes through two

  feet of altitudes

  cf. Fagnano's problem:

  billiard tranjectory in triangle.



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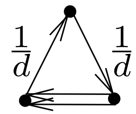
Rank 3 - affine type 

from here: joint with Philipp Lampe

- Away from acyclic belt:

  two feet of altitudes are
  on the belt line.

- Belt (or billiard) line

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  feet of altitudes

  cf. Fagnano's problem:
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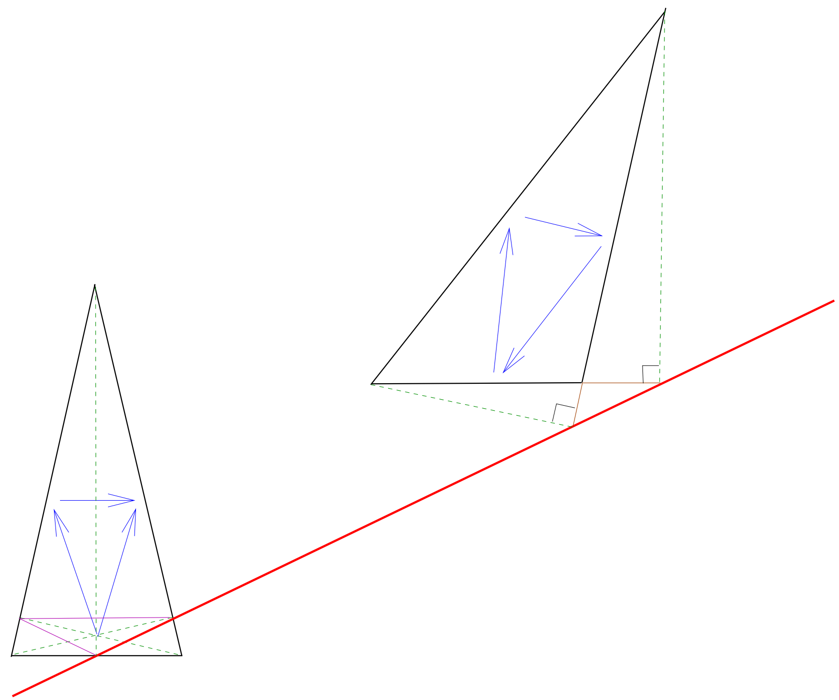

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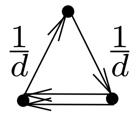

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- Away from acyclic belt:

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$\Rightarrow$ If there are shifts,
$\boxed{\text{they are parallel to the belt line}}$

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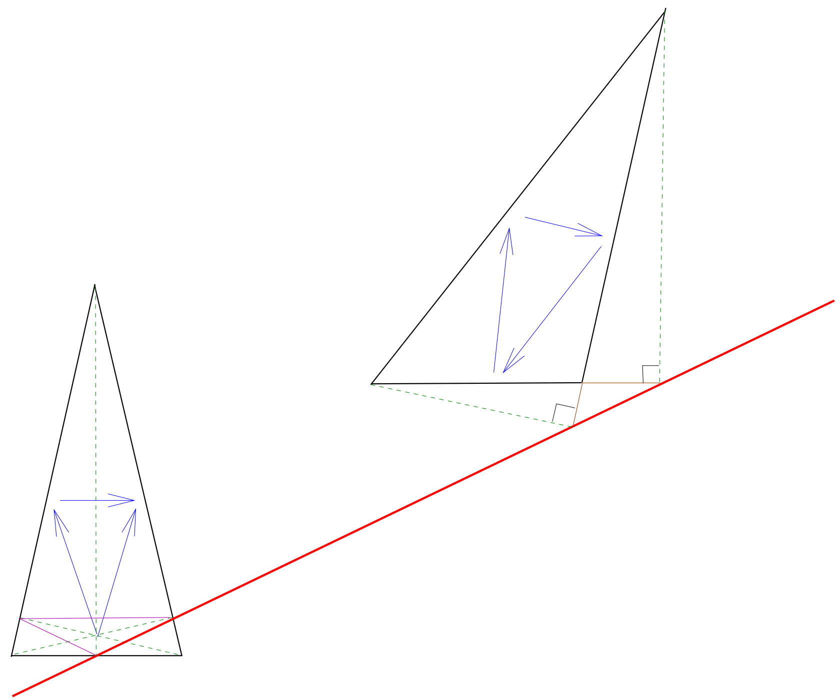

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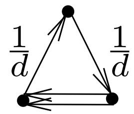


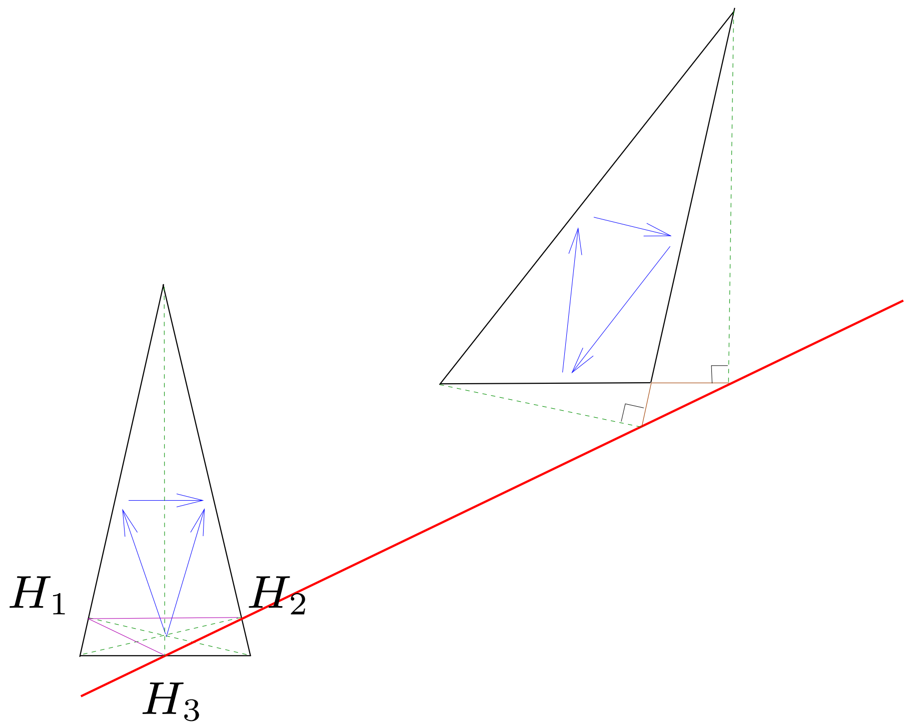
from here: joint with Philipp Lampe

- Invariant under mutation:

  $T(\Delta) = a_i \sin(A_j) \sin(A_k)$

If there are shifts,

they are parallel to the belt line



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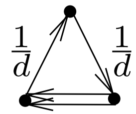

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  Triangles with same angles

  are congruent

If there are shifts,

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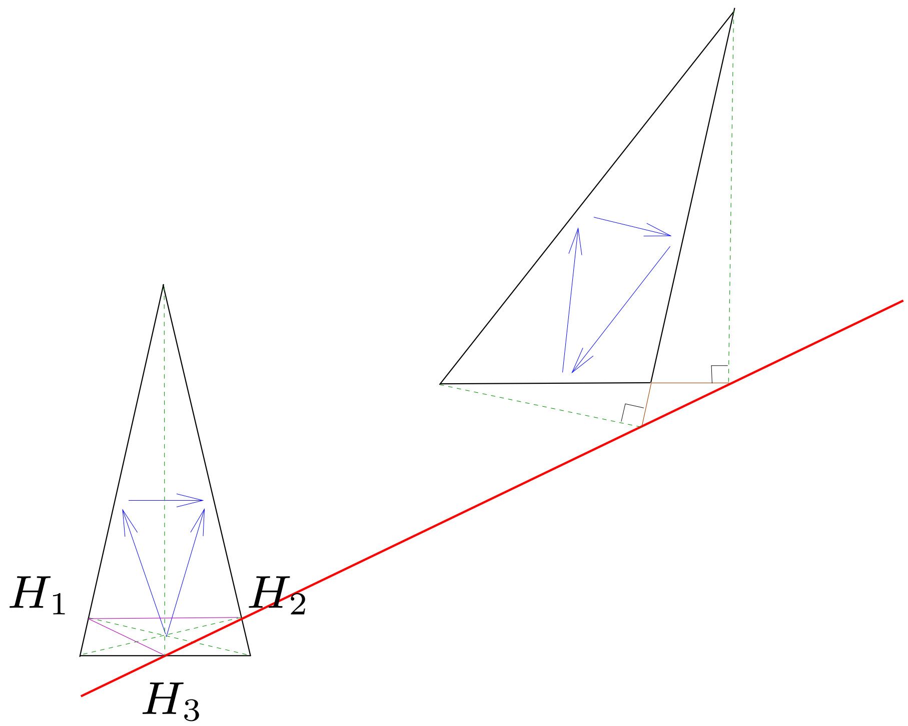

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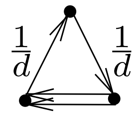
Rank 3 - affine type 

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  Triangles with same angles
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  Need: to find all shifts!

If there are shifts,
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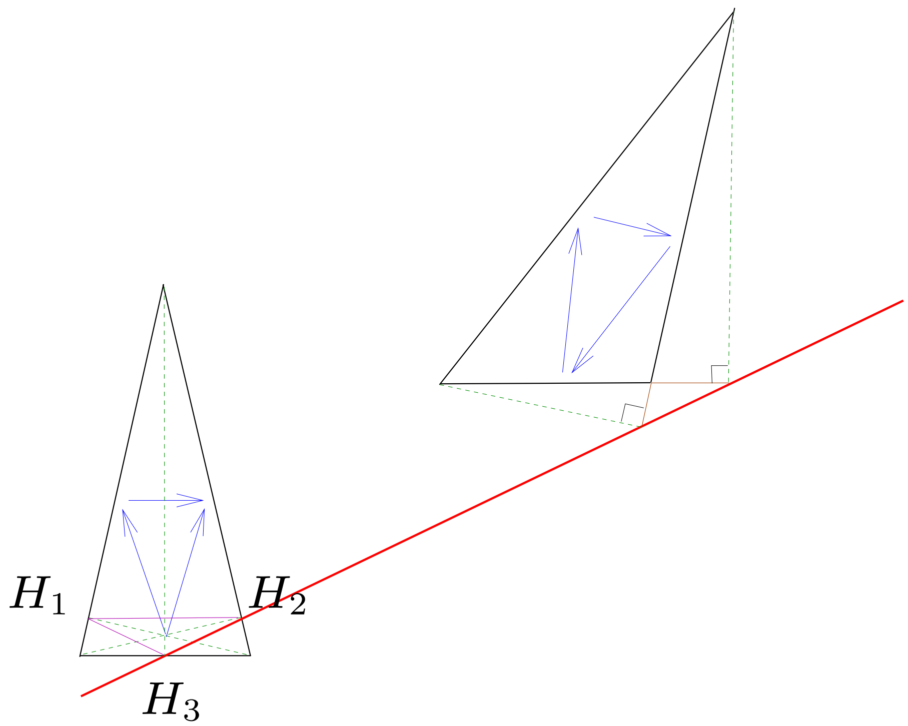

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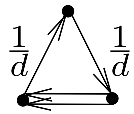

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are congruent

Need: to find all shifts!

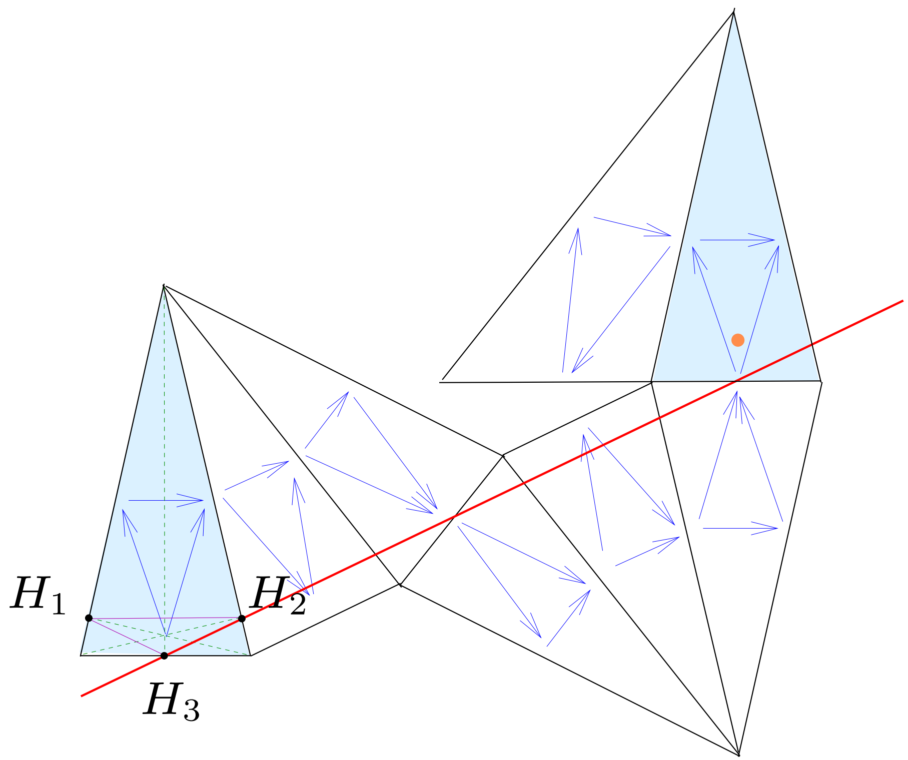
- $T(\Delta) = \frac{1}{2}(|H_1H_2| + |H_2H_3| + |H_3H_1|)$

- $4T(\Delta) =$ shift along the belt line

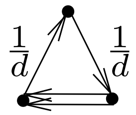
(in every acyclic belt)

If there are shifts,

they are parallel to the belt line



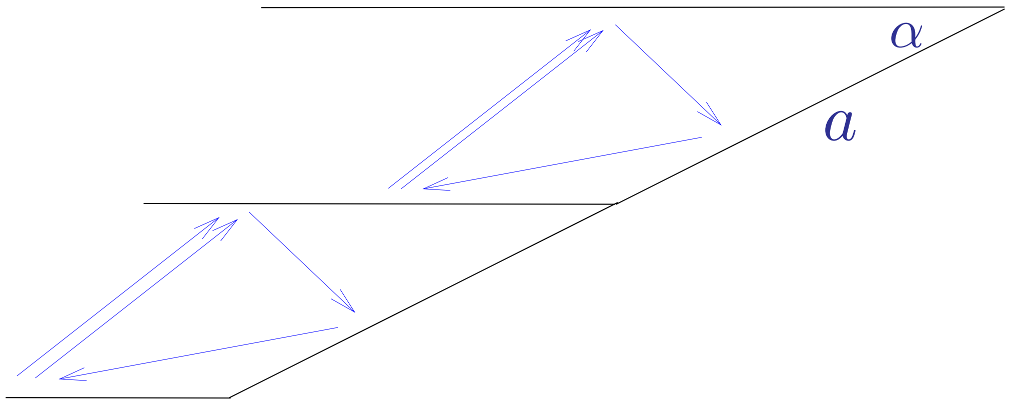
4. Exchange graph:

Rank 3 - affine type 

from here: joint with Philipp Lampe

- There are more shifts:

  each infinite region induces a shift:



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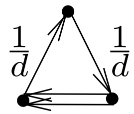

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- There are more shifts:

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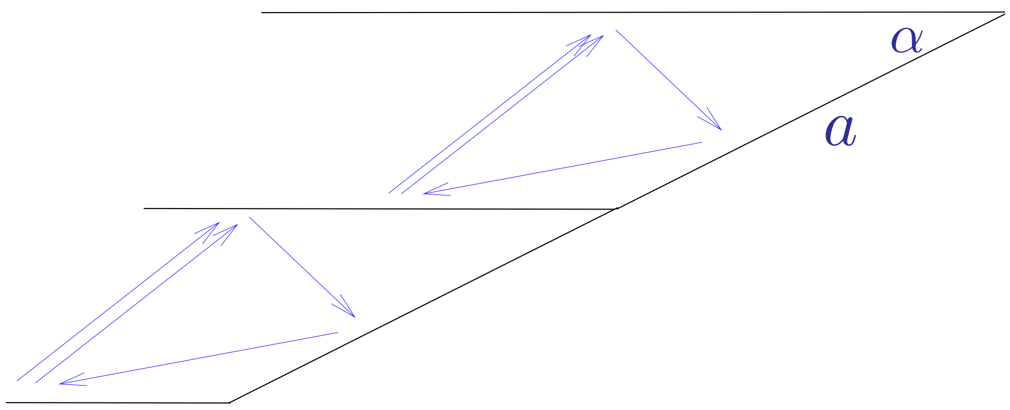
- If a is the finite side, α the angle

  then $T(\Delta) = a \sin^2 \alpha$

  So, $a = \frac{T(\Delta)}{\sin^2 \alpha}$



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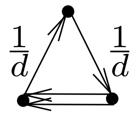

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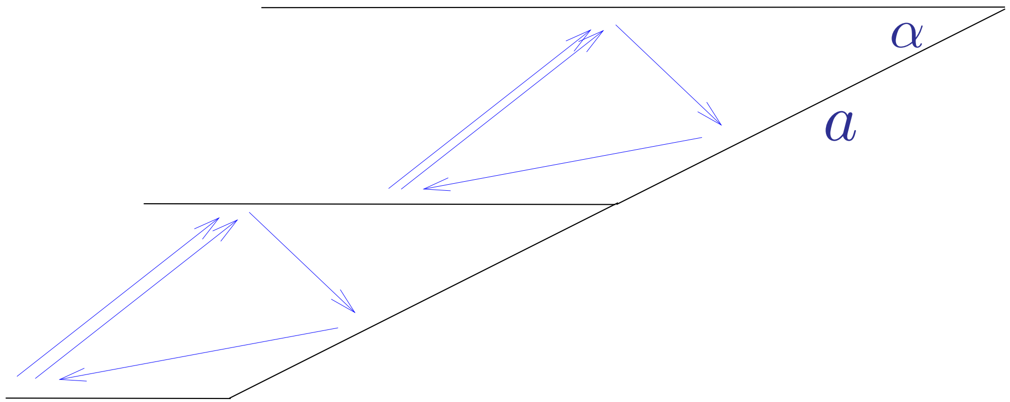

- If $\alpha = \frac{\pi}{d}$, then we have shifts:

$$4T, \ \frac{T}{\sin^2(\pi/d)}, \ \frac{T}{\sin^2(2\pi/d)}, \ \frac{T}{\sin^2(3\pi/d)}, \ \cdots$$

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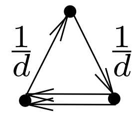
Rank 3 - affine type 

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Theorem. [FL'2018]
Let Q be an affine type rank 3 mutation-finite quiver.
Then the exchange graph of Q grows polynomially
and is quasi-isometric to some lattice L.

$$rk_{\mathbb{Z}}(L) = \begin{cases} \varphi(d), & \text{for some } d \in 2\mathbb{Z}; \\ \frac{1}{2}\varphi(d), & \text{otherwise.} \end{cases}$$

Here, $\varphi(d) = \#\{k \in \{1, 2, \ldots, d\} \mid gcd(k, d) = 1\}$

is the Euler's totient function.

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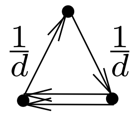

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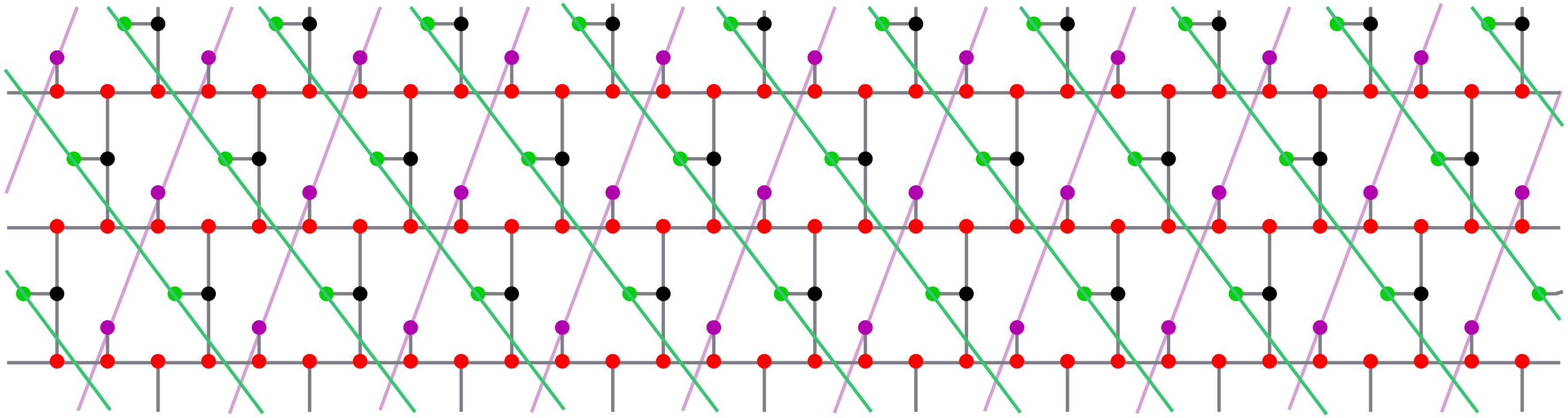
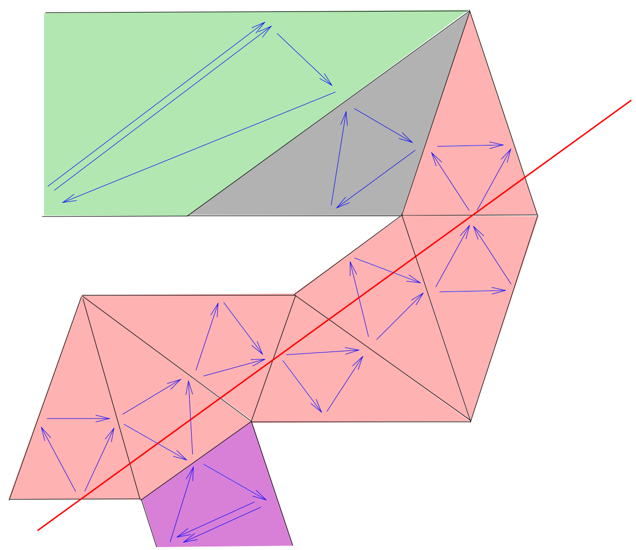


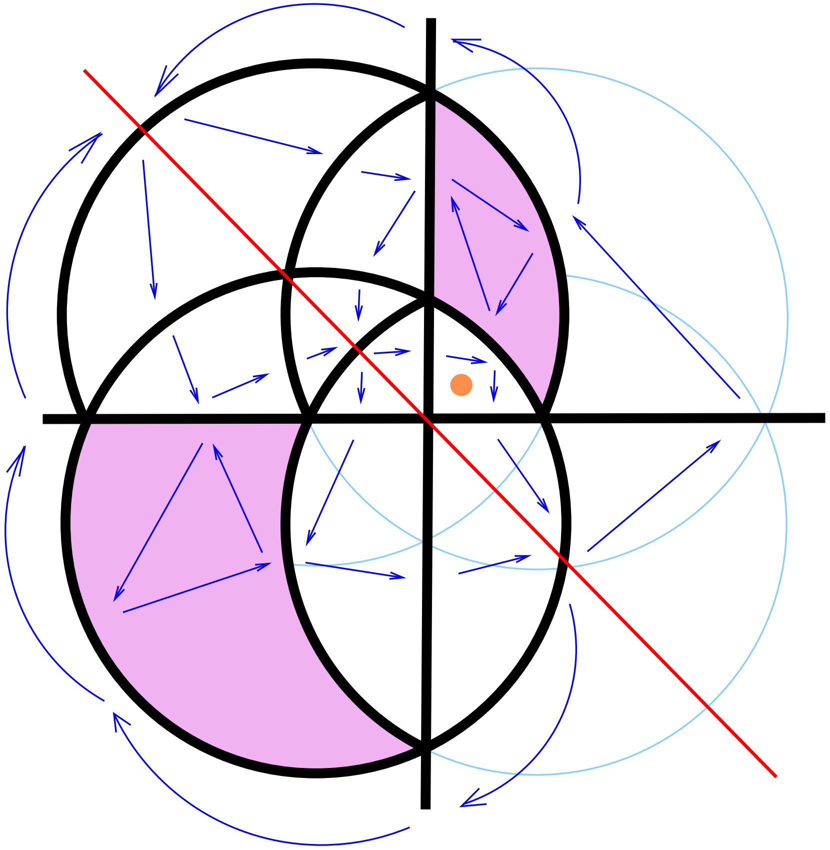
Example: exchange graph for $d = 5$

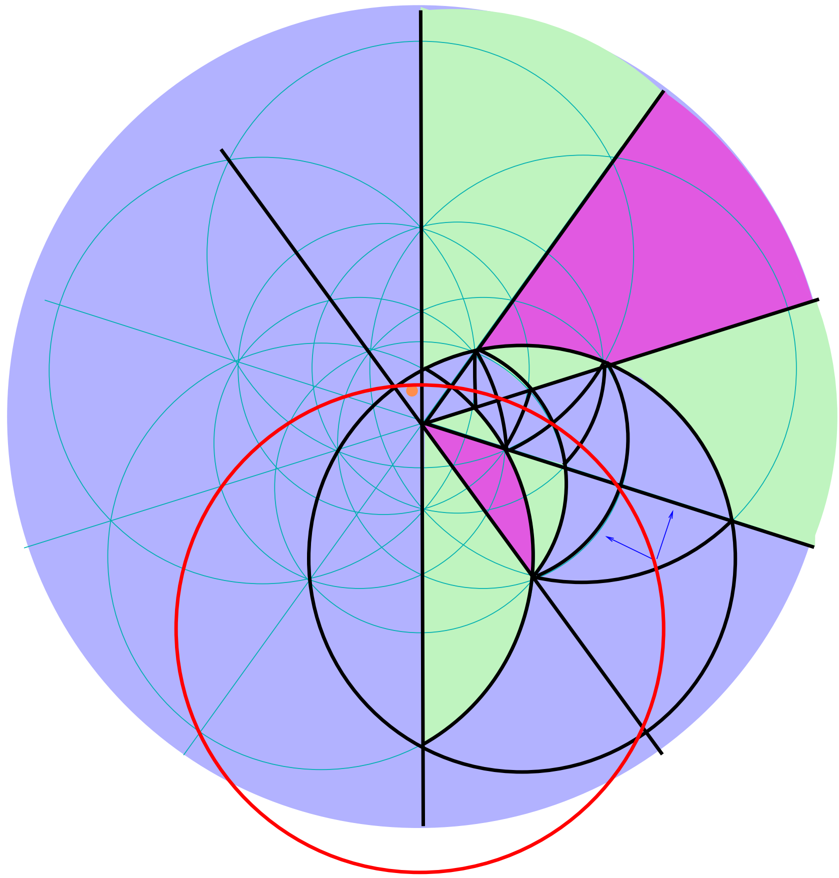


4. Exchange graph:

Rank 3 - affine type

Remark: Similar belt line in finite type:





5. Remarks on definition of mutation

We defined:

Mutation $\rightsquigarrow$ Partial reflection

$$\mu_k(v_i) = \begin{cases} v_i - \langle v_i, v_k \rangle v_k, & \text{if } k \to i \text{ in } Q \\ -v_k, & \text{if } i = k \\ v_i, & \text{otherwise} \end{cases}$$

Not

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Not an involution!

Define:

if v_k is positive:

$$\mu_k(v_i) = \begin{cases} v_i - \langle v_i, v_k \rangle v_k, & \text{if } k \rightarrow i \text{ in } Q \\ -v_k, & \text{if } i = k \\ v_i, & \text{otherwise} \end{cases}$$

if v_k is negative:

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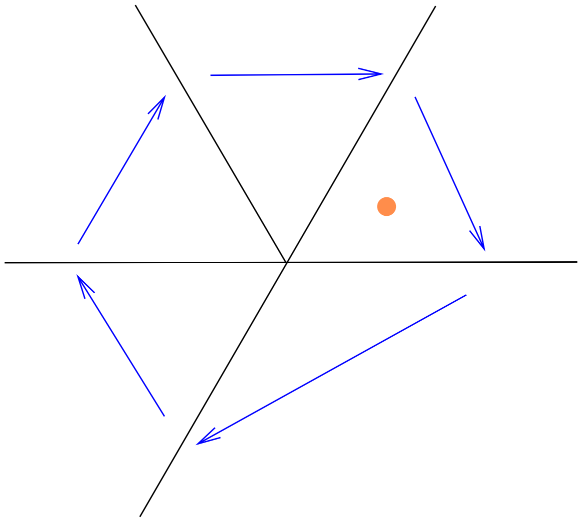
What to mean by positive / negative?

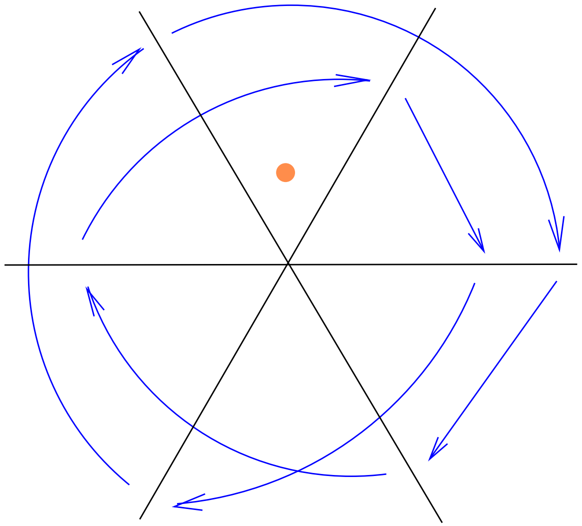
5. Remarks on definition of mutation

What to mean by positive / negative?

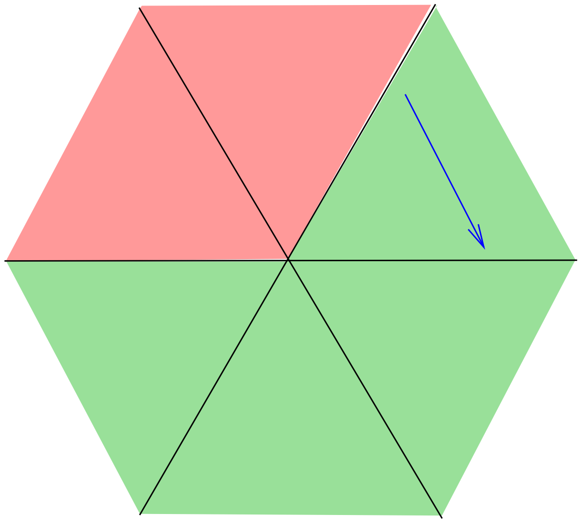
In rank 2:



Period 5



Period 7



- admissible positions
- forbidden positions

of the reference point

5. Remarks on definition of mutation

What to mean by positive / negative?

In rank 3: Reference point is in admissible position,
if it is in admissible position
for every rank 2 subquiver in every cluster.

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Theorem [FL'19] For every rank 3 finite type quiver,

(1) there are geometric realisations
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(4) every choice of reference point in an acute-angled (i.e. acyclic) domain gives such a realisation.

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Theorem [FL'20] For every finite type quiver,

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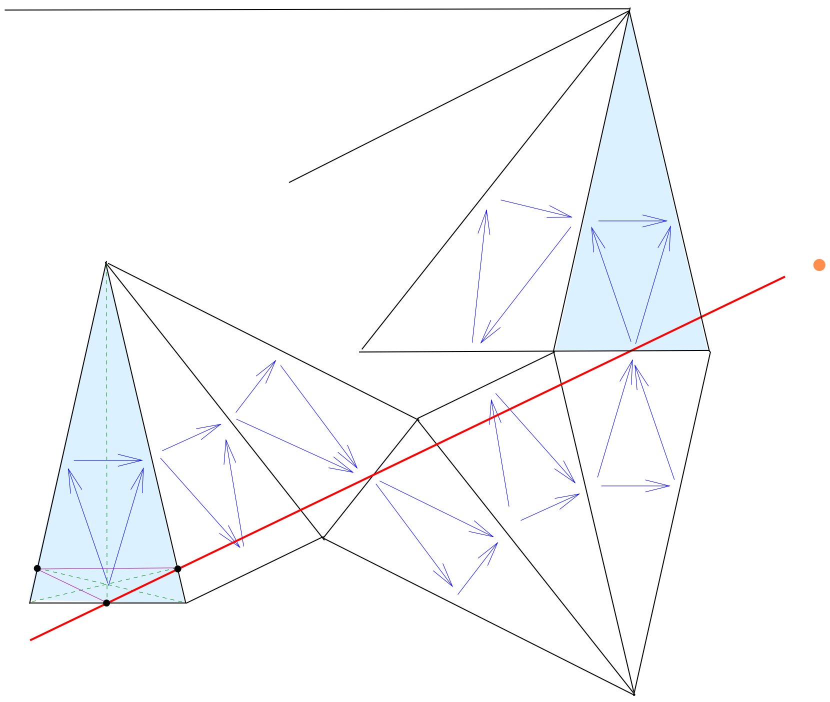
5. Remarks on definition of mutation

What to mean by positive / negative?

Theorem [FL'19] For every rank 3 affine quiver,

there exists a unique admissible position of the reference point:

it is the limit point at the end of the belt line.



Non-integer quivers: geometry and mutation-finiteness

Anna Felikson
Pavel Tumarkin
Durham University

Introduction

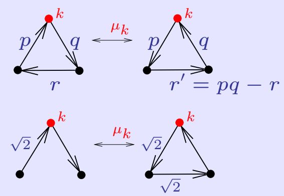
- $B = \{b_{ij}\}$ a skew-symmetric matrix with $b_{ij} \in \mathbb{R}$.
- Mutate B by usual mutation rule:

$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } i = k \text{ or } j = k \\ b_{ij} + \frac{1}{2}(|b_{ik}|b_{kj} + b_{ik}|b_{kj}|), & \text{otherwise} \end{cases}$$

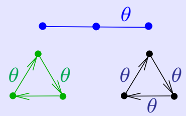
- B defines a **non-integer quiver** (with arrows of real weights $b_{ij} = -b_{ji}$).

- **Mutation** μ_k of a non-integer quiver:
  1) reverse all arrows incident to k;
  2) for every path $i \xrightarrow{p} k \xrightarrow{q} j$ with $p, q > 0$ apply:

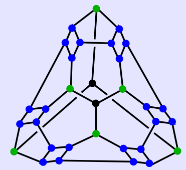
- Example: B_3



- **Question:** when a real quiver Q is mutation-finite?
- Example: H_3 (mutation class and "exchange graph")



where $\theta = 2\cos\frac{\pi}{5}$

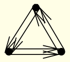


(cf. generalised associahedron in [FR])

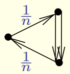
Quivers of rank 3

Theorem [FT1]. Any mutation-finite rank 3 quiver is mutation-equivalent to one of

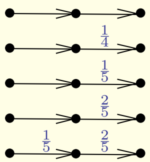
- Markov quiver:



- Affine quivers:



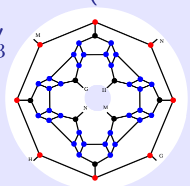
- Finite type quivers:



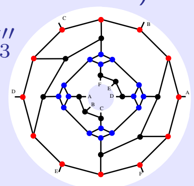
Here, a label $\frac{k}{m}$ stays for the weight $|b_{ij}| = 2\cos\frac{k\pi}{m}$.

- All of these (but Markov) are mutation-acyclic.
- "Exchange graphs" for H'_3 and H''_3 are graphs on a torus (with two acyclic belts each):

H'_3

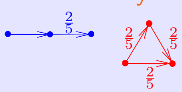


H''_3



- Each of the mutation classes H'_3 and H''_3 has two different acyclic representatives:

H'_3:



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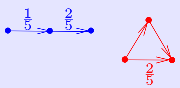


Geometric realisation by reflections (GR)

- GR of a quiver of rank n: vectors $v_1,\dots,v_n$ in a quadratic vector space V s.t. $(v_i, v_i) = 2$ and $(v_i, v_j) = -|b_{ij}|$.
- Mutation = partial reflection:

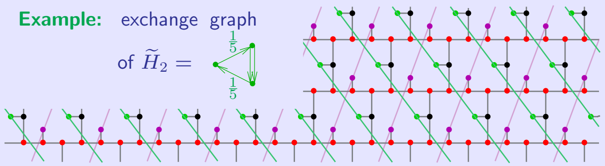
$$\mu_k(v_i) = \begin{cases} v_i, & \text{if } b_{ki} \geq 0, i \neq k \\ -v_i, & \text{if } i = k \\ v_i - (v_i, v_k)v_k, & \text{if } b_{ki} < 0 \end{cases}$$

- Mutation class has a GR if GR of quivers commute with mutations.

Theorem [FT1,FT2]. Mutation class of any real acyclic quiver with $|b_{ij}| \geq 2\ \forall i,j$ admits a GR.

- When GR exists, we define (geometric) Y-seeds (n-tuples of vectors in V) and "exchange graphs".

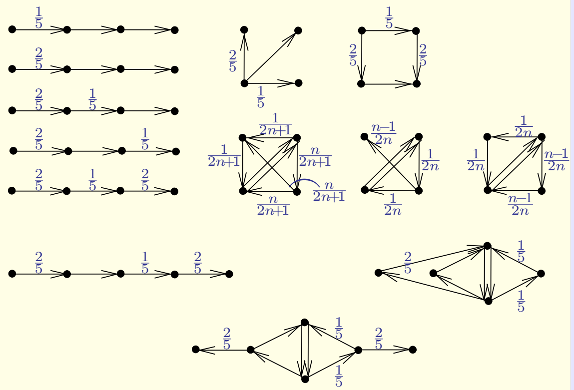
Example: exchange graph of $\widetilde{H}_2 =$



Theorem [FL]. Let Q be an affine type rank 3 mut.-fin. quiver. Then "exchange graph" of Q grows polynomially and is quasi-isometric to a lattice of some dimension.

Finite mutation type: classification

Theorem [FT3]. Any mut.-fin. non-integer quiver of rank $n > 3$ is mutation-equivalent to either one of B_n, F_4, $\widetilde{B}_n$, $\widetilde{C}_n$, $\widetilde{F}_4$ or one of



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References

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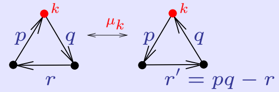

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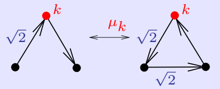

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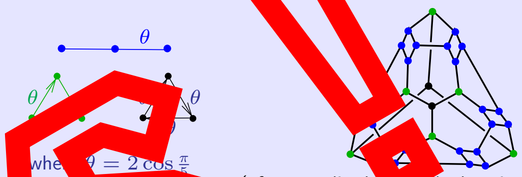

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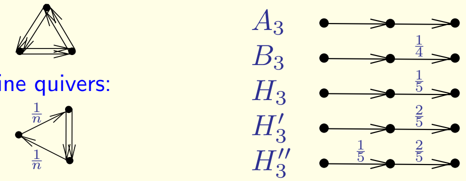

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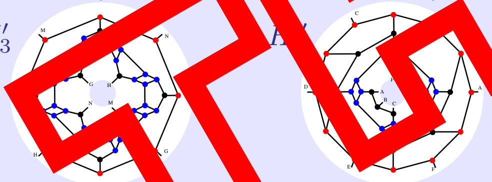

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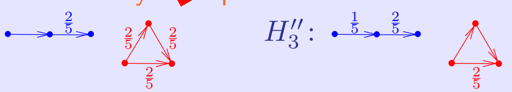

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- Mutation = ...rtial ...flectio...
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- Mu...tion c...ss ha... GR if GR ...f quivers ...omm...te w... muta...o...

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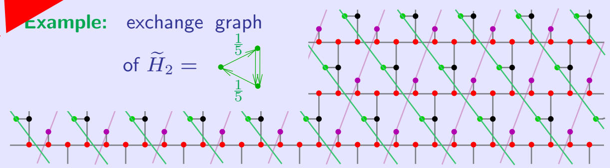

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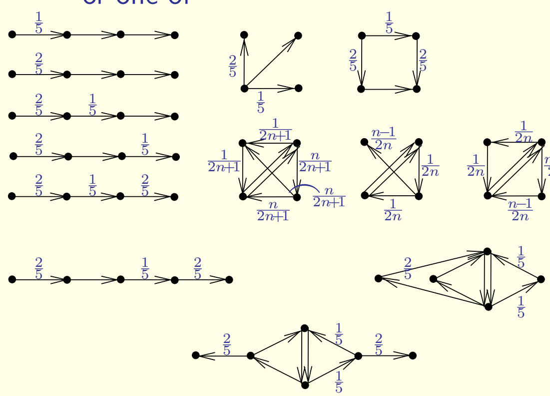

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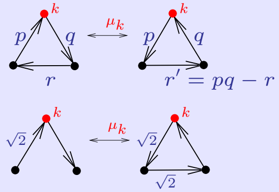
Introduction

- $B = \{b_{ij}\}$ a skew-symmetric matrix with $b_{ij} \in \mathbb{R}$.
- Mutate B by usual mutation rule:

$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } i = k \text{ or } j = k \\ b_{ij} + \frac{1}{2}(|b_{ik}|b_{kj} + b_{ik}|b_{kj}|), & \text{otherwise} \end{cases}$$

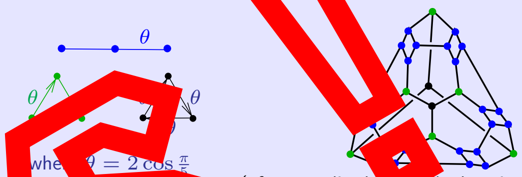
- B defines a **non-integer quiver** (with arrows of real weights $b_{ij} = -b_{ji}$).

- **Mutation** μ_k of a non-integer quiver:
  1) reverse all arrows incident to k;
  2) for every path $i \xrightarrow{p} k \xrightarrow{q} j$ with $p, q > 0$ apply:



- Example: B_3

- **Question:** ... al quiver Q is mutation-finite?
- Example: H_3 (mu... ion c... s and "exchange graph")

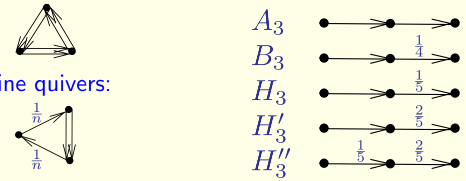


whe... $\theta = 2\cos\frac{\pi}{5}$ ... (cf. generalised a... iahedron in [FR])

Quivers of rank 3

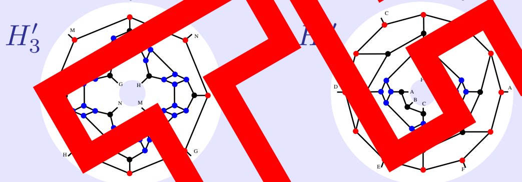
Theorem [FT1]. Any mutation-finite rank 3 quiver is mutation-equivalent to one of

- Markov quiver:
- Finite type quivers: A_3, B_3, H_3, H_3', H_3''
- Affine quivers:



Here, a label $\frac{k}{m}$ stays for the weight $|b_{ij}| = 2\cos\frac{k\pi}{m}$.

- All of these (but Markov) are muta...on-ac...lic.
- "Exchange graphs" for ... a... H_3'' ...e gra...s on a torus (with t... cy...ic be... each):



- Each of the muta...on c...ses H_3' and H_3'' has two different acycl... representatives:

Geometric realisation by reflecti... (GR)

- GR of a quiver of rank n: vectors $v_1, \ldots, v_n$ in a q...ratic ...ector ...ac... V s.t. $(v_i, v_i) = 2$ an...
- Mutation = ...tial ...

$\mu_k(v_i)$ ... v_i, $-v_i$, $v_i - (v_i, v$...

Mu...tion c...ss ha... c...mm...te w... mu...

T...re... [FT1, ...2] ... acy...c qu...r wit...

W...en ... exists, w... n-t...ples of vectors i...

Example: exchange graph of $\widetilde{H}_2 =$

Theorem [FL]. Let ... mut.-fin. quiver. T... grows polynomially ... lattice of some dim...

Finite mut...on type: classification

Th...rem [...T3]. Any mut.-fin. non-integer quiver of rank $n > 3$ is mutation-equivalent

THANKS!