Double pants decompositions of 2-surfaces

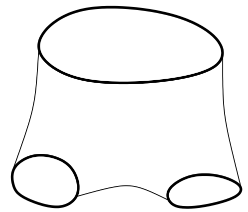
Anna Felikson

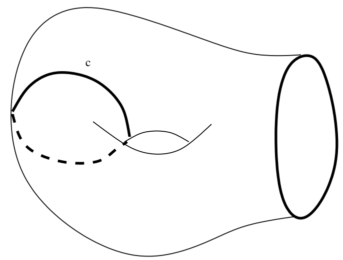
joint work with Sergey Natanzon

27.05.2010, Moscow

1. Pants decompositions

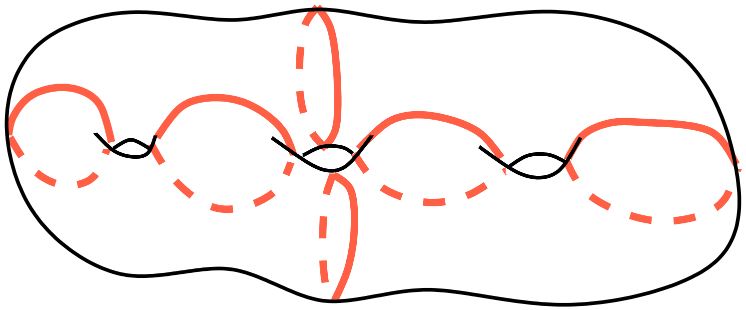
$S = S_{g,n}$ 2-surface, genus g, n holes;
P pants decomposition of $S =$
$=$ dec. into pairs of pants





$$P = \langle c_1, \ldots, c_{3g-3+n} \rangle$$

- $c_i \cap c_j = \emptyset$
- maximal system of curves

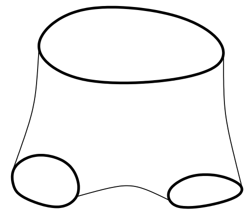


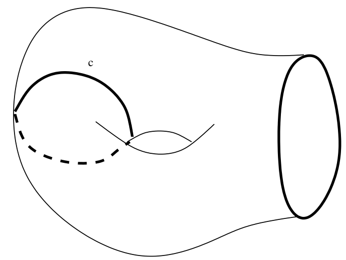
1. Pants decompositions

$S = S_{g,n}$ 2-surface, genus g, n holes;

P pants decomposition of $S =$

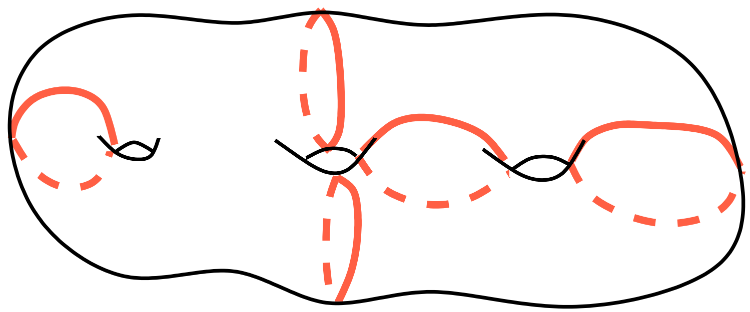
$=$ dec. into pairs of pants





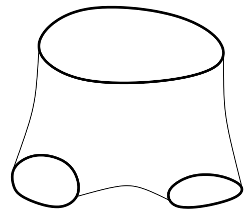
$$P = \langle c_1, \ldots, c_{3g-3+n} \rangle$$

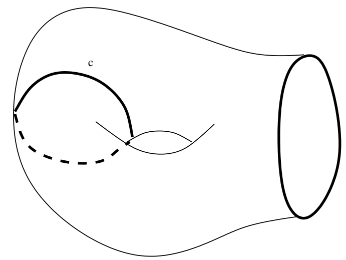
- $c_i \cap c_j = \emptyset$
- maximal system of curves



1. Pants decompositions

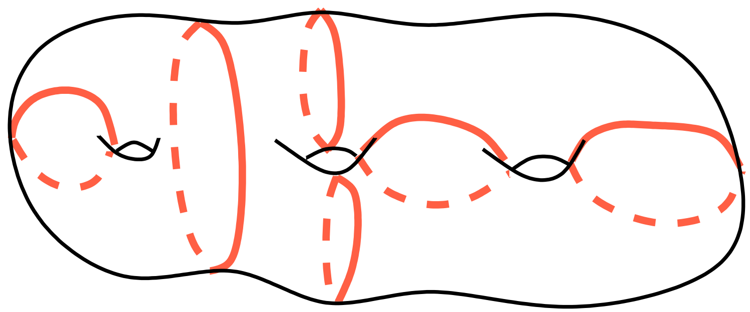
$S = S_{g,n}$ 2-surface, genus g, n holes;
P pants decomposition of $S =$
$=$ dec. into pairs of pants



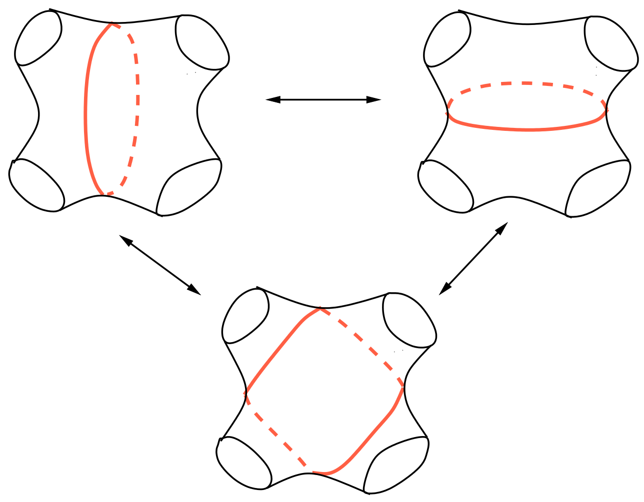


$$P = \langle c_1, \ldots, c_{3g-3+n} \rangle$$

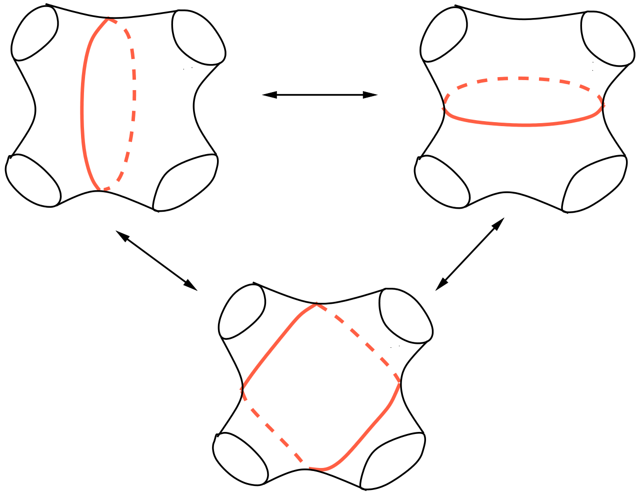
- $c_i \cap c_j = \emptyset$
- maximal system of curves



Flip:

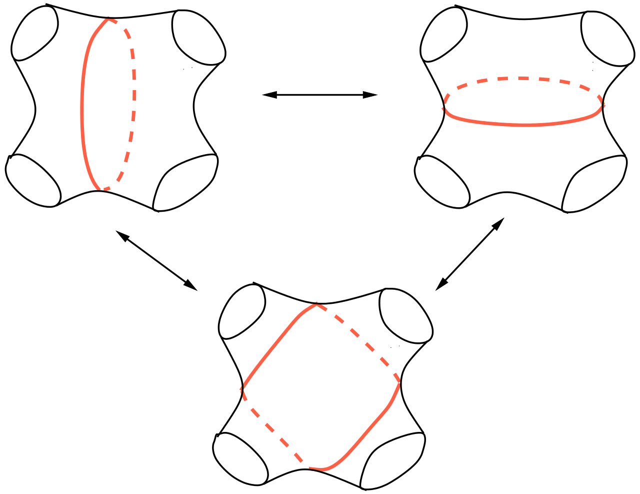


Flip:



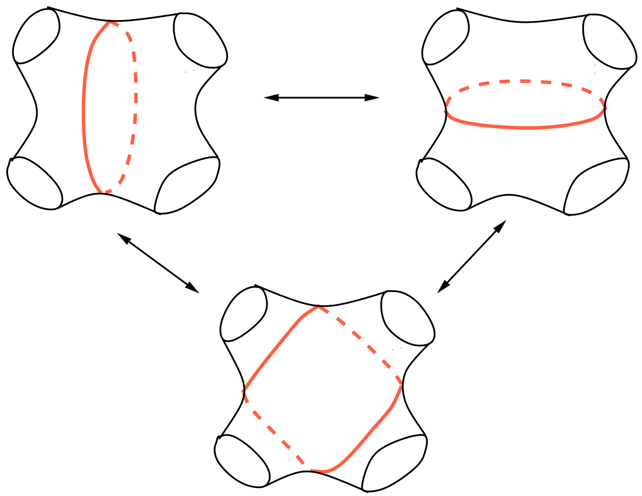
- Flips are sufficient to get to any "combinatorial type" of dec.

Flip:



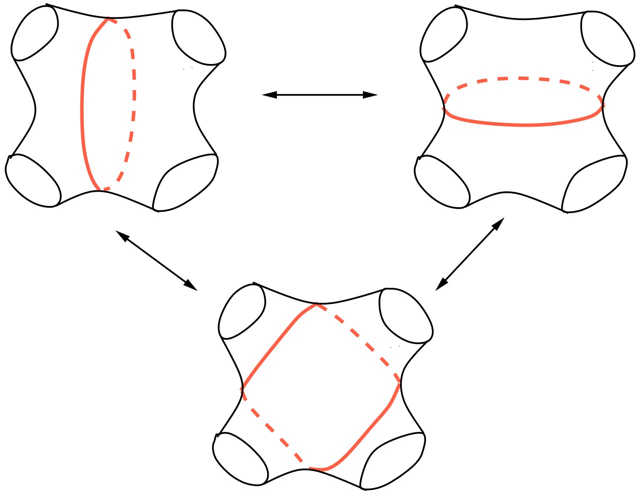
- Flips are sufficient to get to any "combinatorial type" of dec.
- Flip results in one of three possible classes in homology.

Flip:



- Flips are sufficient to get to any "combinatorial type" of dec.
- Flip results in one of three possible classes in homology.
- Flips do not act transitively on all pants decompositions:

Flip:



- Flips are sufficient to get to any "combinatorial type" of dec.
- Flip results in one of three possible classes in homology.
- Flips do not act transitively on all pants decompositions:

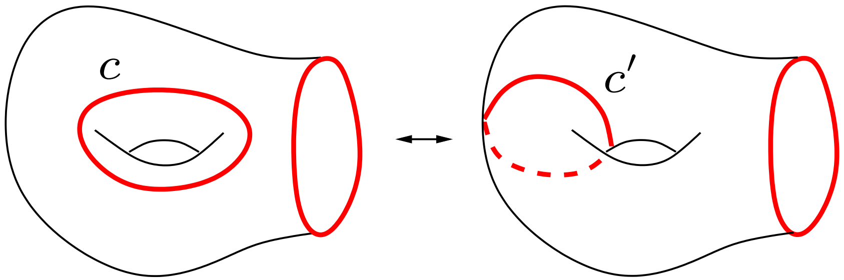
$$P \longrightarrow \mathcal{L}(P) \subset H_1(S, \mathbb{Z})$$
$$\mathcal{L}(P) = \langle h(c_1), \ldots, h(c_{3g-3+n}) \rangle$$

Lagrangian plane.

- Flips preserve $\mathcal{L}(P)$.

How to get out of $\mathcal{L}(P)$?

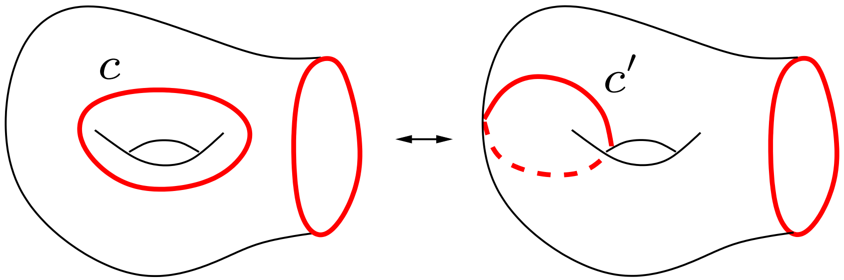
- Classical approach: add S-moves:



$$|c \cap c'| = 1.$$

How to get out of $\mathcal{L}(P)$?

- Classical approach: add S-moves:

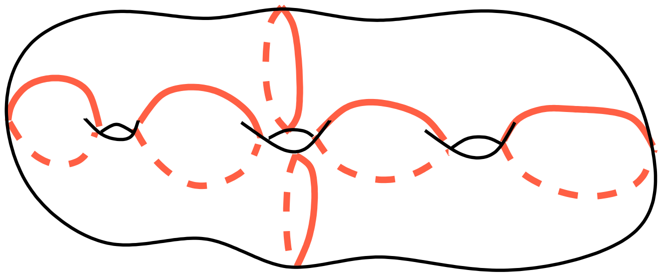


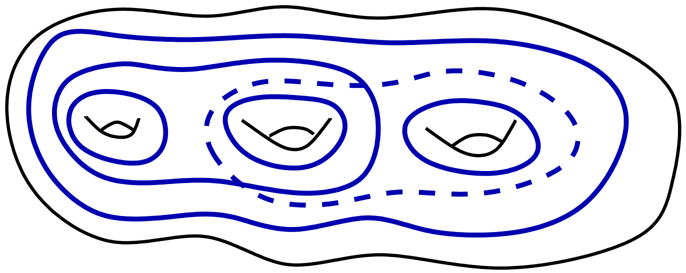
$$|c \cap c'| = 1.$$

Theorem.(Hatcher, Thurston' 80)
Flips and S-moves act transitively on pants dec.

2. Double pants decompositions

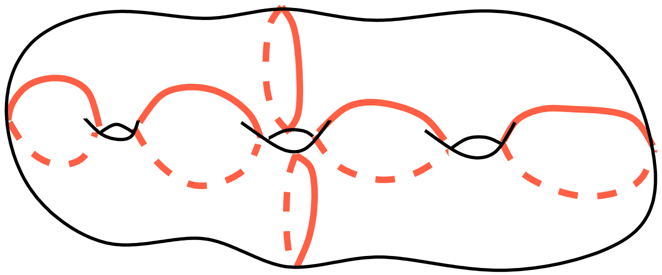
$DP = (P_a, P_b) \qquad \langle \mathcal{L}(P_a), \mathcal{L}(P_b) \rangle = H_1(S, \mathbb{Z}).$

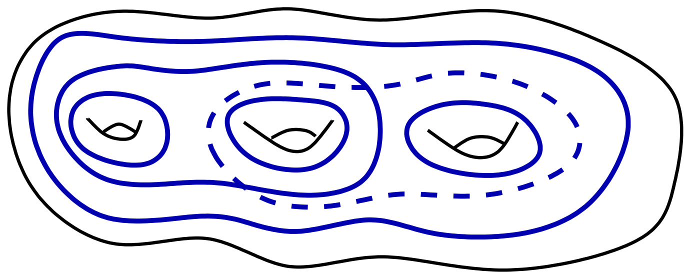




2. Double pants decompositions

$DP = (P_a, P_b) \quad \langle \mathcal{L}(P_a), \mathcal{L}(P_b) \rangle = H_1(S, \mathbb{Z}).$

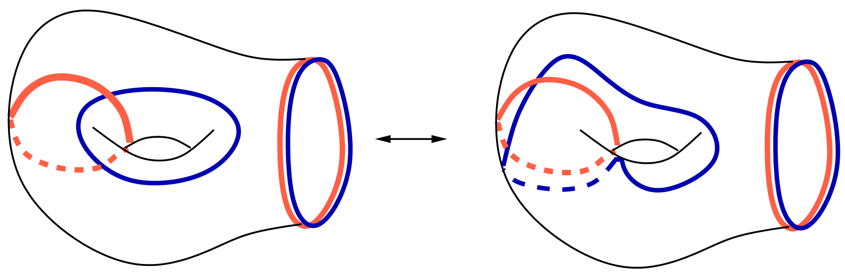




Operations:

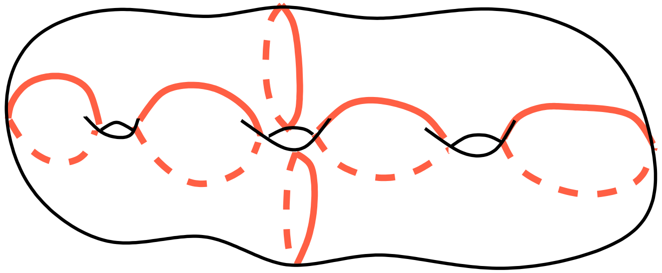
1. Flips in P_a, P_b

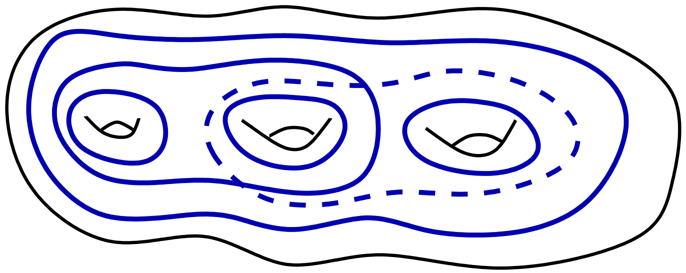
2. Handle-twists:



2. Double pants decompositions

$DP = (P_a, P_b) \quad \langle \mathcal{L}(P_a), \mathcal{L}(P_b) \rangle = H_1(S, \mathbb{Z}).$

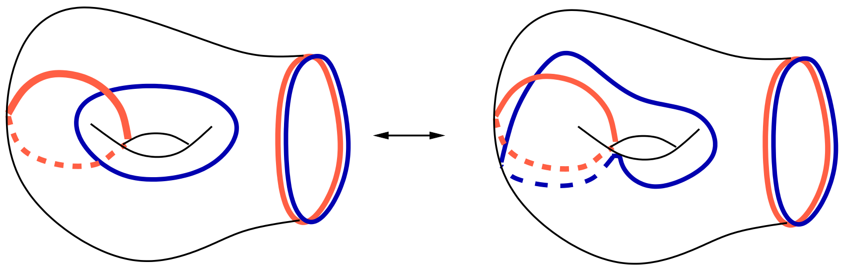




Operations:

1. Flips in P_a, P_b

2. Handle-twists:



Prop. 1. Flips and handle-twists act transitively on the pairs of Lagrangian planes.

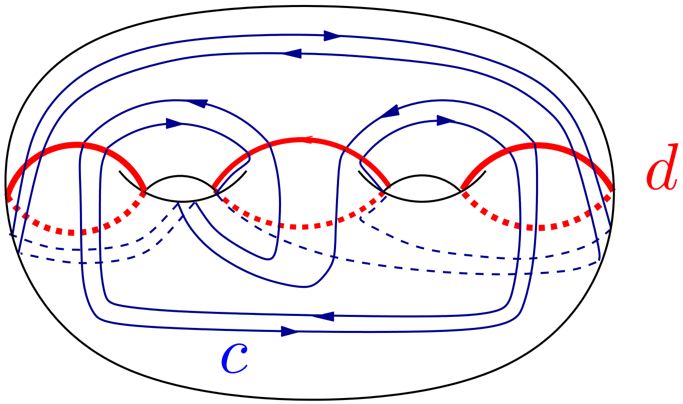
Problem: Do flips and handle-twists act transitively on DP's?

Problem: Do flips and handle-twists act transitively on DP's?

Natural way to solve: Prop. 1 + Transitivity of flips on $\mathcal{L}(P)$

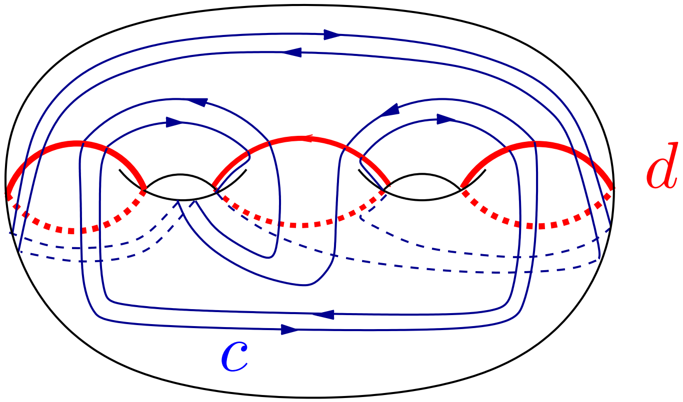
Problem: Do flips and handle-twists act transitively on DP's?

Natural way to solve: Prop. 1 + Transitivity of flips on $\mathcal{L}(P)$



Problem: Do flips and handle-twists act transitively on DP's?

Natural way to solve: Prop. 1 + Transitivity of flips on $\mathcal{L}(P)$

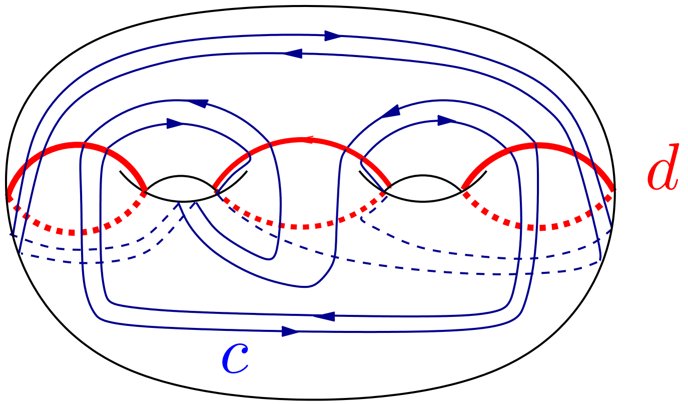


$S \hookrightarrow \mathbb{R}^3$ defines an inner handlebody S_+.

- If all $c \in P$ are contractible in S_+ then $flip(c)$ is...
- c is not contr. in S_+ (linked with d).

Problem: Do flips and handle-twists act transitively on DP's?

Natural way to solve: Prop. 1 + ~~Transitivity of flips on $\mathcal{L}(P)$~~



$S \hookrightarrow \mathbb{R}^3$ defines an inner handlebody S_+.

- If all $c \in P$ are contractible in S_+ then $flip(c)$ is...
- c is not contr. in S_+ (linked with d).

Problem: Do flips and handle-twists act transitively on DP's?

New strategy: use Hatcher-Thurston's theorem:

Need: $(P_a, P_b) \longrightarrow (P'_a, P'_b)$ (by flips and handle-twists).

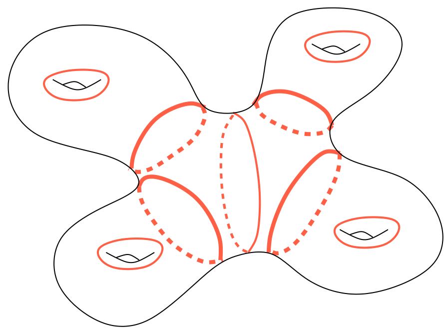
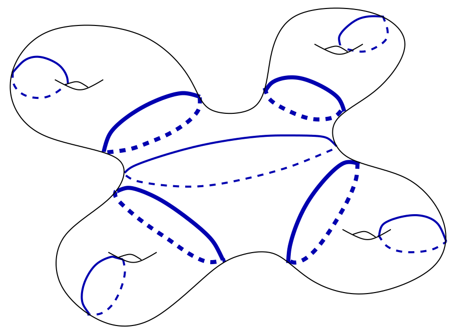
HT treats P_a (by flip and S-moves).

We take care of P_b.

Price: restrictions on DP.

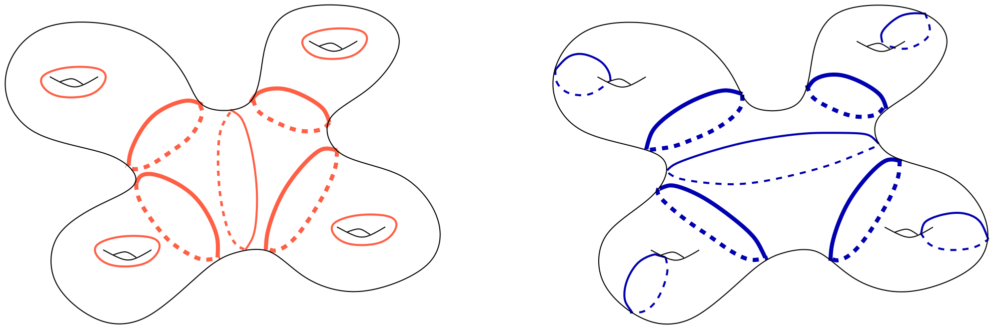
3. Admissible double pants decompositions

- Standard DP:
  g common handles.

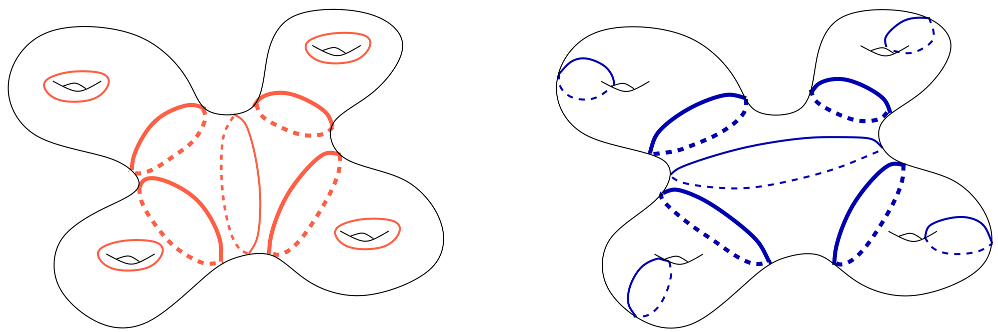
3. Admissible double pants decompositions

- Standard DP:
  g common handles.



- Admissible DP:
  may be transformed to a standard one by flips.

3. Admissible double pants decompositions



- Standard DP:
  g common handles.

- Admissible DP:
  may be transformed to a standard one by flips.

Theorem 1.

Flips and handle-twists act transitively on admissible DP's.

Question: How wide is the class of admissible DP's?

Question: How wide is the class of admissible DP's?

Flip-equivalence class of pants decomp. $\longrightarrow$ a handlebody.
(Luo Feng' 97: Flips act transitively on pants dec. contractible in the same handlebody).

Question: How wide is the class of admissible DP's?

Flip-equivalence class of pants decomp. $\longrightarrow$ a handlebody.
(Luo Feng' 97: Flips act transitively on pants dec. contractible in the same handlebody).

$DP \longrightarrow$ two handlebodies $\longrightarrow$ Heegard splitting of a 3-manifold

Question: How wide is the class of admissible DP's?

Flip-equivalence class of pants decomp. $\longrightarrow$ a handlebody.
(Luo Feng' 97: Flips act transitively on pants dec. contractible in the same handlebody).

$DP \longrightarrow$ two handlebodies $\longrightarrow$ Heegard splitting of a 3-manifold

Property: If $DP = (P_a, P_b)$ is standard
then S may be embedded to $\mathbb{S}^3$ so that
P_a is contractible in S_+ and P_b is contractible in S_-.

Question: How wide is the class of admissible DP's?

Flip-equivalence class of pants decomp. $\longrightarrow$ a handlebody.
(Luo Feng' 97: Flips act transitively on pants dec. contractible in the same handlebody).

$DP \longrightarrow$ two handlebodies $\longrightarrow$ Heegard splitting of a 3-manifold

Property: If $DP = (P_a, P_b)$ is standard
then S may be embedded to $\mathbb{S}^3$ so that
P_a is contractible in S_+ and P_b is contractible in S_-.

DP is admissible iff handlebodies defined by P_a and P_b induce a Heegard splitting of $\mathbb{S}^3$.

Theorem 1.

Flips and handle-twists act transitively on admissible DP's (i.e. on DP's inducing Heegard splittings of $\mathbb{S}^3$).

Theorem 1.
Flips and handle-twists act transitively on admissible DP's (i.e. on DP's inducing Heegard splittings of $\mathbb{S}^3$).

Idea of proof.

1. Sufficient to prove transitivity on standard DP's.

Theorem 1.
Flips and handle-twists act transitively on admissible DP's (i.e. on DP's inducing Heegard splittings of $\mathbb{S}^3$).

Idea of proof.

1. Sufficient to prove transitivity on standard DP's.

2. For two standard DP's with the same set of handles: is easy.

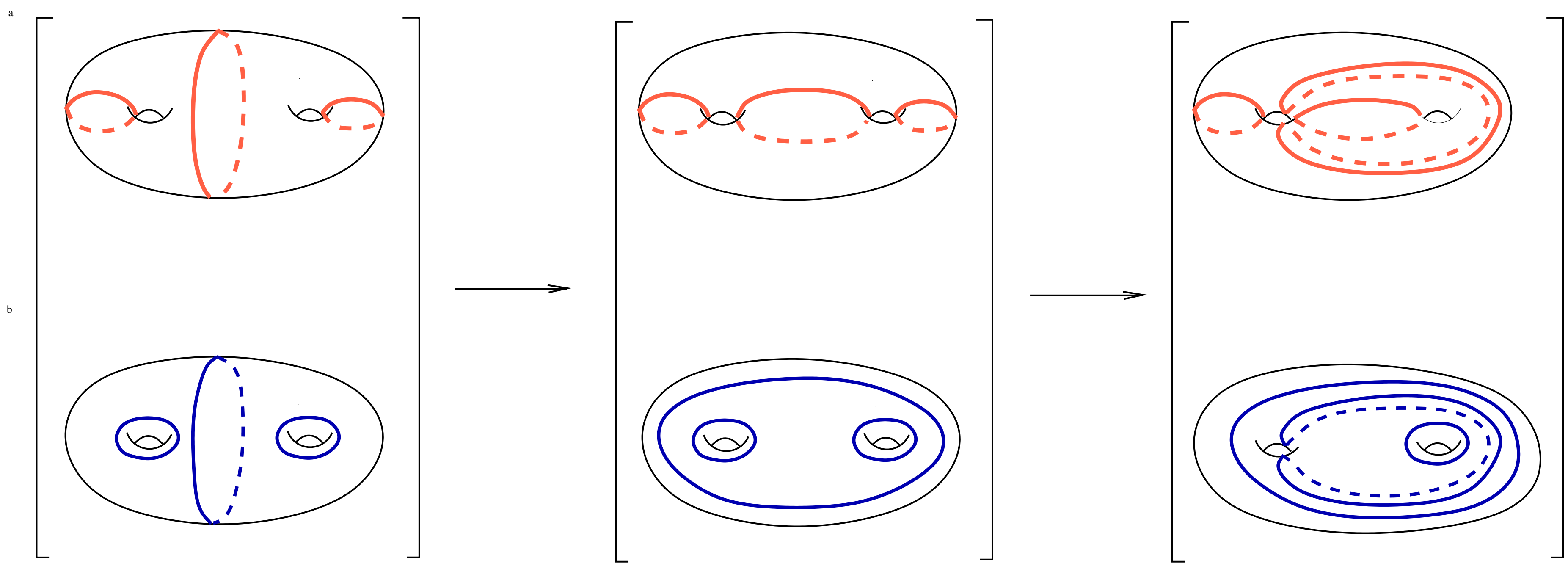
Theorem 1.
Flips and handle-twists act transitively on admissible DP's (i.e. on DP's inducing Heegard splittings of $\mathbb{S}^3$).

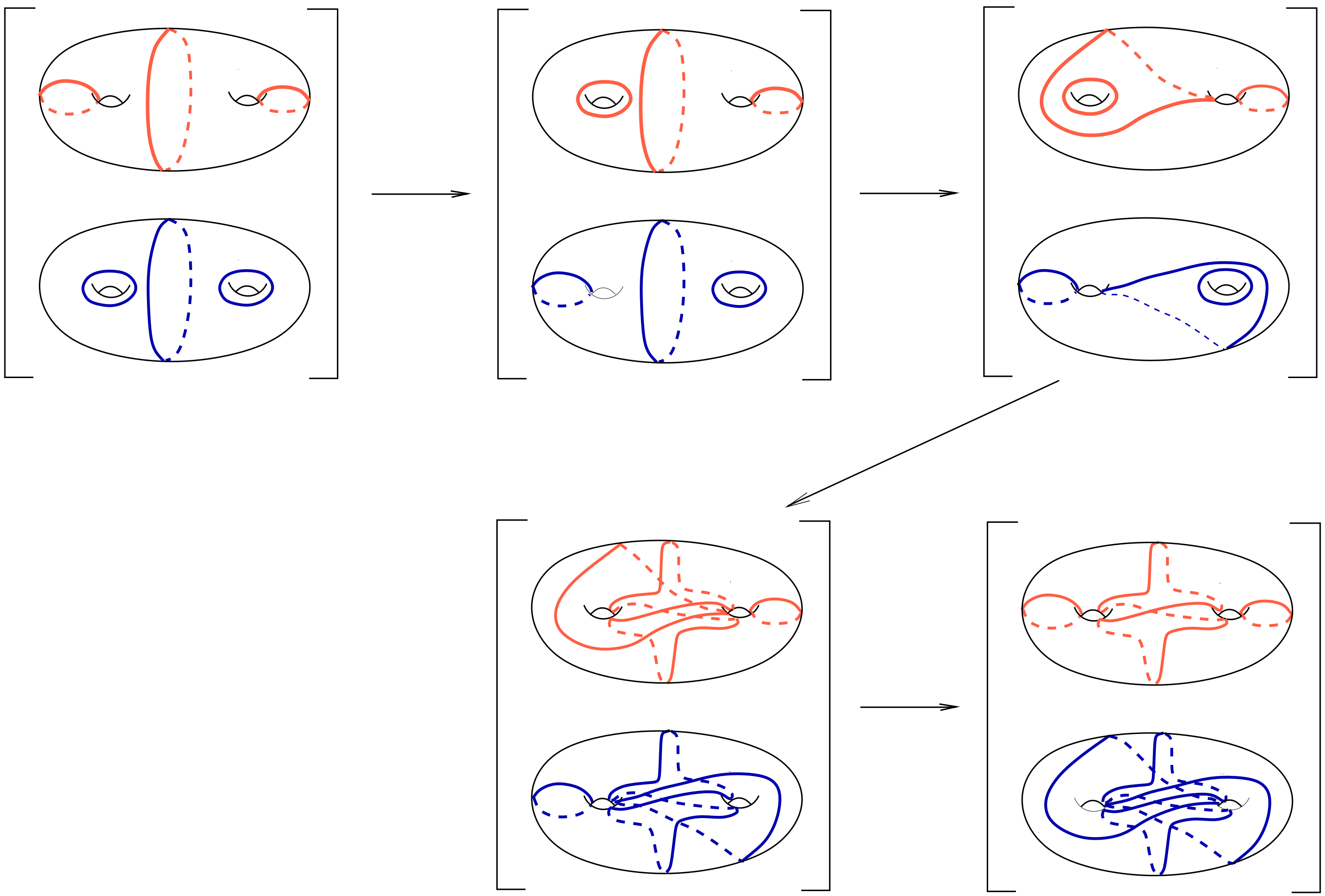
Idea of proof.

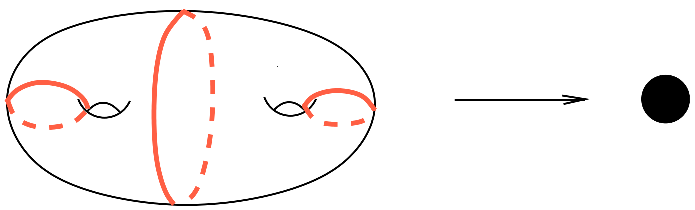
1. Sufficient to prove transitivity on standard DP's.

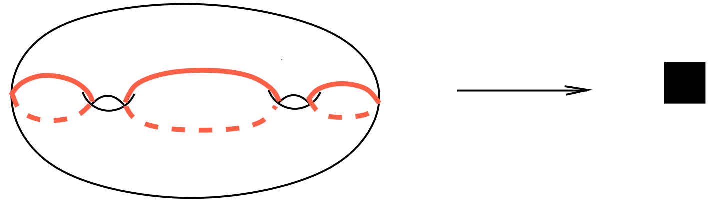
2. For two standard DP's with the same set of handles: is easy.

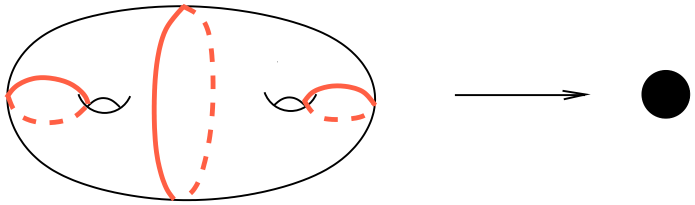
3. Change the set of handles in small steps.

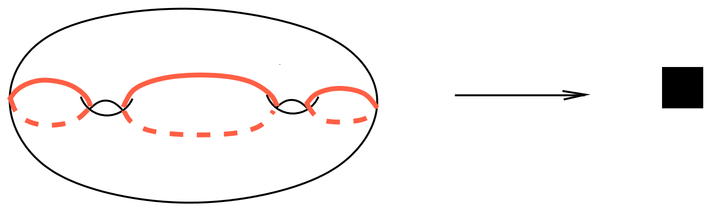

a
b

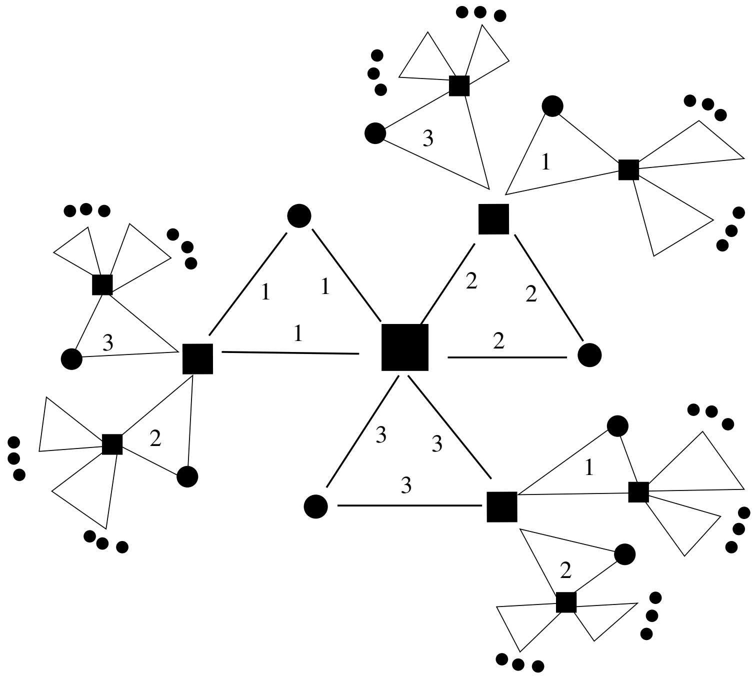


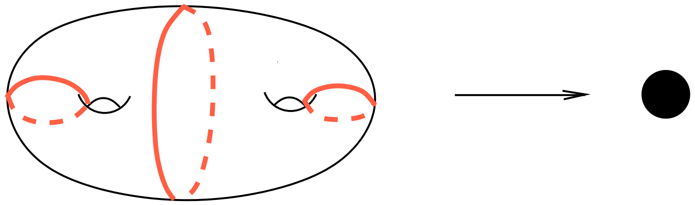


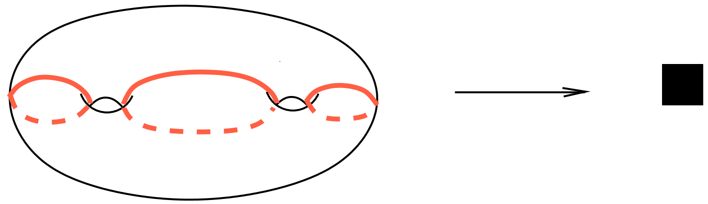


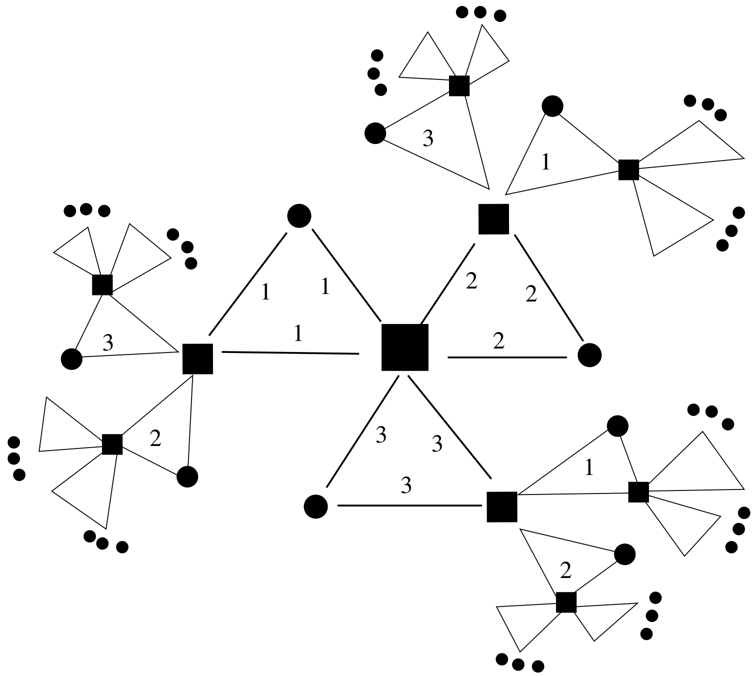




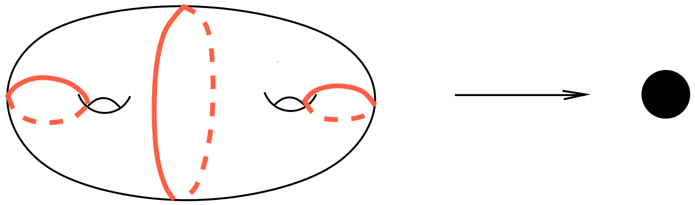


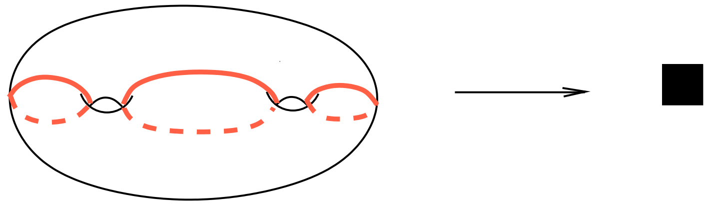


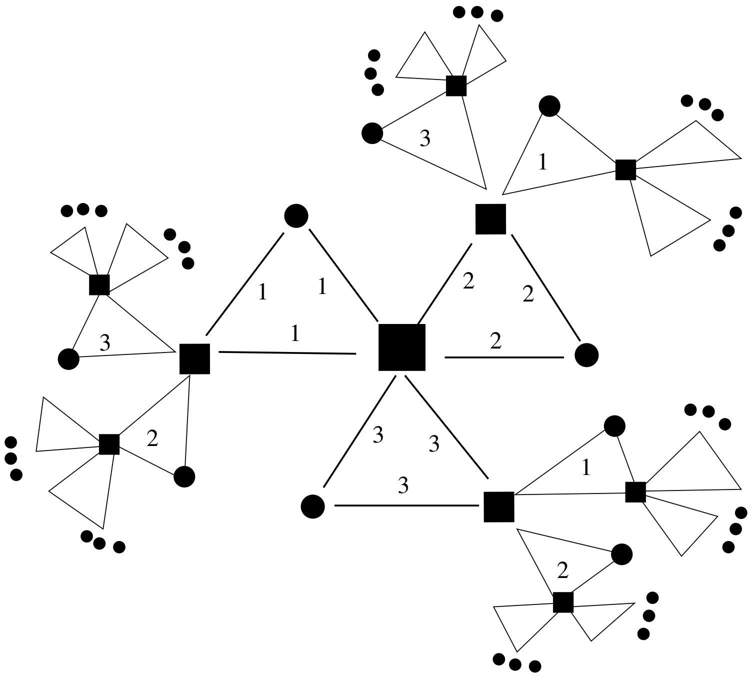




- For any two ●'s
  ∃ an alternating connecting path:
  ● ⟶ ■ ⟶ ● ⟶ ■ ⟶ ●







- For any two ●'s
  ∃ an alternating connecting path:
  ● ⟶ ■ ⟶ ● ⟶ ■ ⟶ ●
- "Small step" = two flips

Double pants decompositions and mapping class group

Double pants decompositions and mapping class group

- $MCG(S)$ acts on DP's

Double pants decompositions and mapping class group

- $MCG(S)$ acts on DP's

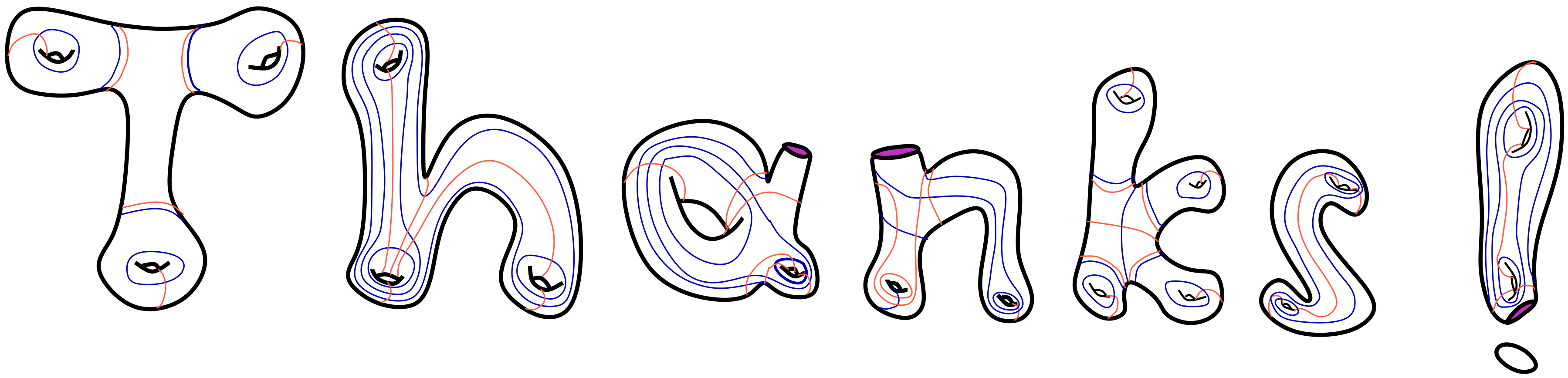
Theorem 2. Let DP be an admissible double pants dec. of S without double curves and let $g \in MCG(S)$.
If $g(DP) = DP$ then $g = id$.

Double pants decompositions and mapping class group

- $MCG(S)$ acts on DP's

Theorem 2. Let DP be an admissible double pants dec. of S without double curves and let $g \in MCG(S)$.
If $g(DP) = DP$ then $g = id$.

Theorem 2'. A groupoid generated by flips and twists contains the mapping class group.


Thanks!