Geometric realisations of quiver mutations

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(joint with Pavel Tumarkin)

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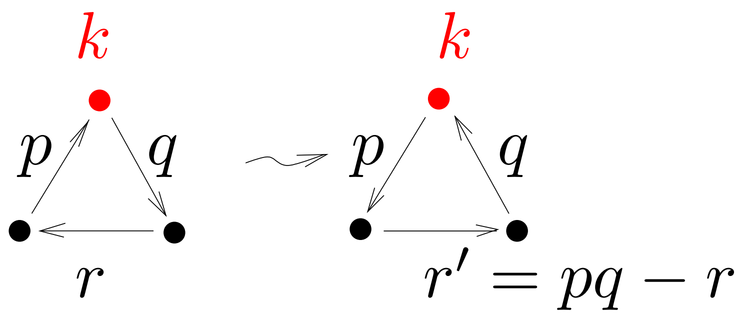

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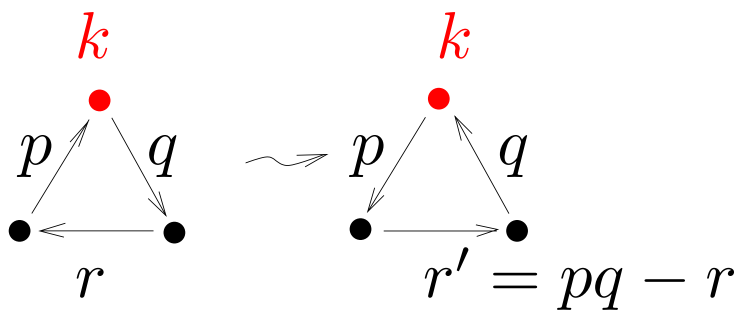

Quiver mutation is used in cluster algebras and connected to: representation theory, geometry of triangulated surfaces, Grassmannians, root systems, integrable systems, tropical geometry, Poisson geometry, combinatorics of polytopes...

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- π-rotation groups [cyclic mutation types]

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- π-rotation groups [cyclic mutation types]

Q is of acyclic mut. type
iff its mutation class contains a quiver without oriented cycles.

Q is if cyclic mut. type
otherwise.

1. Cyclic mutation classes via π-rotations

$Q = (p, q, r)$, mutation-cyclic $\rightsquigarrow$ $\begin{pmatrix} -2 & p & q \\ p & -2 & r \\ q & r & -2 \end{pmatrix} = (v_i, v_j)$

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$\langle v_1, v_2, v_3 \rangle = \mathbb{R}^{2,1}$:

$$\begin{aligned} x &= (x_1, x_2, x_3) \\ y &= (y_1, y_2, y_3) \end{aligned} \quad \Rightarrow \quad (x, y) = x_1 y_1 + x_2 y_2 - x_3 y_3$$

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linear model of $\mathbb{H}^2 = \{x \in \mathbb{R}^{2,1} \mid (x, x) = -2\}$

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Q $\rightsquigarrow$ three points x, y, z on distances $\operatorname{arcosh}\frac{p}{2}$, $\operatorname{arcosh}\frac{q}{2}$, $\operatorname{arcosh}\frac{r}{2}$.

Why exist?

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Lemma. (Beineke, Brüstle, Hille)

Q mutation-cyclic $\Rightarrow$ $p, q, r \geq 2$.

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Mutation: "partial π-rotation".

π-rotation $\quad R_y(x) =$ "rotation of x around y by π" $= -x - (x, y)y$

$$\mu_k(v_i) = \begin{cases} -v_i - (v_i, v_k)v_k, & \text{if } i \to k \text{ in } Q \\ v_i, & \text{otherwise} \end{cases}$$

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Thm 1. If $v_1, v_2, v_3 \in \mathbb{H}^2$, then the values $2\cosh d_{v_i,v_j}$ change under mutations in the same way as the weights of the arrows in Q, i.e.

$$r' + r = pq, \qquad\qquad 2\cosh d_{r'} + 2\cosh d_r = 2\cosh d_p \cdot 2\cosh d_q$$

2. Acyclic mutation classes via reflections

$Q = (p, q, -r)$,
acyclic

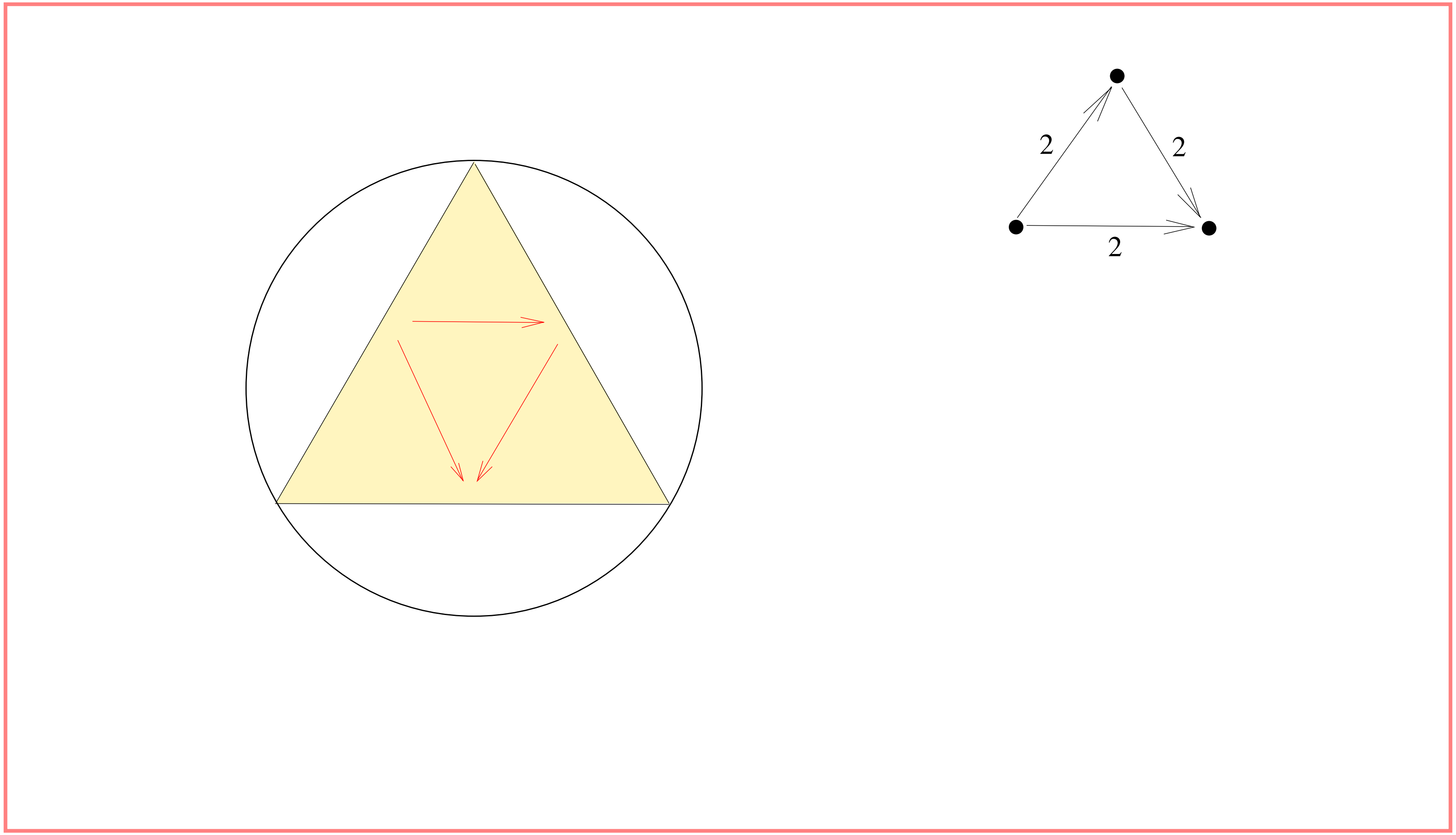
$$\rightsquigarrow \quad \begin{pmatrix} 2 & -p & -q \\ -p & 2 & -r \\ -q & -r & 2 \end{pmatrix} = (v_i, v_j)$$

$\langle v_1, v_2, v_3 \rangle = \mathbb{H}^2, \mathbb{E}^2, \mathbb{S}^2$ (proj model)

$$|(v_i, v_j)| = \begin{cases} 2\cosh d_{ij}, & \text{if } v_i^\perp \cap v_j^\perp = \emptyset, \\ 2\cos \alpha_{ij}, & \text{if } v_i^\perp \cap v_j^\perp \neq \emptyset, \end{cases}$$

Mutation: "partial reflection": $\mu_k(v_i) = \begin{cases} v_i - (v_i, v_k)v_k, & \text{if } i \to k \text{ in } Q \\ -v_k, & \text{if } i = k \\ v_i, & \text{otherwise} \end{cases}$

Thm 2. (Barot, Geiss, Zelevinsky' 2006)
The values (v_i, v_j) change under mutations
in the same way as the weights of the arrows in Q.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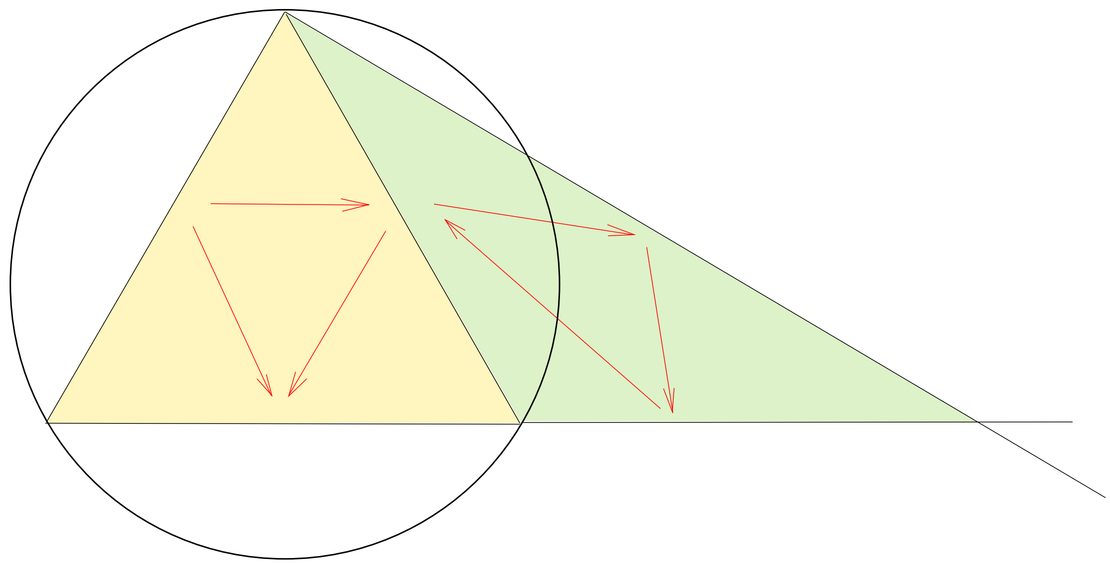

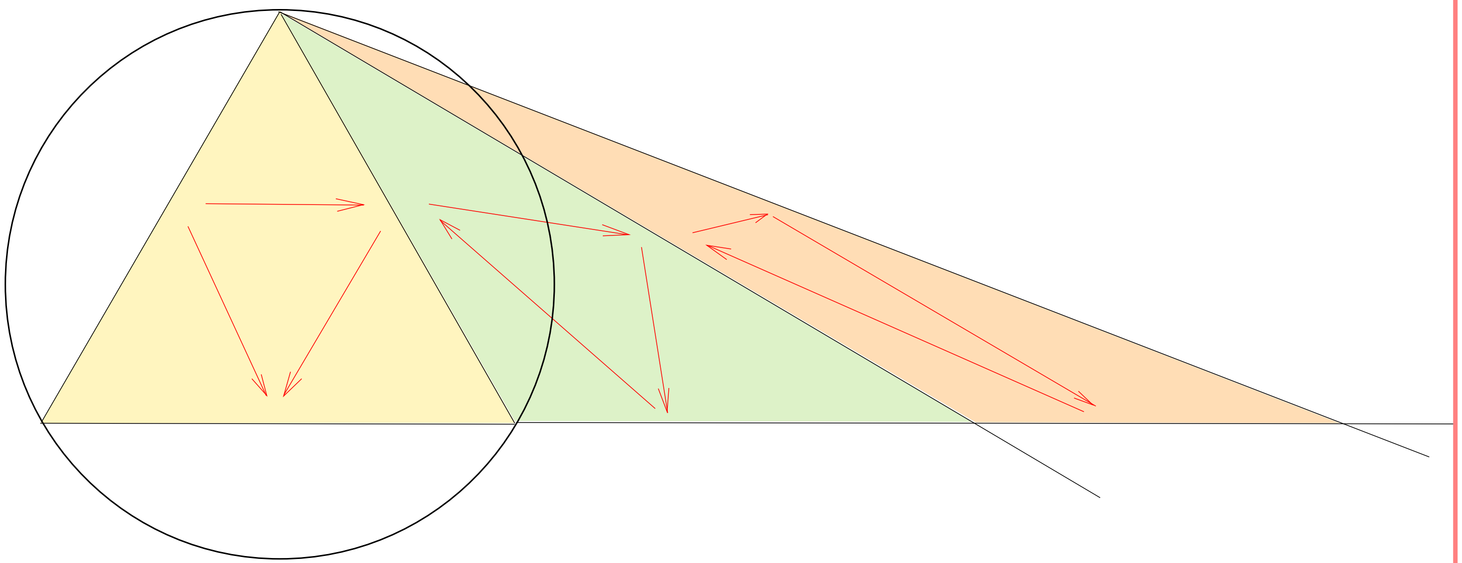

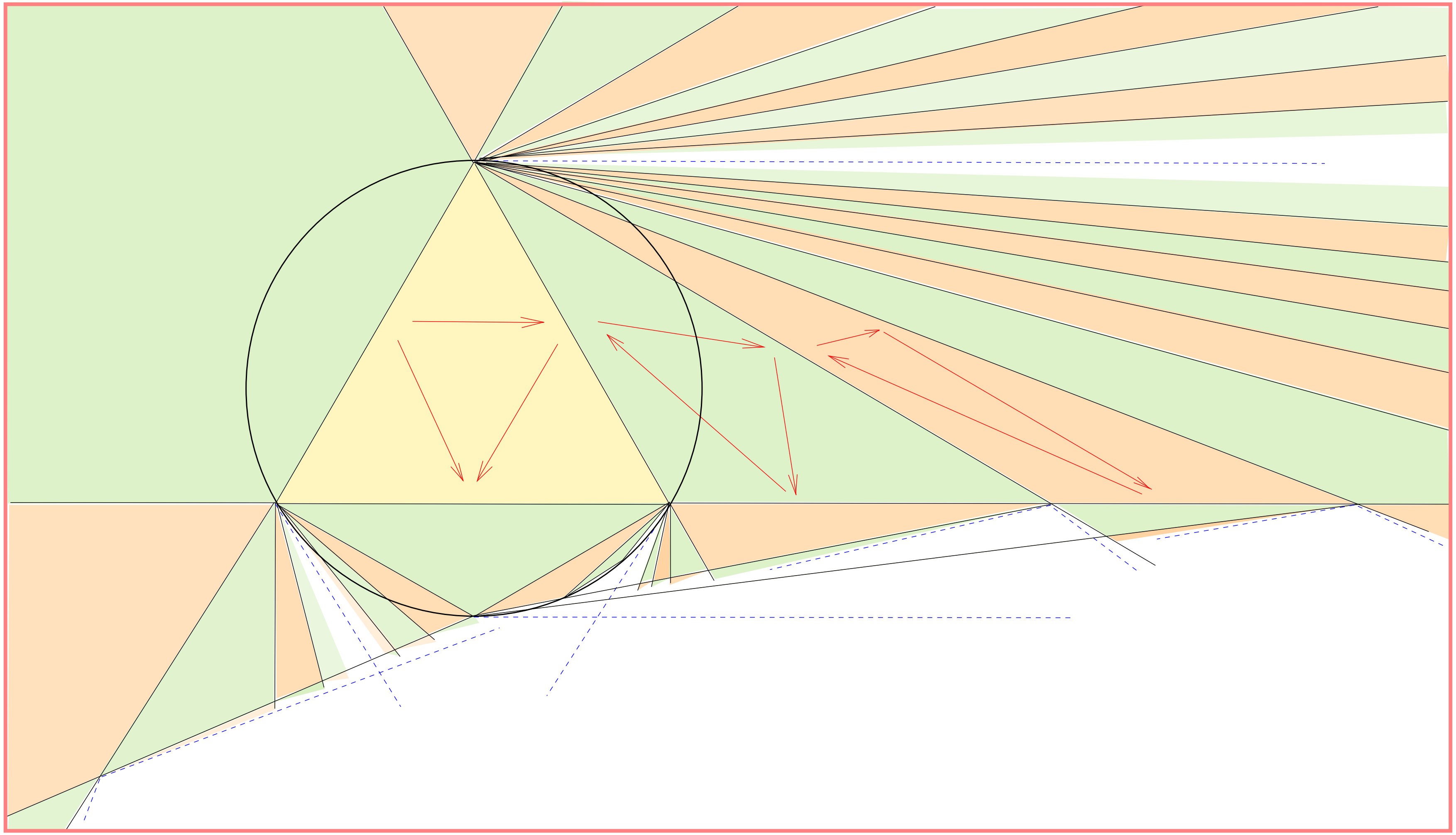

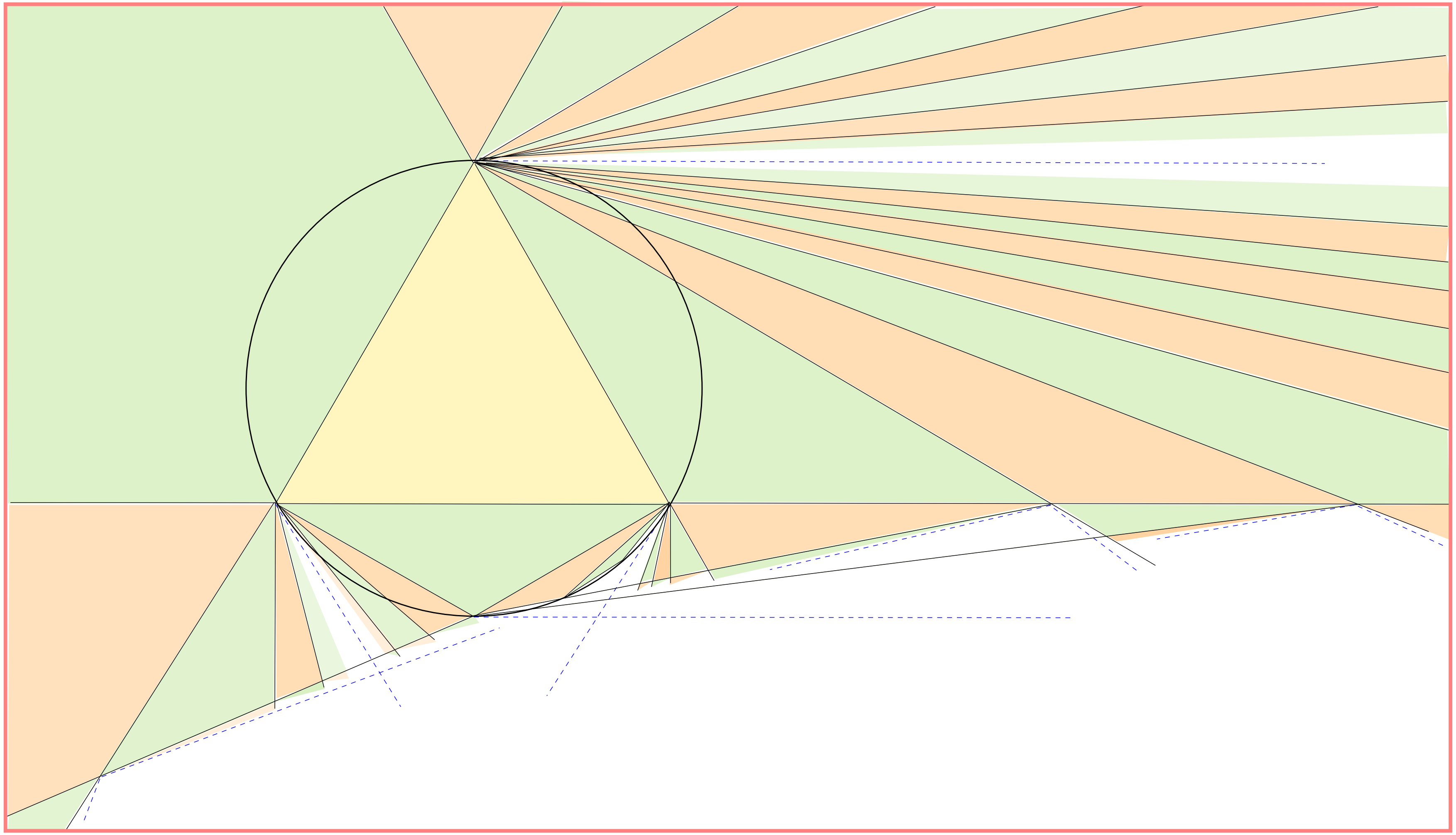

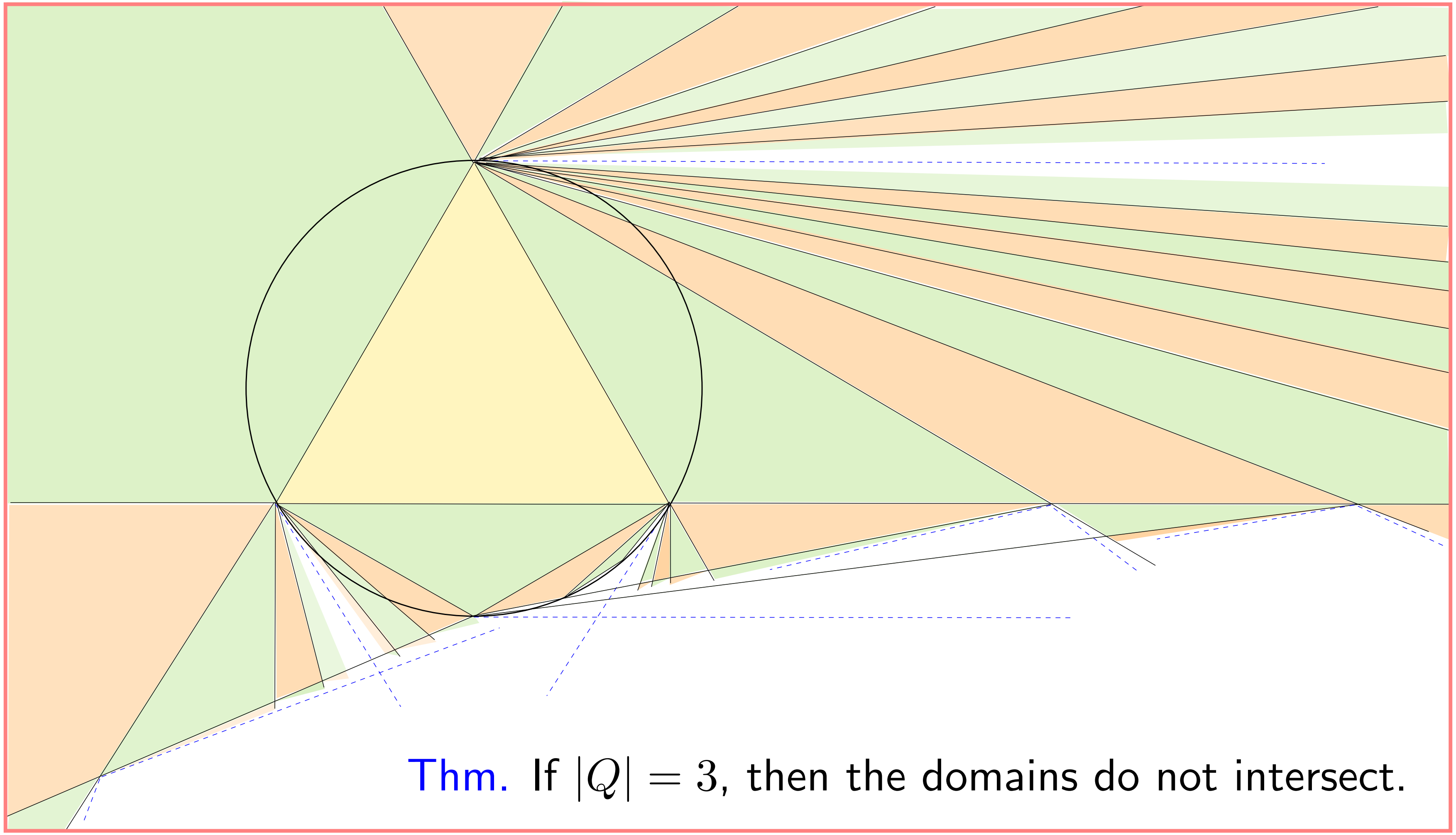

Thm. If $|Q| = 3$, then the domains do not intersect.

Thm 1,2: "If Q has a geometric realization
then it works for the whole mutation class"

Thm 3. Every Q of rank 3 has a realization.

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- if Q is mut.-acyclic $\to$ by reflections [Seven; Speyer-Thomas]
- if Q is mut.-cyclic $\Rightarrow p, q, r \geq 2 \Rightarrow$
  there are 3 pts in $\mathbb{H}^2$ iff $d_p + d_q \geq d_r$
  .... what if....... $d_p + d_q < d_r$?

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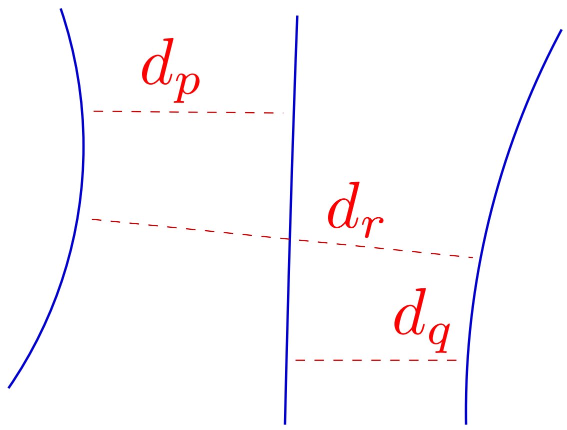
  Three lines in $\mathbb{H}^2$:
  realization by reflections!



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Thm 3'.

1. Q mut.-acyclic $\Rightarrow$ Q has realization by reflections.
2. Q mut.-cyclic $\Rightarrow$ Q has realization by π-rotations.
3. Q has both realizations $\Leftrightarrow$
   $Q = (p, q, r)$ with $p, q, r \geq 2$ and $d_p + d_q = d_r$.

3. Non-integer quivers

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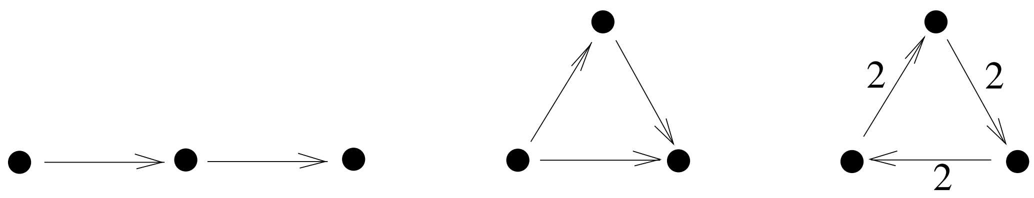
Def. A quiver is of finite mutation type
if it is mutation equivalent to fin. many other quivers.

In integer case:



A_3 $\qquad \widetilde{A}_2$ $\qquad$ Markov

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- $(0, 0, 0)$;
- $(\frac{1}{n}, \frac{1}{n}, 0)$, where $n \in \mathbb{Z}_+$;
- $(\frac{1}{3}, \frac{1}{3}, \frac{1}{2})$, $(\frac{1}{3}, \frac{1}{4}, \frac{1}{2})$, $(\frac{1}{3}, \frac{1}{5}, \frac{1}{2})$, $(\frac{1}{5}, \frac{2}{5}, \frac{1}{2})$, $(\frac{1}{3}, \frac{2}{5}, \frac{1}{2})$.

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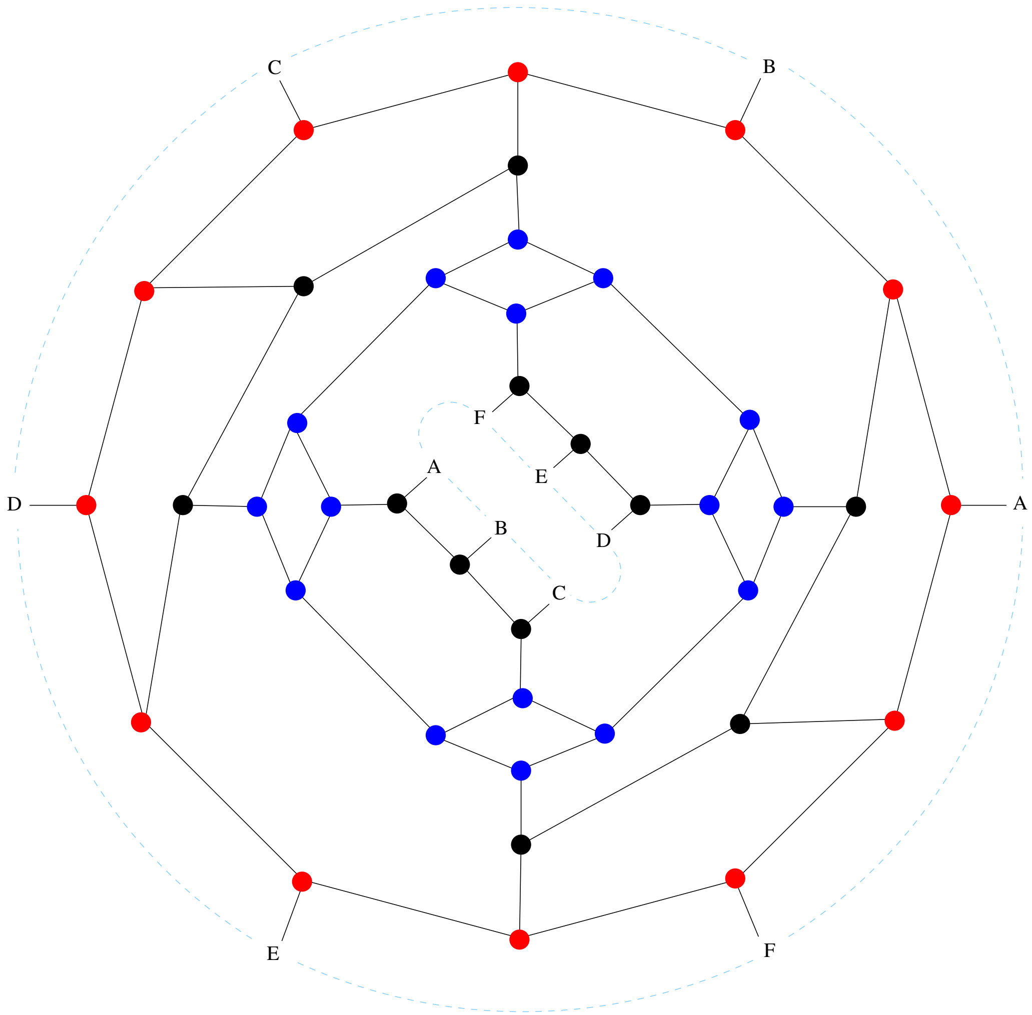
  A_3 $\quad B_3$ $\quad H_3$ $\quad H_3^{(1)}$ $\quad H_3^{(2)}$

Two finite type mutation classes:

	Acyclic	Cyclic
$H_3^{(1)}$	$(2\cos\frac{\pi}{5}, 2\cos\frac{2\pi}{5}, 0)$ $(1, 1, -2\cos\frac{2\pi}{5})$	$(2\cos\frac{2\pi}{5}, 2\cos\frac{2\pi}{5}, 1)$
$H_3^{(2)}$	$(2\cos\frac{\pi}{3}, 2\cos\frac{2\pi}{5}, 0)$ $(2\cos\frac{2\pi}{5}, 2\cos\frac{2\pi}{5}, -2\cos\frac{2\pi}{5})$	$(2\cos\frac{1\pi}{5}, 2\cos\frac{2\pi}{5}, 1)$ $(1, 1, 2\cos\frac{\pi}{5})$

Exchange graph for $H_3^{(1)}$:



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- $C(Q)$ is mutation-invariant;

- $C(Q)$ controls geometry of the realization:

  - if $p, q, r \geq 2$, triangle ineq. $\Leftrightarrow C(Q) \leq 4$;
  - if Q mut.-acyclic, $C(Q) < 4 / = 4 / > 4 \Leftrightarrow$ refl. in $\mathbb{S}^2/\mathbb{E}^2/\mathbb{H}^2$.
  - if Q is mut.-cyclic, $C(Q)$ controls geometry of $g = R_1 \circ R_2 \circ R_3$:
    $C(Q) < 0 / = 0 / > 0 \Leftrightarrow g$ is hyperbolic/parabolic/elliptic.

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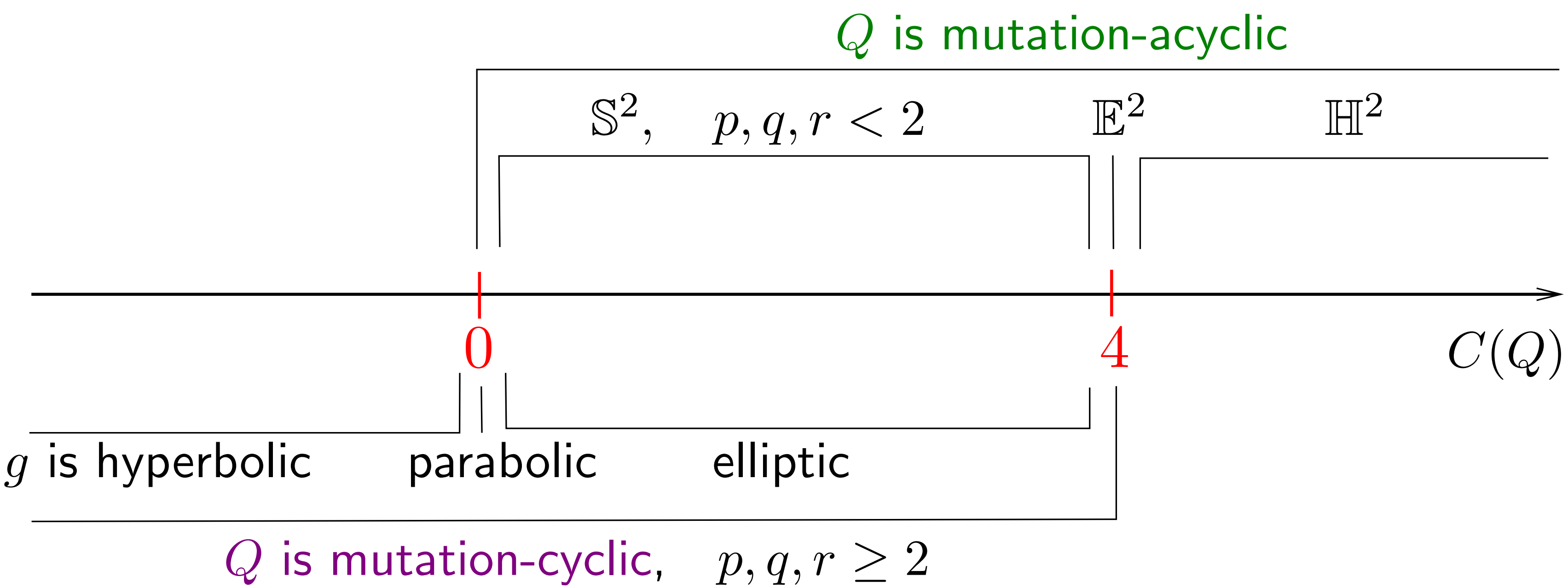

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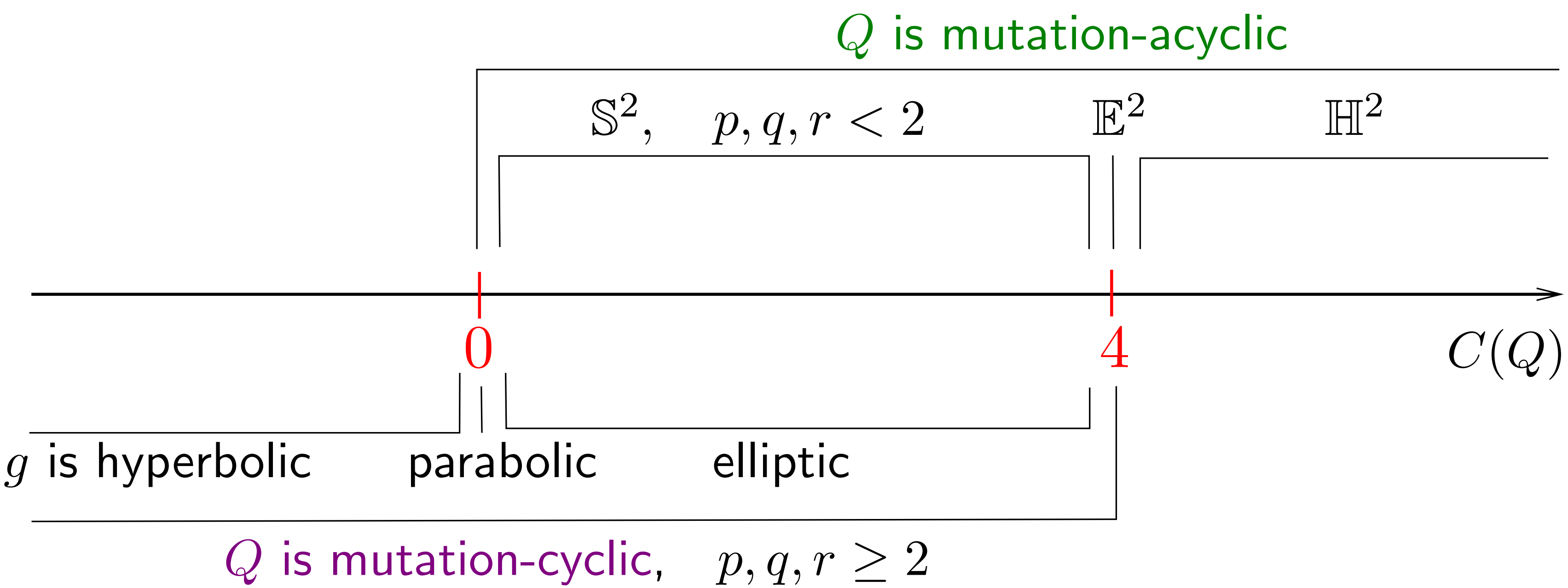

THANKS!