Coxeter groups, quiver mutations and hyperbolic manifolds

Anna Felikson
(joint with Pavel Tumarkin)

Workshop on Galois Covers, Grothendieck-Teichmüller Theory and Dessins d'Enfants

Univerisity of Leicester, June 6, 2018

1. Coxeter group: $G = \langle s_1, \ldots, s_n \mid s_i^2 = (s_i s_j)^{m_{ij}} = e \rangle.$

1. Coxeter group: $G = \langle s_1, \ldots, s_n \mid s_i^2 = (s_i s_j)^{m_{ij}} = e \rangle.$

2. Quiver mutation:

1. Coxeter group: $G = \langle s_1, \ldots, s_n \mid s_i^2 = (s_i s_j)^{m_{ij}} = e \rangle.$

2. Quiver mutation:

- Quiver is an oriented graph without loops and 2-cycles.

Agreement: $\bullet \xrightarrow{p} \bullet \quad = \quad \bullet \xleftarrow{-p} \bullet$

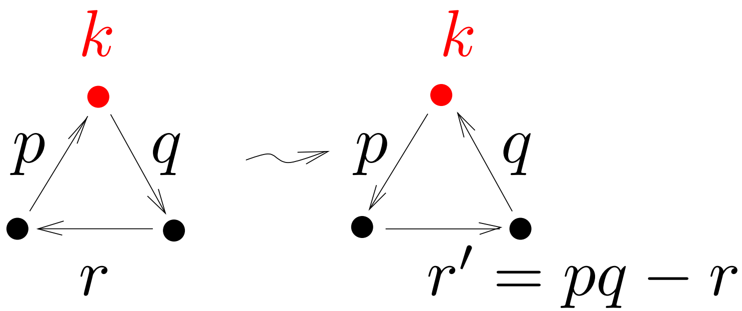
1. Coxeter group: $G = \langle s_1, \ldots, s_n \mid s_i^2 = (s_i s_j)^{m_{ij}} = e \rangle.$

2. Quiver mutation:

- Quiver is an oriented graph without loops and 2-cycles.

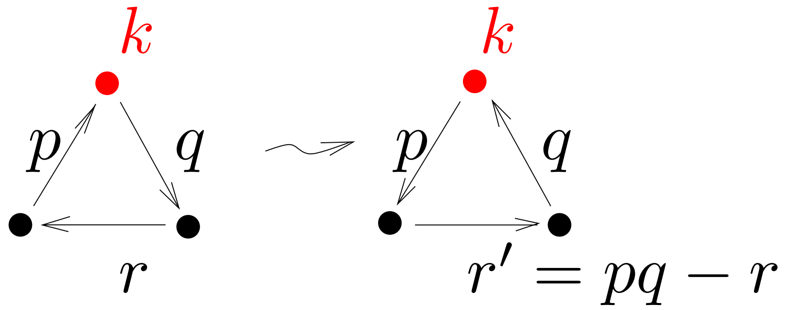
  Agreement: $\bullet \xrightarrow{p} \bullet \quad = \quad \bullet \xleftarrow{-p} \bullet$

- Mutation μ_k of quivers:
  - reverse all arrows incident to k;
  - for every oriented path through k do



1. Coxeter group: $G = \langle s_1, \ldots, s_n \mid s_i^2 = (s_i s_j)^{m_{ij}} = e \rangle.$

2. Quiver mutation:



Plan:

Quiver $Q \longrightarrow$

$\longrightarrow$ (Quotient of) Coxeter group $G \longrightarrow$

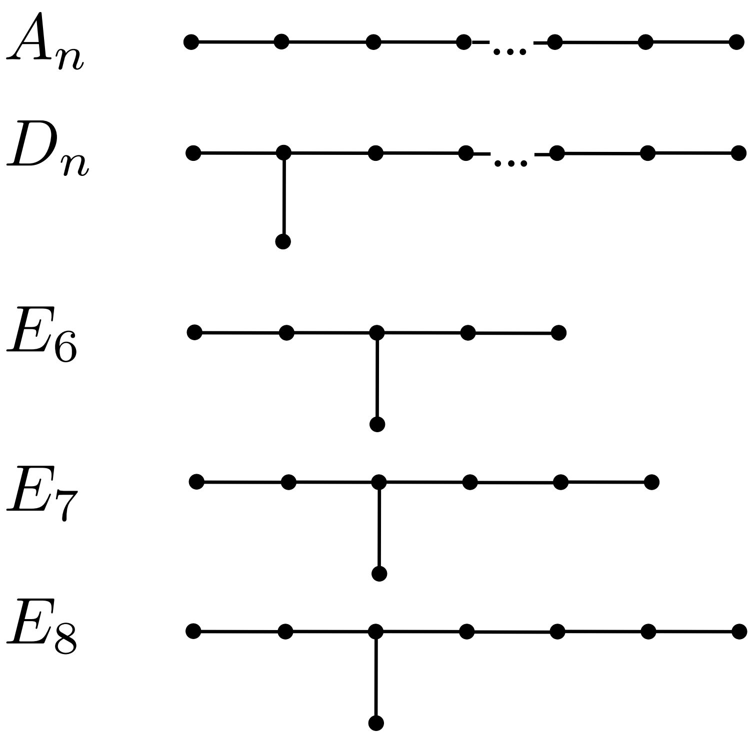
$\longrightarrow$ Action of G on $X \longrightarrow$

Hyperbolic manifold X
with symmetry group G

3. Construction by Barot - Marsh: quiver $Q \longrightarrow$ group $G(Q)$.

3. Construction by Barot - Marsh: quiver $Q \longrightarrow$ group $G(Q)$.

Let Q be a quiver of finite type,
i.e. mutation-equivalent to an orientation of A_n, D_n or E_6, E_7, E_8:



3. Construction by Barot - Marsh: quiver $Q \longrightarrow$ group $G(Q)$.

Let Q be a quiver of finite type,
i.e. mutation-equivalent to an orientation of A_n, D_n or E_6, E_7, E_8.

- Generators of G – nodes of Q.

- Relations of G – (R1) $s_i^2 = e$

  (R2) $(s_i s_j)^{m_{ij}} = e$,

$$m_{ij} = \begin{cases} 2, & \bullet \qquad \bullet \\ 3, & \bullet \!-\!\!-\!\!-\!\!-\! \bullet \\ \infty, & otherwise. \end{cases}$$

  (R3) Cycle relation:

  for each chordless cycle $1 \to 2 \to \cdots \to n \to 1$

  $(s_1 \;\; s_2 s_3 \dots s_n \dots s_3 s_2)^2 = e.$

3. Construction by Barot - Marsh: quiver $Q \longrightarrow$ group $G(Q)$.

Let Q be a quiver of finite type,
i.e. mutation-equivalent to an orientation of A_n, D_n or E_6, E_7, E_8.

Theorem 1. [Barot-Marsh'2011]. Given a quiver Q of finite type, $G(Q)$ is invariant under mutations of Q, i.e. $G(Q) = G(\mu_k(Q))$.

3. Construction by Barot - Marsh: quiver $Q \longrightarrow$ group $G(Q)$.

Let Q be a quiver of finite type,
i.e. mutation-equivalent to an orientation of A_n, D_n or E_6, E_7, E_8.

Theorem 1. [Barot-Marsh'2011]. Given a quiver Q of finite type, $G(Q)$ is invariant under mutations of Q, i.e. $G(Q) = G(\mu_k(Q))$.

- In particular, $G(Q)$ is a finite Coxeter group.

3. Construction by Barot - Marsh: quiver $Q \longrightarrow$ group $G(Q)$.

Let Q be a quiver of finite type,
i.e. mutation-equivalent to an orientation of A_n, D_n or E_6, E_7, E_8.

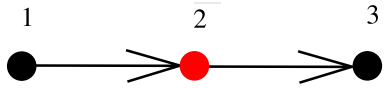
Theorem 1. [Barot-Marsh'2011]. Given a quiver Q of finite type, $G(Q)$ is invariant under mutations of Q, i.e. $G(Q) = G(\mu_k(Q))$.

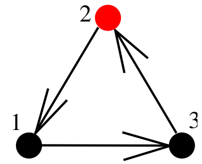
- In particular, $G(Q)$ is a finite Coxeter group.

- If $Q_2 = \mu_k(Q_1)$, s_i - generators of $G(Q_1)$, t_i generators of $G(Q_2)$, then

$$t_i = \begin{cases} s_k s_i s_k, & \overset{i}{\bullet} \longrightarrow \overset{k}{\bullet} \text{ in } Q_1 \\ s_i, & \text{otherwise} \end{cases}$$

4. Geometric interpretation.

Example: $Q_1 = A_3 =$ 

$\xrightarrow{\mu_2}$ $Q_2 =$ 

4. Geometric interpretation.

Example: $Q_1 = A_3 = \overset{1}{\bullet} \longrightarrow \overset{2}{\bullet} \longrightarrow \overset{3}{\bullet} \quad \xrightarrow{\mu_2} \quad Q_2 =$

$$\begin{tikzcd} & 2 \arrow[dl] & \\ 1 \arrow[rr] & & 3 \arrow[ul] \end{tikzcd}$$

$$G(Q_1) = \langle s_1, s_2, s_3 \mid s_i^2 = (s_1 s_2)^3 = (s_2 s_3)^3 = (s_1 s_2)^2 = e \rangle$$

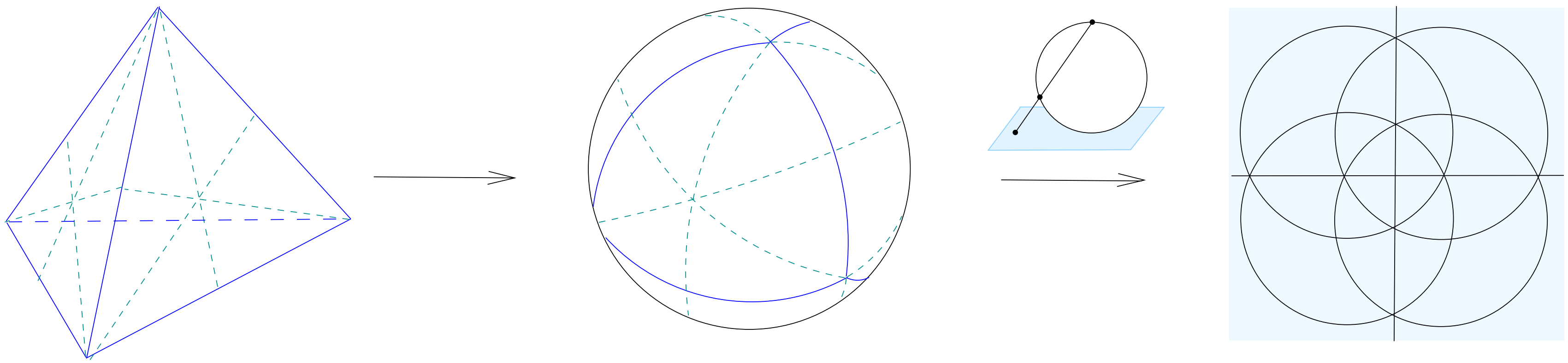
4. Geometric interpretation.

Example: $Q_1 = A_3 =$

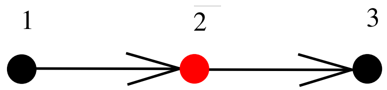
$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \end{tikzcd} \quad \xrightarrow{\mu_2} \quad Q_2 = \begin{tikzcd} & 2 \arrow[dl] & \\ 1 \arrow[rr] & & 3 \arrow[ul] \end{tikzcd}$$

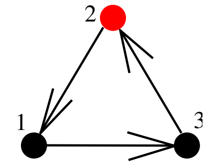
$$G(Q_1) = \langle s_1, s_2, s_3 \mid s_i^2 = (s_1 s_2)^3 = (s_2 s_3)^3 = (s_1 s_2)^2 = e \rangle$$

finite Coxeter group A_3, acts on S^2 by reflections, 24 elements:



4. Geometric interpretation.

Example: $Q_1 = A_3 =$ 

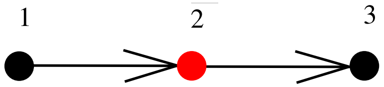
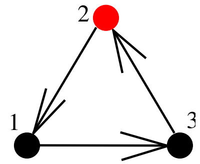
$\xrightarrow{\mu_2}$ $Q_2 =$ 

$$G(Q_1) = \langle s_1, s_2, s_3 \mid s_i^2 = (s_1 s_2)^3 = (s_2 s_3)^3 = (s_1 s_2)^2 = e \rangle$$

finite Coxeter group A_3, acts on S^2 by reflections, 24 elements.

$$G(Q_2) = \langle\ t_1, t_2, t_3 \mid t_i^2 = (t_i t_j)^3 = (t_1\ t_2 t_3 t_2)^2 = e \rangle$$

4. Geometric interpretation.

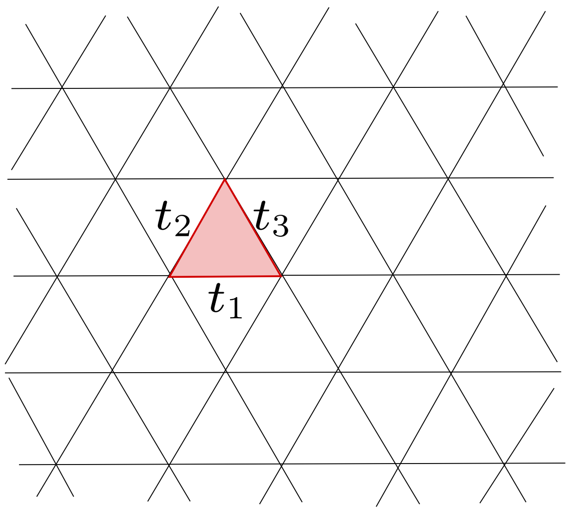
Example: $Q_1 = A_3 =$  $\xrightarrow{\mu_2}$ $Q_2 =$ 

$$G(Q_1) = \langle s_1, s_2, s_3 \mid s_i^2 = (s_1 s_2)^3 = (s_2 s_3)^3 = (s_1 s_2)^2 = e \rangle$$

finite Coxeter group A_3, acts on S^2 by reflections, 24 elements.

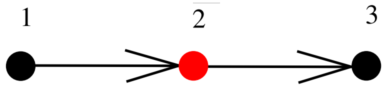
$$G(Q_2) = \langle \underbrace{t_1, t_2, t_3 \mid t_i^2 = (t_i t_j)^3}_{} = (t_1\ t_2 t_3 t_2)^2 = e \rangle$$

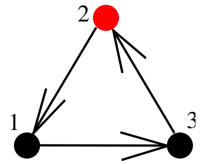
G_0 – affine Coxeter group $\widetilde{A}_2$, acts on $\mathbb{E}^2$ by reflections.



$$(t_1\ t_2 t_3 t_2)^2 = ?$$

4. Geometric interpretation.

Example: $Q_1 = A_3 =$ 

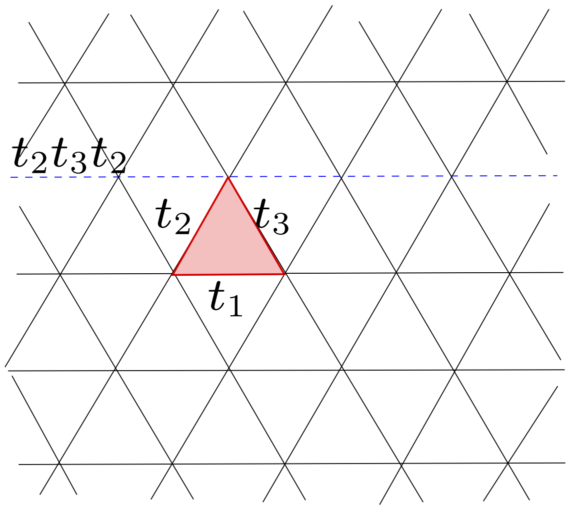
$\xrightarrow{\mu_2}$ $Q_2 =$ 

$$G(Q_1) = \langle s_1, s_2, s_3 \mid s_i^2 = (s_1s_2)^3 = (s_2s_3)^3 = (s_1s_2)^2 = e \rangle$$

finite Coxeter group A_3, acts on S^2 by reflections, 24 elements.

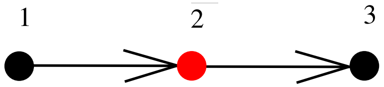
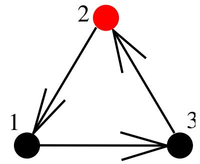
$$G(Q_2) = \langle \underbrace{t_1, t_2, t_3 \mid t_i^2 = (t_it_j)^3} = (t_1\ t_2t_3t_2)^2 = e \rangle$$

G_0 – affine Coxeter group $\widetilde{A}_2$, acts on $\mathbb{E}^2$ by reflections.



$(t_1\ t_2t_3t_2)^2 =?$

4. Geometric interpretation.

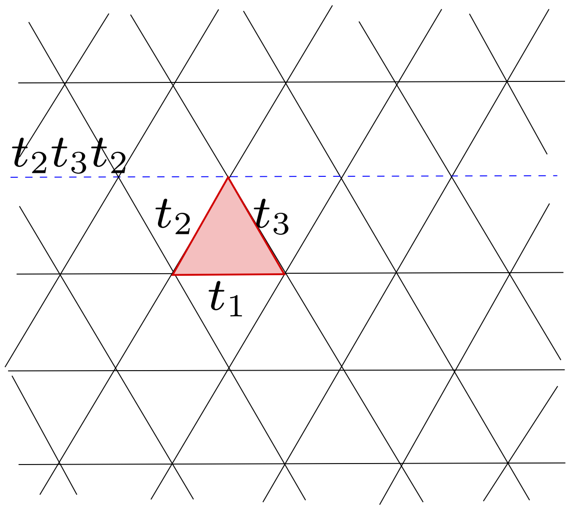
Example: $Q_1 = A_3 =$  $\xrightarrow{\mu_2}$ $Q_2 =$ 

$$G(Q_1) = \langle s_1, s_2, s_3 \mid s_i^2 = (s_1 s_2)^3 = (s_2 s_3)^3 = (s_1 s_2)^2 = e \rangle$$

finite Coxeter group A_3, acts on S^2 by reflections, 24 elements.

$$G(Q_2) = \langle \underbrace{t_1, t_2, t_3 \mid t_i^2 = (t_i t_j)^3}_{} = (t_1 \ t_2 t_3 t_2)^2 = e \rangle$$

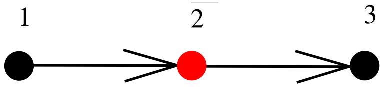
G_0 – affine Coxeter group $\widetilde{A}_2$, acts on $\mathbb{E}^2$ by reflections.

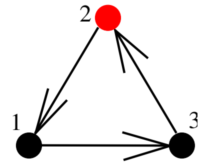


$(t_1 \ t_2 t_3 t_2)^2 =?$

$t_1 \ t_2 t_3 t_2$ - translation by 2 levels

4. Geometric interpretation.

Example: $Q_1 = A_3 =$ 

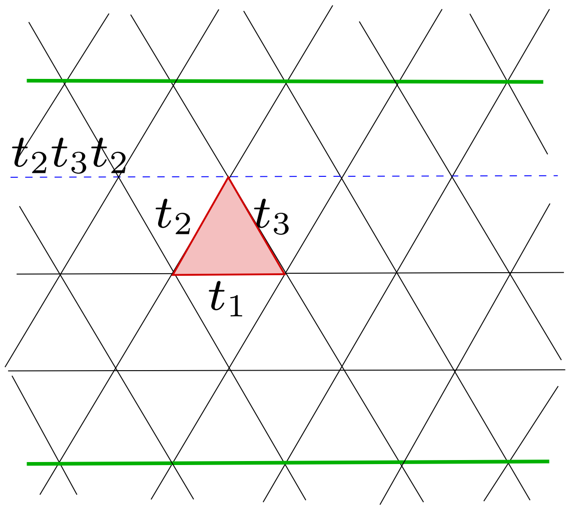
$\xrightarrow{\mu_2}$ $Q_2 =$ 

$$G(Q_1) = \langle s_1, s_2, s_3 \mid s_i^2 = (s_1 s_2)^3 = (s_2 s_3)^3 = (s_1 s_2)^2 = e \rangle$$

finite Coxeter group A_3, acts on S^2 by reflections, 24 elements.

$$G(Q_2) = \langle \underbrace{t_1, t_2, t_3 \mid t_i^2 = (t_i t_j)^3}_{} = (t_1 \ t_2 t_3 t_2)^2 = e \rangle$$

G_0 – affine Coxeter group $\widetilde{A}_2$, acts on $\mathbb{E}^2$ by reflections.

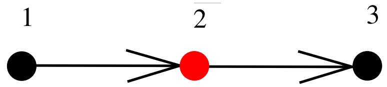


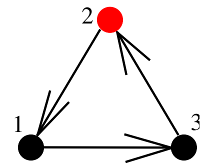
$(t_1 \ t_2 t_3 t_2)^2 = ?$

$t_1 \ t_2 t_3 t_2$ - translation by 2 levels

$(t_1 \ t_2 t_3 t_2)^2$ - translation by 4 levels

4. Geometric interpretation.

Example: $Q_1 = A_3 =$ 

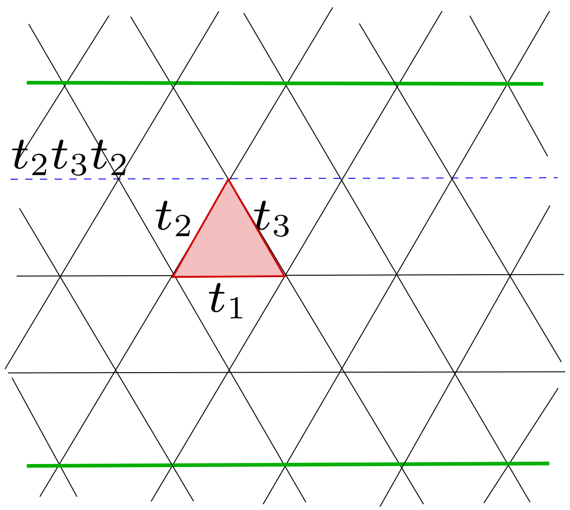
$\xrightarrow{\mu_2}$ $Q_2 =$ 

$$G(Q_1) = \langle s_1, s_2, s_3 \mid s_i^2 = (s_1 s_2)^3 = (s_2 s_3)^3 = (s_1 s_2)^2 = e \rangle$$

finite Coxeter group A_3, acts on S^2 by reflections, 24 elements.

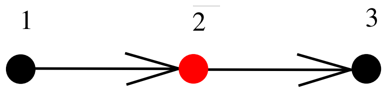
$$G(Q_2) = \langle \underbrace{t_1, t_2, t_3 \mid t_i^2 = (t_i t_j)^3}_{} = (t_1\ t_2 t_3 t_2)^2 = e \rangle$$

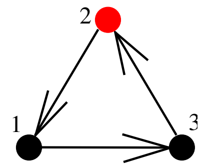
G_0 – affine Coxeter group $\widetilde{A}_2$, acts on $\mathbb{E}^2$ by reflections.



$(t_1\ t_2 t_3 t_2)^2 = e =$ transl. by 4 levels – Identify!

4. Geometric interpretation.

Example: $Q_1 = A_3 =$ 

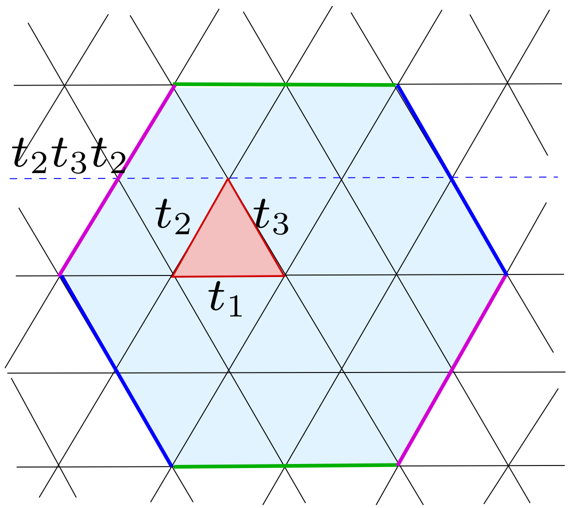
$\xrightarrow{\mu_2}$ $Q_2 =$ 

$$G(Q_1) = \langle s_1, s_2, s_3 \mid s_i^2 = (s_1 s_2)^3 = (s_2 s_3)^3 = (s_1 s_2)^2 = e \rangle$$

finite Coxeter group A_3, acts on S^2 by reflections, 24 elements.

$$G(Q_2) = \langle \underbrace{t_1, t_2, t_3 \mid t_i^2 = (t_i t_j)^3}_{G_0} = (t_1\ t_2 t_3 t_2)^2 = e \rangle$$

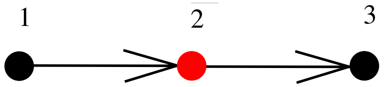
G_0 – affine Coxeter group $\widetilde{A}_2$, acts on $\mathbb{E}^2$ by reflections.

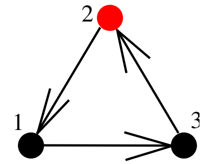


$(t_1\ t_2 t_3 t_2)^2 = e =$ transl. by 4 levels – Identify!

$G = G_0 / NCl((t_1\ t_2 t_3 t_2)^2)$ – Identify! Identify!

4. Geometric interpretation.

Example: $Q_1 = A_3 =$ 

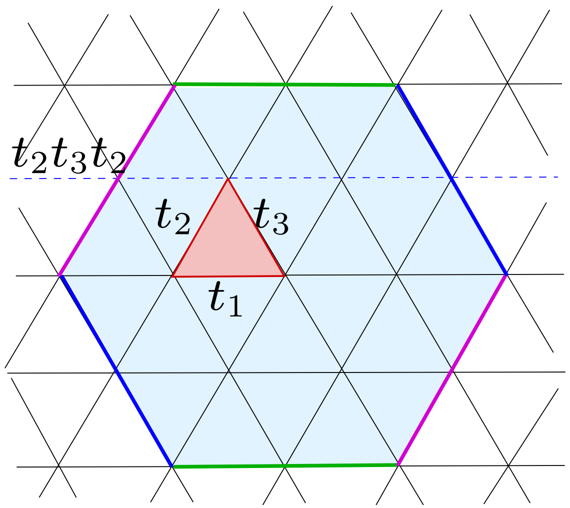
$\xrightarrow{\mu_2}$ $Q_2 =$ 

$$G(Q_1) = \langle s_1, s_2, s_3 \mid s_i^2 = (s_1 s_2)^3 = (s_2 s_3)^3 = (s_1 s_2)^2 = e \rangle$$

finite Coxeter group A_3, acts on S^2 by reflections, 24 elements.

$$G(Q_2) = \langle \underbrace{t_1, t_2, t_3 \mid t_i^2 = (t_i t_j)^3}_{G_0} = (t_1\ t_2 t_3 t_2)^2 = e \rangle$$

G_0 – affine Coxeter group $\widetilde{A}_2$, acts on $\mathbb{E}^2$ by reflections.



$(t_1\ t_2t_3t_2)^2 = e =$ transl. by 4 levels – Identify!

$G = G_0/NCl((t_1\ t_2t_3t_2)^2)$ – Identify! Identify!

$G = G(Q_2)$ acts on a torus T^2.

5. More generally:

- $G_0 =$ a Coxeter group defined by (R1) and (R2).

5. More generally:

- $G_0 =$ a Coxeter group defined by (R1) and (R2).
- Each Coxeter group G_0 acts on its Davis complex $\Sigma(G_0)$
  (contractible, piecewise Euclidean, with $CAT(0)$ metric).

5. More generally:

- $G_0 =$ a Coxeter group defined by (R1) and (R2).
- Each Coxeter group G_0 acts on its Davis complex $\Sigma(G_0)$
  (contractible, piecewise Euclidean, with $CAT(0)$ metric).
- Take its quotient by cycle relations:
  Denote $G_{rel} := NCl(R3)$,
  consider $X = \Sigma(G_0)/G_{rel}$,
  then $G : X$.

5. More generally:

- G_0 = a Coxeter group defined by (R1) and (R2).
- Each Coxeter group G_0 acts on its Davis complex $\Sigma(G_0)$
  (contractible, piecewise Euclidean, with $CAT(0)$ metric).
- Take its quotient by cycle relations:
  Denote $G_{rel} := NCl(R3)$,
  consider $X = \Sigma(G_0)/G_{rel}$,
  then $G : X$.

Theorem 2 [F-Tumarkin'14] (Manifold property)
The group G_{rel} is torsion free,
i.e. if $\Sigma(G_0)$ is a manifold then X is a manifold.

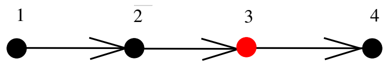
Taking the quotient, we are not introducing any new singularities!

5. More generally:

Corollary from Manifold Property: can cook hyperbolic manifolds with large symmetry groups.

5. More generally:

Corollary from Manifold Property: can cook hyperbolic manifolds with large symmetry groups.

Example: $A_4 =$ 

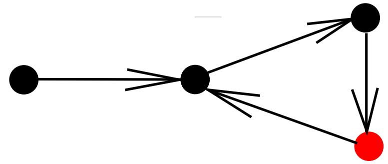
$\xrightarrow{\mu_3}$ 

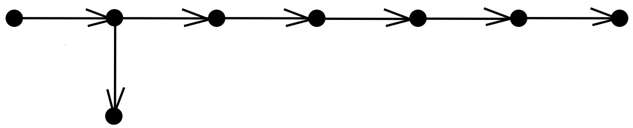
diagram of hyperbolic simplex

$\Rightarrow$ Hyperbolic 3-manifold with action of the group A_4.

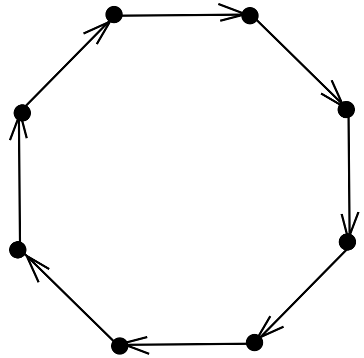
5. More generally:

Corollary from Manifold Property: can cook hyperbolic manifolds with large symmetry groups.

Another example:



$\xrightarrow{\mu}$



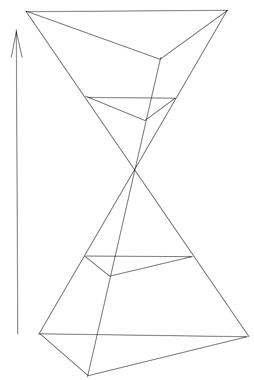
$D_n : S^n$

$\widetilde{A}_{n-1} : \mathbb{E}^{n-1}$

$$G_{rel} = NCl(\ (s_1\ \ s_2 s_3 \ldots s_n \ldots s_3 s_2)^2)$$

$\mathbb{E}^{n-1}/(n$ translations$) = \mathbb{T}^{n-1}$

tiled by simplices

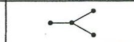
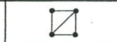
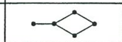
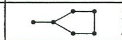
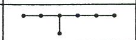
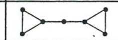
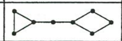


5. More generally:

Corollary from Manifold Property: can cook hyperbolic manifolds with large symmetry groups.

More hyperbolic examples:

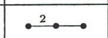
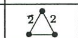
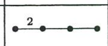
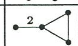
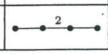
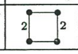
TABLE 5.1. Actions on hyperbolic manifolds.

W	Q	Q_1	$\lvert W\rvert$	dim X	vol X approx.	number of cusps	$\chi(X)$
A_4			$5!$	3	$\lvert W\rvert \cdot 0.084578$	5	
D_4			$2^3 \cdot 4!$	3	$\lvert W\rvert \cdot 0.422892$	16	
D_5			$2^4 \cdot 5!$	4	$\lvert W\rvert \cdot 0.013707$	10	2
E_6			$2^7 \cdot 3^4 \cdot 5$	5	$\lvert W\rvert \cdot 0.002074$	27	
E_7			$2^{10} \cdot 3^4 \cdot 5 \cdot 7$	6	$\lvert W\rvert \cdot 2.962092 \times 10^{-4}$	126	-52
E_8			$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$	7	$\lvert W\rvert \cdot 4.110677 \times 10^{-5}$	2160	
A_7			$8!$	5		70	
D_8			$2^7 \cdot 8!$	6	$\lvert W\rvert \cdot 0.002665$	1120	-832

simplices

pyramids over a product of 2 simplices

TABLE 7.1. Actions on hyperbolic manifolds, non-simply-laced case.

W	$\mathcal{G}$	$\mathcal{G}_1$	$\lvert W\rvert$	dim (X)	vol X approx.	number of cusps	$\chi(X)$ (dim X even)
B_3			$2^3 \cdot 3!$	2	8π	compact	-4
B_4			$2^4 \cdot 4!$	3	$\lvert W\rvert \cdot 0.211446$	16	
F_4			$2^7 \cdot 3^2$	3	$\lvert W\rvert \cdot 0.222228$	compact	

6. Beyond finite type:

- Q is of finite mutation type if $\quad \sharp|Q' \sim_\mu Q| < \infty$.

6. Beyond finite type:

- Q is of finite mutation type if $\quad \sharp|Q' \sim_\mu Q| < \infty.$

Classification [F, P.Tumarkin, M.Shapiro'2008]:
Connected quiver is of finite mutation type iff

(a) Q has 2 vertices, or
(b) Q arises from a triangulated surface, or
(c) Q is mutation-equivalent to one of 11 exceptional quivers:

6. Beyond finite type:

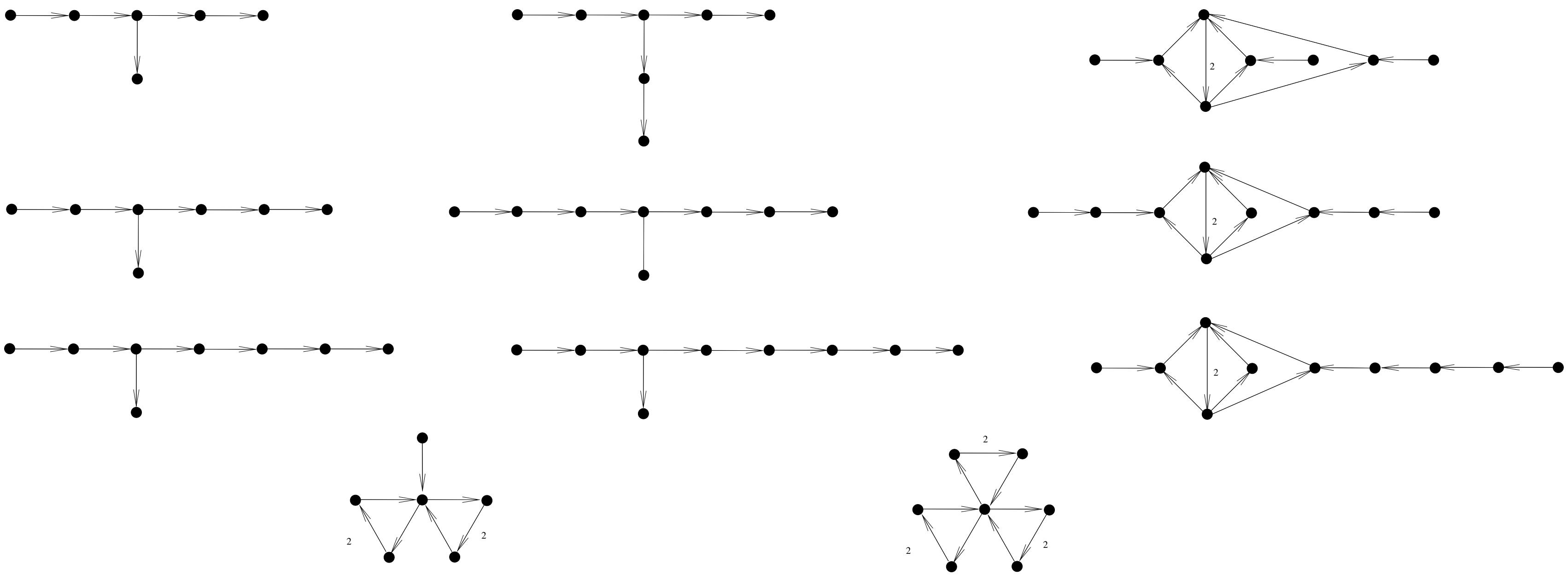
- Q is of finite mutation type if $\qquad \sharp|Q' \sim_\mu Q| < \infty$.

Classification [F, P.Tumarkin, M.Shapiro'2008]:
Connected quiver is of finite mutation type iff

(a) Q has 2 vertices, or

(b) Q arises from a triangulated surface, or

(c) Q is mutation-equivalent to one of 11 exceptional quivers:



6. Beyond finite type:

• Q is of finite mutation type if $\sharp|Q' \sim_\mu Q| < \infty$.

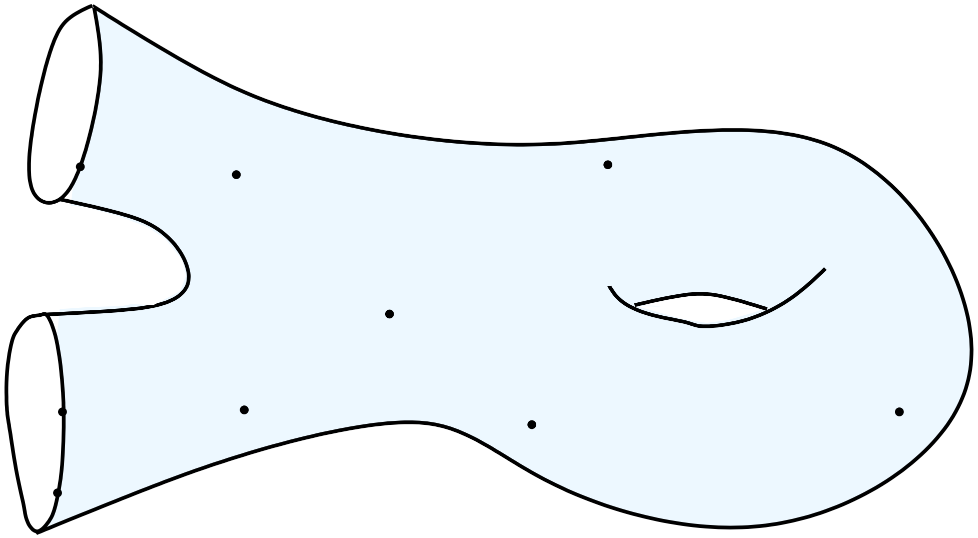
Classification [F, P.Tumarkin, M.Shapiro'2008]:
Connected quiver is of finite mutation type iff

(a) Q has 2 vertices, or
(b) Q arises from a triangulated surface, or
(c) Q is mutation-equivalent to one of 11 exceptional quivers.

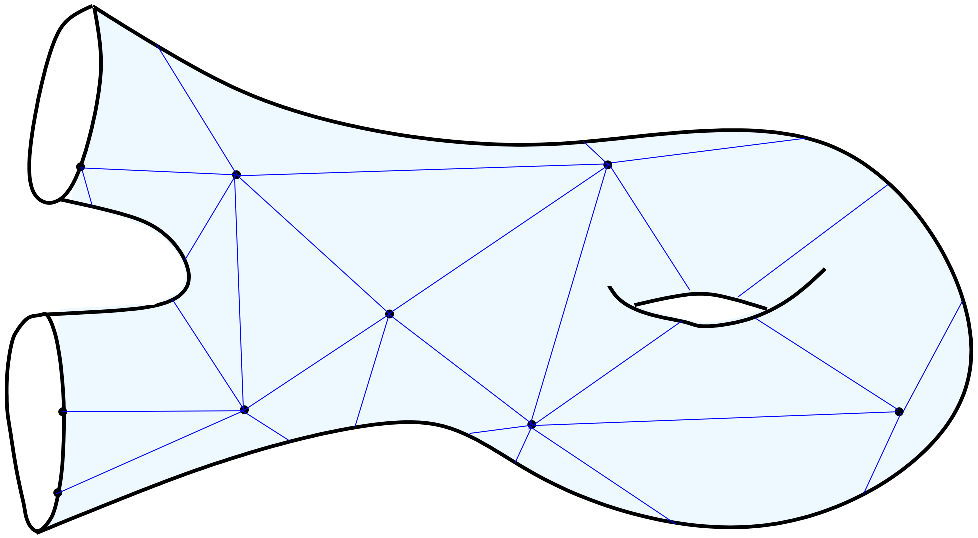
Groups $G(Q)$ for them:

(a) trivial
(b) ?????
(c) can construct (with some additional relations).

7. Quivers from triangulated surfaces:

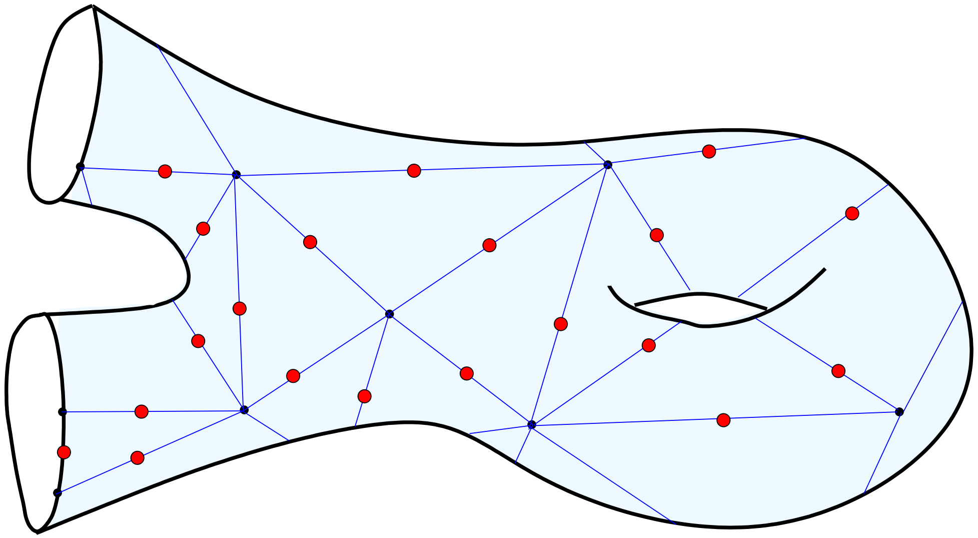


7. Quivers from triangulated surfaces



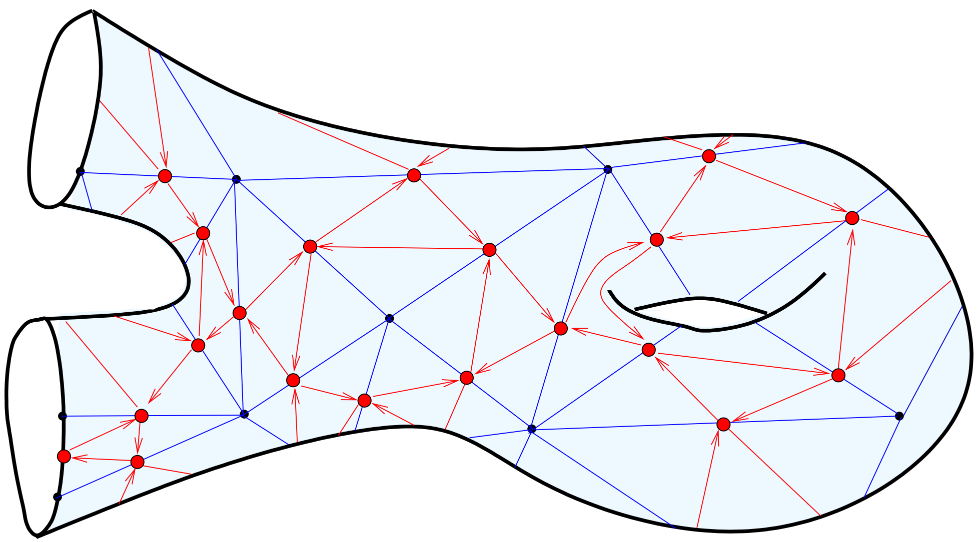
7. Quivers from triangulated surfaces

Triangulated surface $\longrightarrow$ Quiver



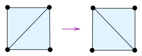
7. Quivers from triangulated surfaces

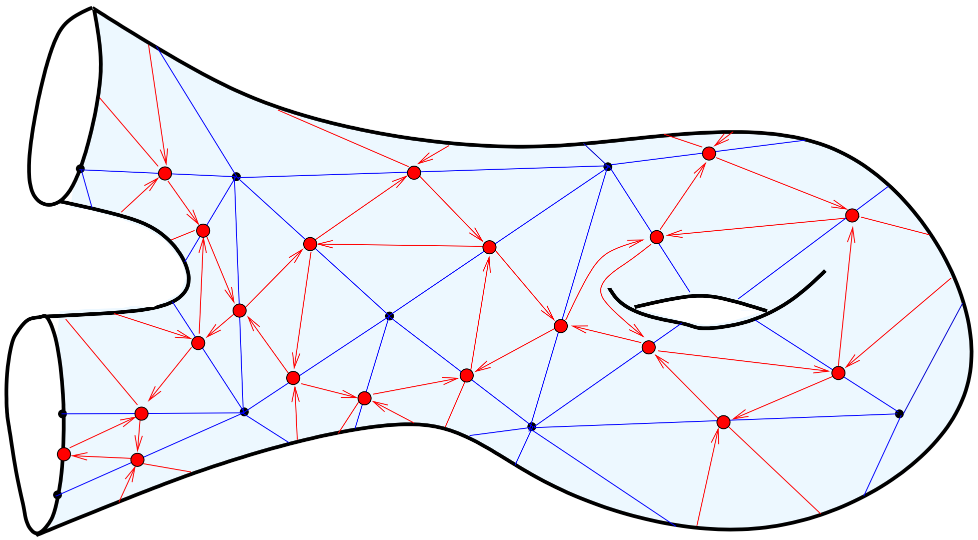
Triangulated surface	$\longrightarrow$	Quiver
edge of triangulation		vertex of quiver
two edges of one triangle		arrow of quiver

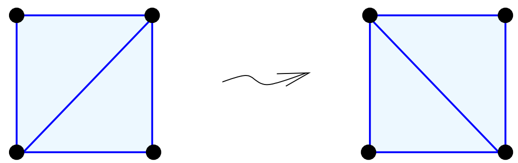


7. Quivers from triangulated surfaces

Triangulated surface	$\longrightarrow$	Quiver
edge of triangulation		vertex of quiver
two edges of one triangle		arrow of quiver

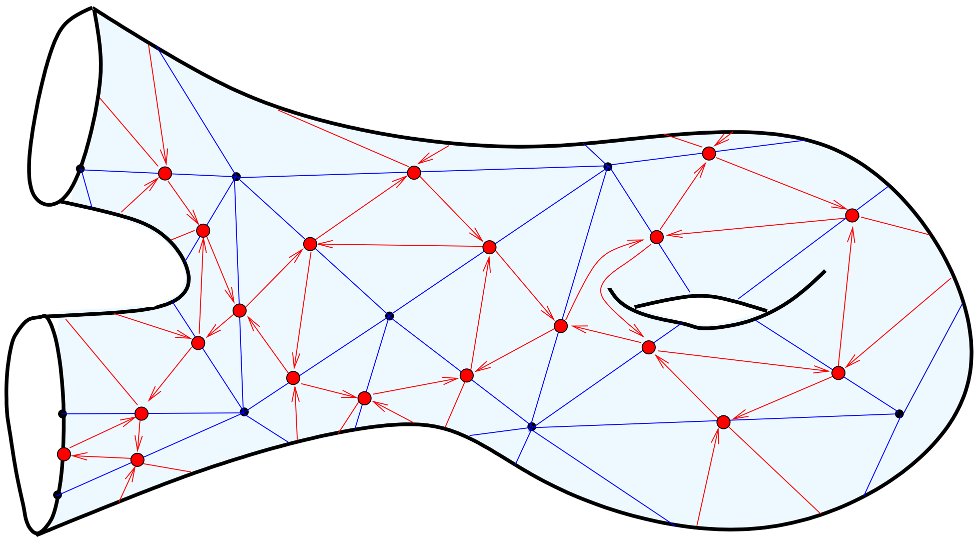
flip of triangulation 

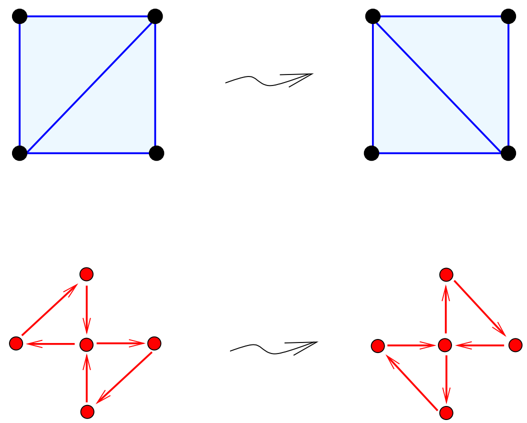




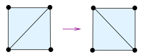
7. Quivers from triangulated surfaces

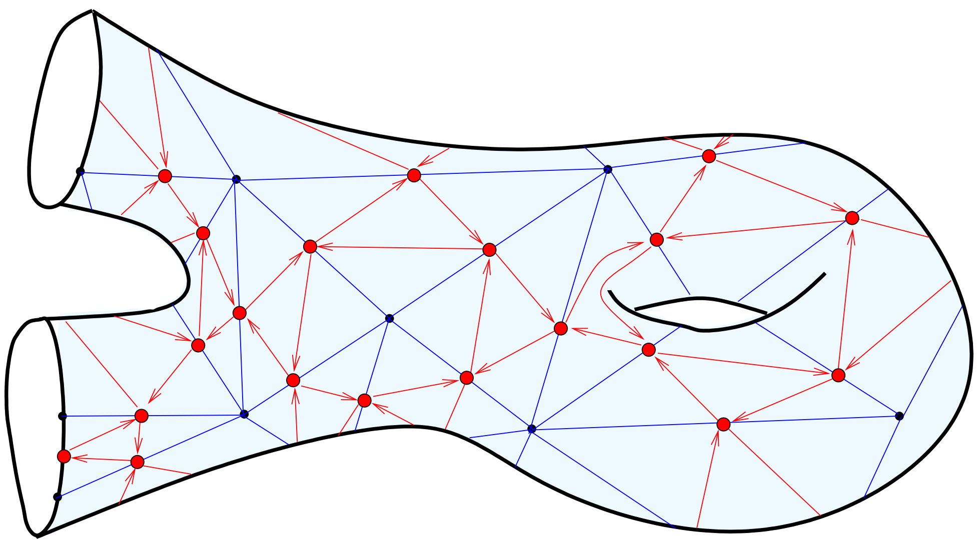
Triangulated surface	$\longrightarrow$	Quiver
edge of triangulation		vertex of quiver
two edges of one triangle		arrow of quiver
flip of triangulation		mutation of quiver

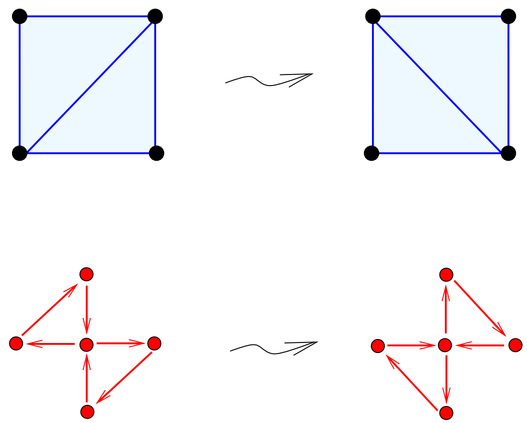




7. Quivers from triangulated surfaces

Triangulated surface	$\longrightarrow$	Quiver
edge of triangulation		vertex of quiver
two edges of one triangle		arrow of quiver
flip of triangulation 		mutation of quiver





Fact. Quivers from triangulations of the same surface are mutation-equivalent (and form the whole mutation class).

7. Quivers from triangulated surfaces

Fact. Quivers from triangulations of the same surface are mutation-equivalent (and form the whole mutation class).

Want: Group G for every mut. class $Q(T)$, i.e. G for every surface.

7. Quivers from triangulated surfaces

Fact. Quivers from triangulations of the same surface are mutation-equivalent (and form the whole mutation class).

Want: Group G for every mut. class $Q(T)$, i.e. G for every surface.

Construction of $G(Q)$ for unpunctured surfaces:

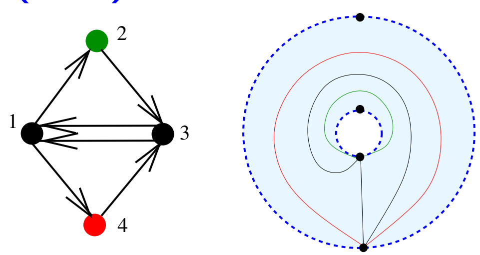
- Generators of G $\leftrightarrow$ arcs of the triangulation of Q.
- Relations of G:

(R1) $s_i = e$

(R2) $(s_i s_j)^{m_{ij}} = e$

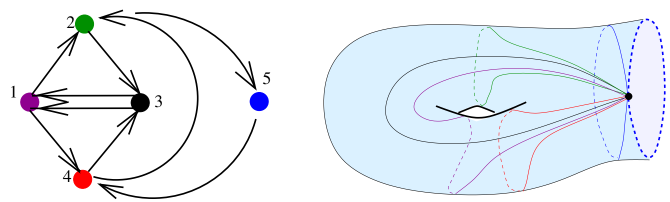
(R3) Cycle relations

(R4) $\widetilde{A}_2$-relations:



$(s_1 \ s_2 s_3 s_4 s_3 s_2)^2 = e$

(R5) Handle relations:



$(s_1 \ s_2 s_3 s_4 s_5 s_4 s_3 s_2)^2 = e$

$(s_1 \ s_4 s_3 s_2 s_5 s_2 s_3 s_4)^2 = e$

7. Quivers from triangulated surfaces: unpunctured case

Theorem [FT'13]
If S is an unpunctured surface, T triangulation of S, $Q = Q(T)$, $G = G(Q)$, then G is mutation invariant, i.e. G does not depend on the choice of triangulation T.

In other words, G is an invariant of a surface.

7. Quivers from triangulated surfaces: unpunctured case

Theorem [FT'13]
If S is an unpunctured surface, T triangulation of S,
$Q = Q(T)$, $G = G(Q)$, then G is mutation invariant,
i.e. G does not depend on the choice of triangulation T.

In other words, G is an invariant of a surface.

Remark. • Now, G may be not a Coxeter group, but a quotient.

7. Quivers from triangulated surfaces: unpunctured case

Theorem [FT'13]

If S is an unpunctured surface, T triangulation of S, $Q = Q(T)$, $G = G(Q)$, then G is mutation invariant, i.e. G does not depend on the choice of triangulation T.

In other words, G is an invariant of a surface.

Remark.
- Now, G may be not a Coxeter group, but a quotient.
- Now, we don't know manifold property.

7. Quivers from triangulated surfaces: unpunctured case

Theorem [FT'13]
If S is an unpunctured surface, T triangulation of S,
$Q = Q(T)$, $G = G(Q)$, then G is mutation invariant,
i.e. G does not depend on the choice of triangulation T.

In other words, G is an invariant of a surface.

Remark.
- Now, G may be not a Coxeter group, but a quotient.
- Now, we don't know manifold property.
- We know nothing about this group!

7. Quivers from triangulated surfaces: unpunctured case

Theorem [FT'13]
If S is an unpunctured surface, T triangulation of S,
$Q = Q(T)$, $G = G(Q)$, then G is mutation invariant,
i.e. G does not depend on the choice of triangulation T.

In other words, G is an invariant of a surface.

Remark.
- Now, G may be not a Coxeter group, but a quotient.
- Now, we don't know manifold property.
- We know nothing about this group!

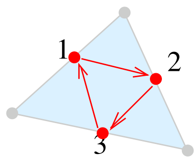
Proposition. G does not depend on the distribution of marked points along boundary components.

7. Quivers from triangulated surfaces: punctured case

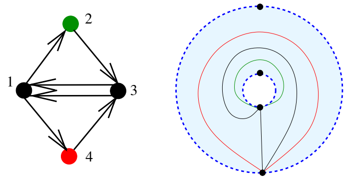
(a) Punctured spheres:

(R1) $s_i = e$

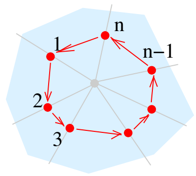
(R2) $(s_i s_j)^{m_{ij}} = e$

(R3) triangle relations: 

$(s_1\ s_2 s_3 s_2)^2 = e$

(R4) annulus relations: 

$(s_1\ s_2 s_3 s_4 s_3 s_2)^2 = e$

(R6) puncture relations: 

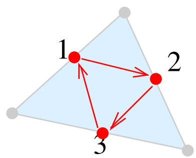
$(s_1\ s_2 s_3 \dots s_n \dots s_3 s_2)^2 = e$

7. Quivers from triangulated surfaces: punctured case

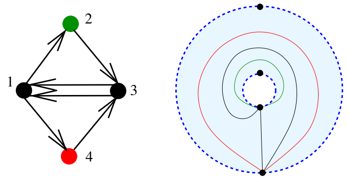
(a) Punctured spheres:

(R1) $s_i = e$

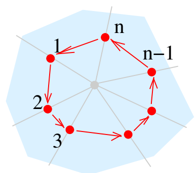
(R2) $(s_i s_j)^{m_{ij}} = e$

(R3) triangle relations: 

$(s_1\ s_2 s_3 s_2)^2 = e$

(R4) annulus relations: 

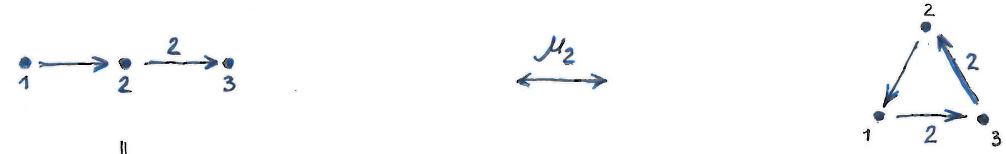
$(s_1\ s_2 s_3 s_4 s_3 s_2)^2 = e$

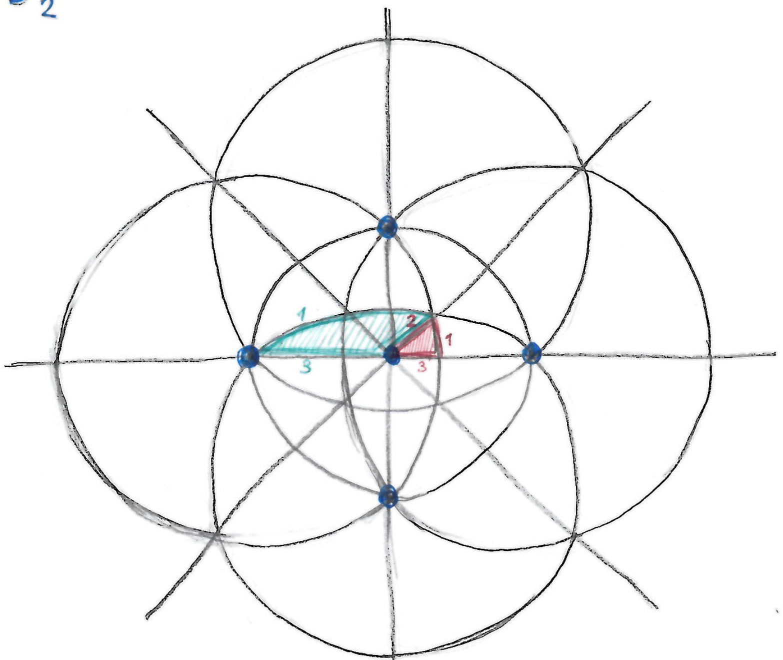
(R6) puncture relations: 

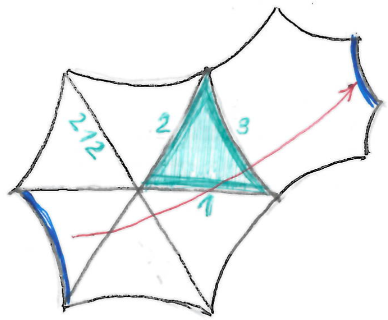
$(s_1\ s_2 s_3 \ldots s_n \ldots s_3 s_2)^2 = e$

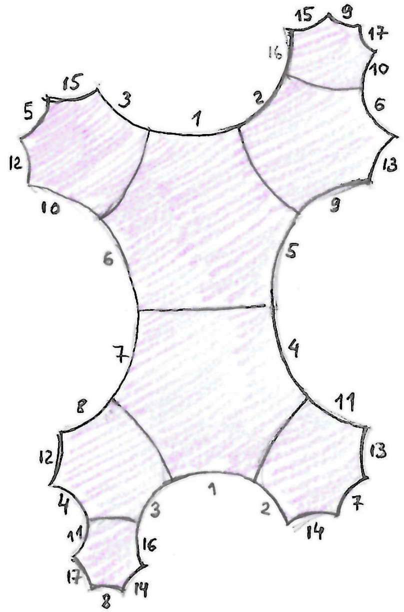
(b) Punctured, $g > 0$???

B_2



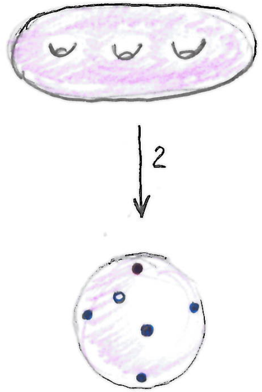






$\langle s_1, s_2, s_3 \mid s_i^2 = (s_1 s_2)^3 = (s_2 s_3)^4 = (s_3 s_1)^2 \rangle$

$\langle t_1, t_2, t_3 \mid t_i^2 = (t_1 t_2)^3 = (t_1 t_3)^4 = (t_2 t_3)^4 = (t_3\, t_2 t_1 t_2)^2 \rangle$



Thanks!