Reflection subgroups of Coxeter groups

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joint work with P. Tumarkin

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1. Introduction
2. Zoo: “Easy to find” reflection subgroups
3. Finite index reflection subgroups
    a. Rank?
    b. Existance?

(G, S) is a Coxeter system if G is a group with a finite set of involutions $S = \{s_1, ..., s_n\}$ and a presentation

$$G = \langle s_i \in S \mid (s_i s_j)^{m_{ij}} \rangle.$$

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Question: About finite index reflection subgroups of G?

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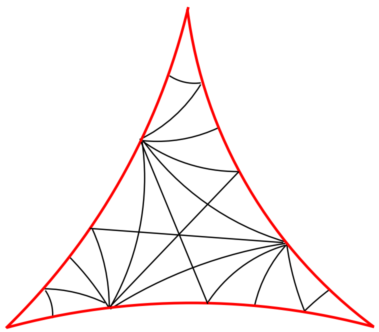

$F = (\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{7})$ $\qquad G = \langle s_1, s_2, s_3 \mid s_i^2 = (s_1 s_2)^2 = (s_2 s_3)^3 = (s_1 s_3)^7 = 1 \rangle$

$P = (\frac{\pi}{7}, \frac{\pi}{7}, \frac{\pi}{7})$ $\qquad H = \langle s_1, s_2, s_3 \mid s_i^2 = (s_i s_j)^7 = 1 \rangle$

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- G is gen. by refl with resp. to facets of F.

- For any Coxeter polytope $P \subset \mathbb{X}^n$,
  a group gen. by refl. with resp. to facets of P is discrete;
- Any discrete refl. gp. in $\mathbb{X}^n$ is a Coxeter group.

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Dyer's theorem is almost evident for geometric groups:
a reflection subgroup $H \subset G$ is generated by reflections with respect to the facets of P (where P is a chamber of H).

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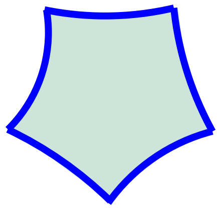

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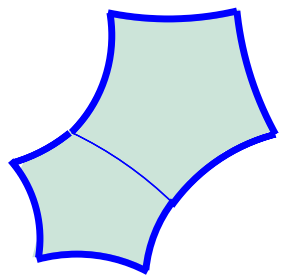

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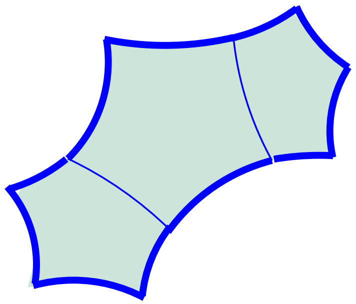

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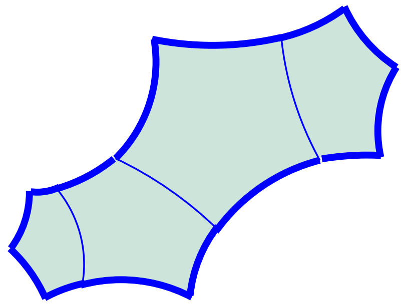

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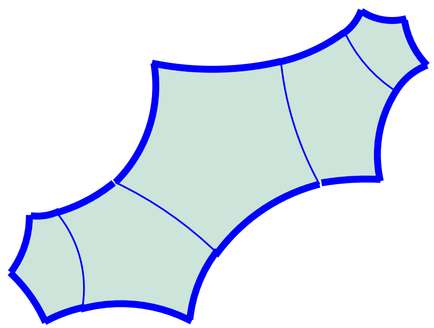

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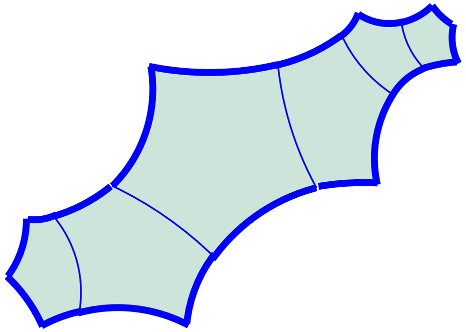

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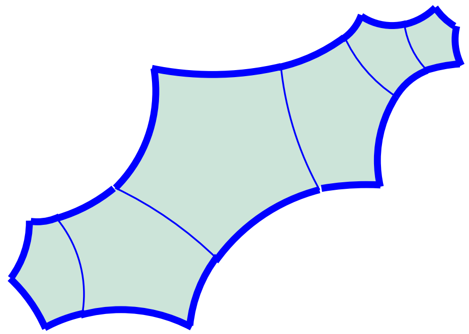
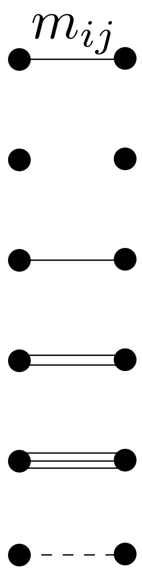


Example 3. $G = \langle s_1, s_2, s_3 \mid s_i^2 = (s_i s_j)^5 = 1\rangle$
has NO finite index subgroups.

Coxeter diagram $C(G)$

- Nodes $\longleftrightarrow$ generating reflections s_i of G

- Edges:



if $(s_i s_j)^{m_{ij}} = 1$

if $(s_i s_j)^2 = 1$

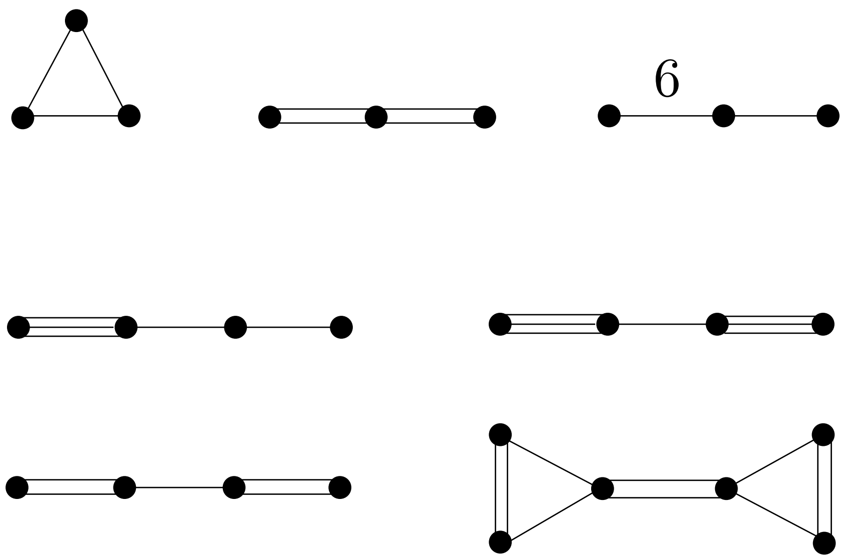
if $(s_i s_j)^3 = 1$

if $(s_i s_j)^4 = 1$

if $(s_i s_j)^5 = 1$

if $s_i s_j$ has infinite order

Examples:



“Visual” subgroups

$$G = \langle s_1, \ldots, s_{n-1}, s_n \rangle. \qquad H = \langle s_1, \ldots, s_{n-2}, s_n s_{n-1} s_n \rangle.$$

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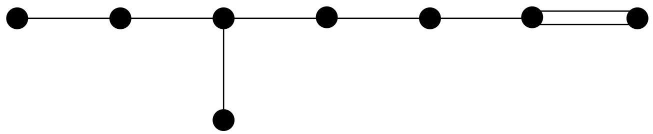

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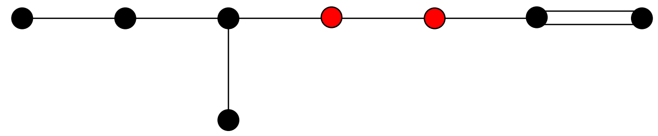

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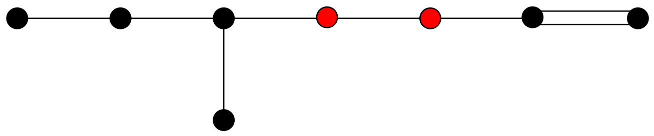

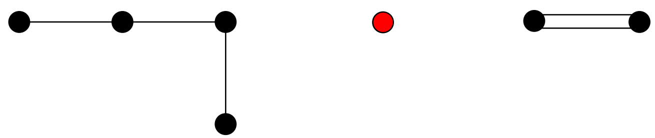

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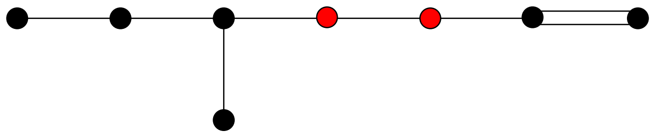

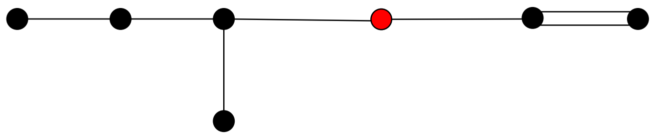

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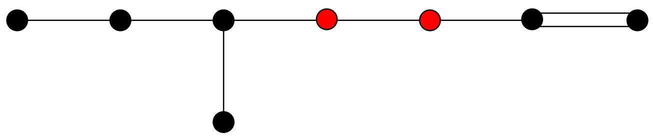

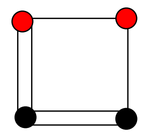

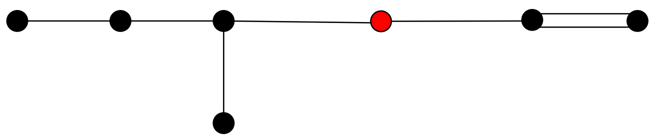

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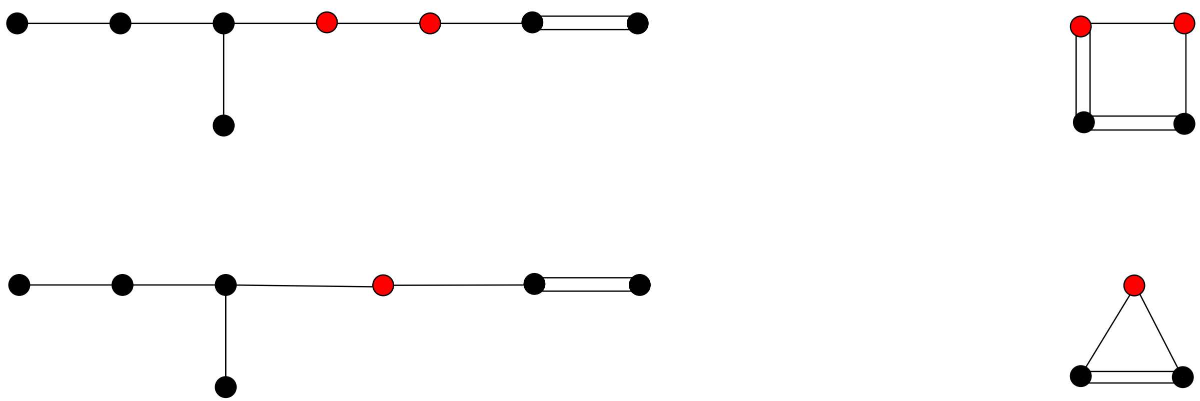

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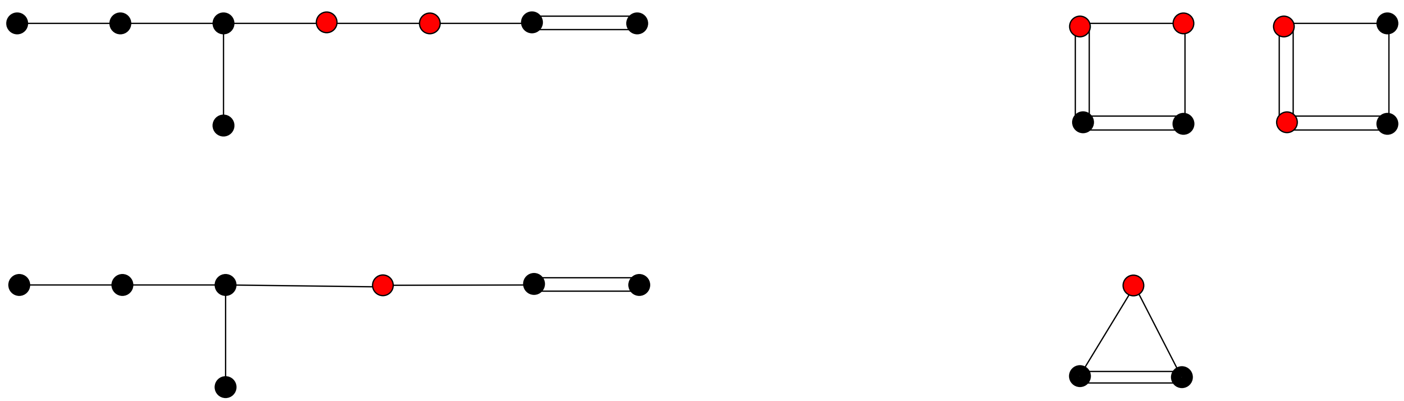

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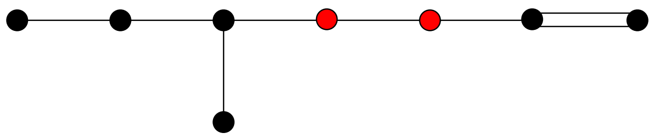
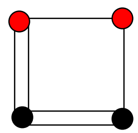
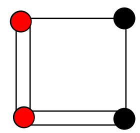
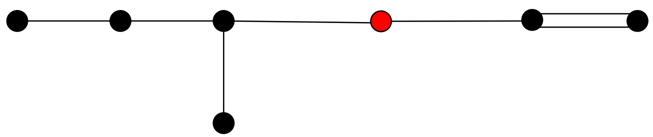
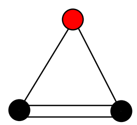
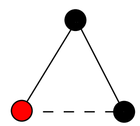






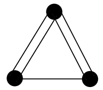


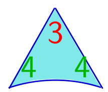




“Rotation” subgroups

Example. $G = \langle s_1, s_2, s_3 \mid s_i^2 = (s_1 s_2)^3 = (s_2 s_3)^4 = (s_1 s_3)^4 = 1 \rangle$





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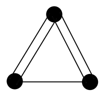

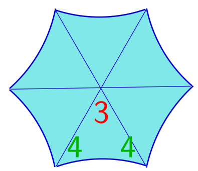

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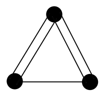

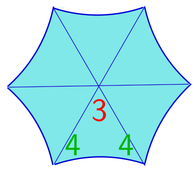

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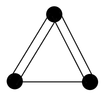

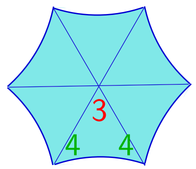

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In general: Let $G : \mathbb{H}^n$, $\mathbb{E}^n$ or $\mathbb{S}^n$ and let $C_1 \subset C(G)$ be an even subdiagram. If C_1 is a diagram of a finite group $K(C_1)$ then G has a finite index subgroup H, $[G : H] = |K(C_1)|$.

Standard subgroups

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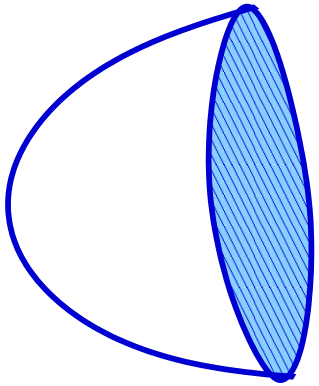
Theorem (Deodhar' 82). Let (G, S) be a Coxeter system and G be an infinite indecomposable group. Any proper standard subgroup $H \subset G$ has infinite index in G.

Indecomposable means "is not a decomposable",

(G, S) is decomposable if $S = I \sqcup J$, and $(s_i s_j)^2 = 1$ for any $s_i \in I$, $s_j \in J$.

Proof of Deodhar's thm, geometric case:



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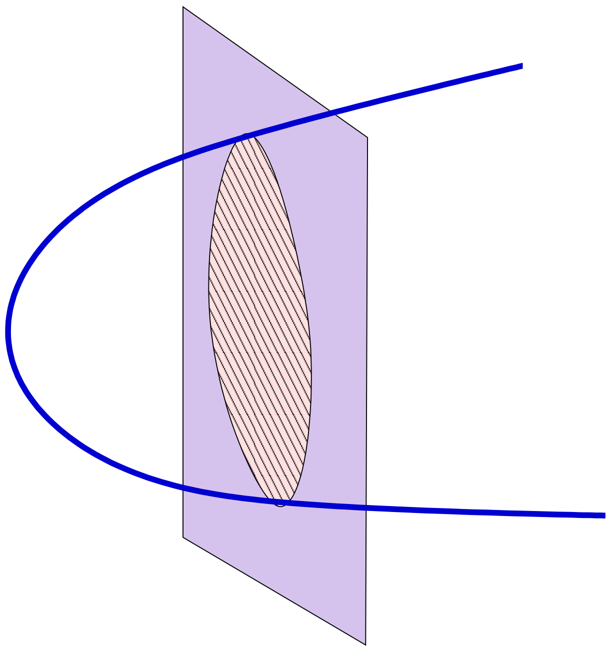

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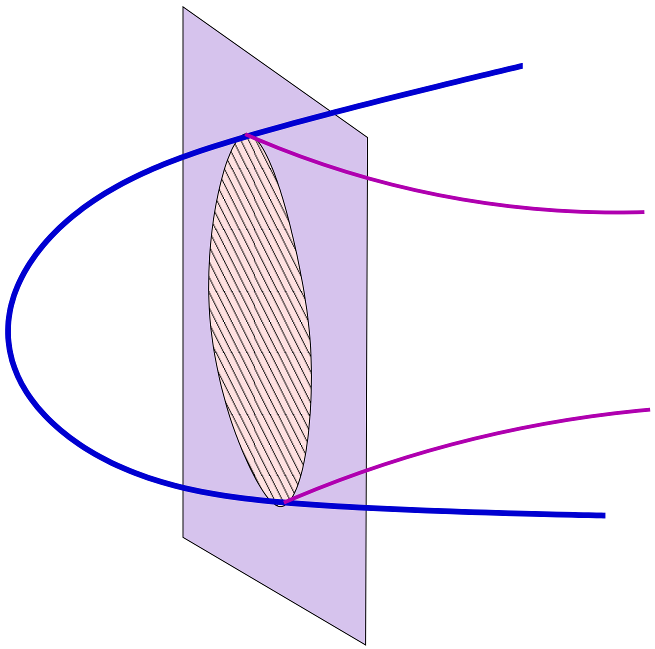

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Thm'
Let G be a cocompact reflection group acting on $\mathbb{H}^n$ or $\mathbb{E}^n$
and P be a finite volume polytope bounded by walls of G.
Then $|P| \geq |F|$, where F is a fundamental chamber of G,
$|P|$ is the number of facets of P.

Proof of Theorem. Suppose that $|P| = k$, $|F| \geq k+1$, and Th. holds for any polytope P' such that $|P'| < k$.

$$M := \{P_1 \mid P_1 \subset P, \quad |P_1| = k\}$$

- finite
- $\neq \emptyset \quad (P \in M\ )$

Take $P_{min} \in M$ minimal by inclusion. P_{min} is a Coxeter polytope.

$$N := \{P_1 \mid P_1 \text{ is bounded by } k \text{ facets of } P_{min} \text{ and one extra mirror}\}$$

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Take $P'_{min} \in N$ minimal by inclusion. P'_{min} is a Coxeter polytope.
Pair (P'_{min}, P_{min}) contradicts to Deodhar's thm.

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Does it hold for arbitrary Coxeter group? No:

Any Coxeter group contains $H = \langle s \mid s^2 = 1 \rangle$.

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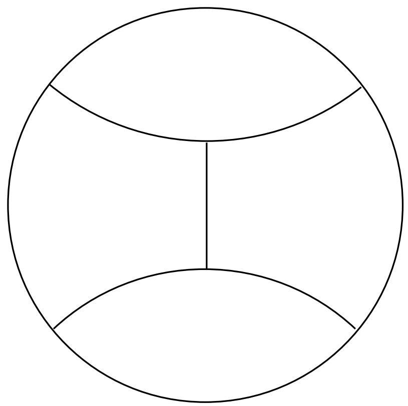

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Yes!

To prove, we need some geometric realization of (G, S).

Tits representation

$$S = \{s_1, \ldots, s_n\} \;\to\; V^n = \langle v_1, \ldots, v_n \rangle, \qquad (*,*):$$

$$(v_i, v_i) = 1, \qquad (v_i, v_j) = \begin{cases} -\cos(\pi/k) & \text{if } \; ord(s_i s_j) = k, \\ -1 & \text{if } \; ord(s_i s_j) = \infty. \end{cases}$$

$$R_{v_j}(v_i) = v_i - 2\frac{(v_i, v_j)}{(v_j, v_j)} v_j.$$

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Prop. $\forall \, (G, S)$, G can not be generated by less than $|S|$ reflections.

Davis complex $\Sigma(G, S)$

- $\Sigma(G, S)$ is a contractible piecewise Euclidean cell complex;

- G acts on $\Sigma(G, S)$ discretely, properly and cocompactly;

- (Moussong' 88)

  $\Sigma(G, S)$ yields a complete piecewise Euclidean $CAT(0)$ metric.

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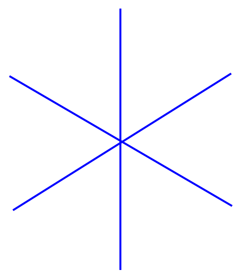
- For a finite group $\Sigma(G,S)$ is just one cell:

  it is a convex hull C of a G-orbit of a suitable point p, s.t. the stabilizer of p is trivial, all edges of C are of length 1.



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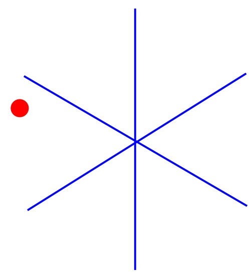

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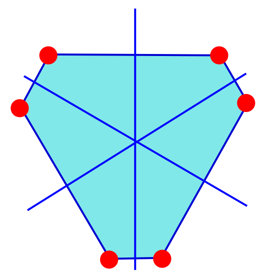

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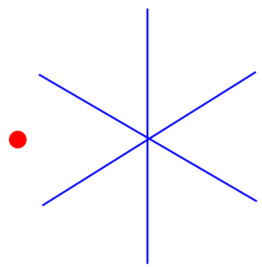

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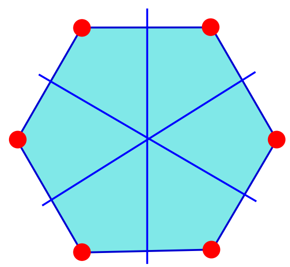

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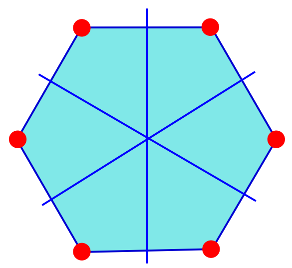

- Faces of C are Davis complexes for the subgroups of G.

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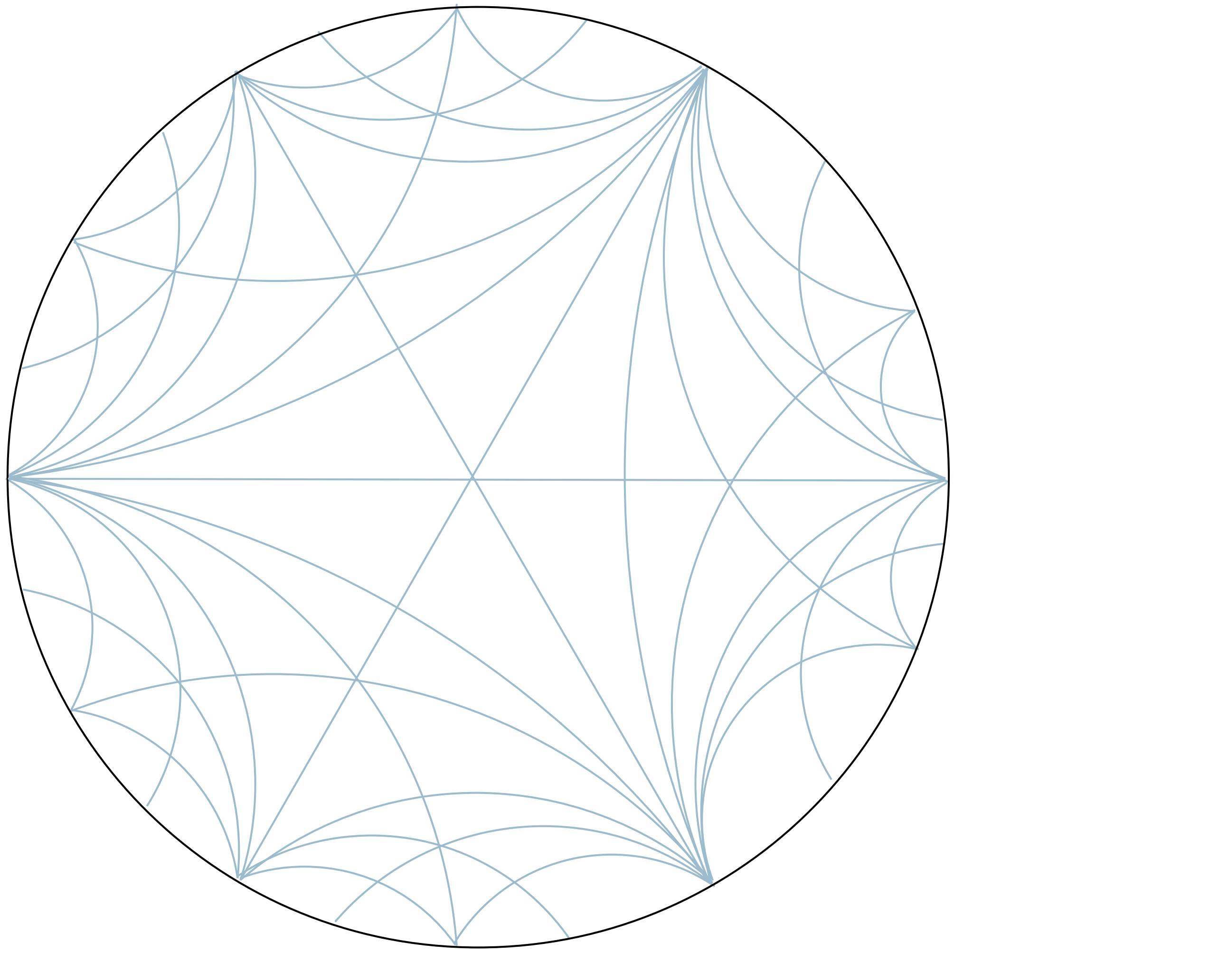
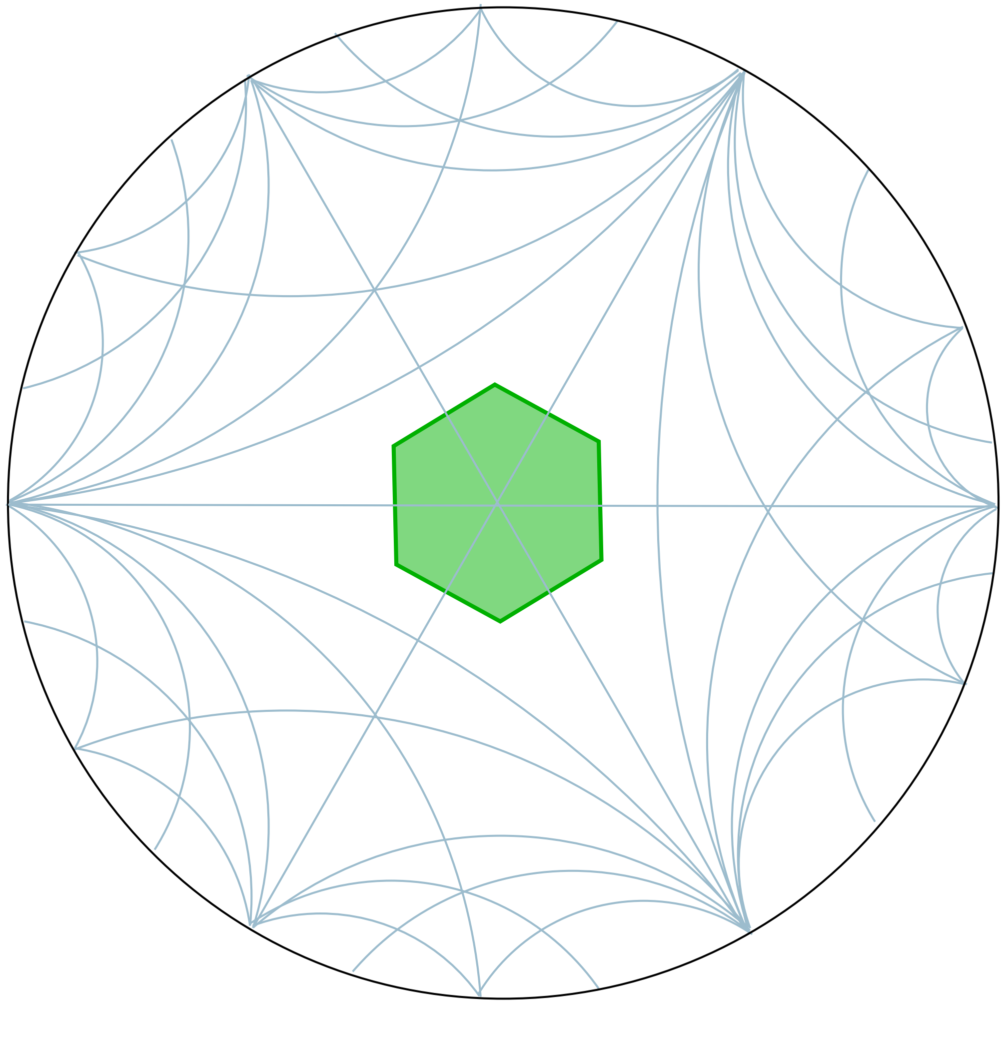
- In more detailes: fin. standard subgp G_J, $J \subset S \to$ cell X_J;

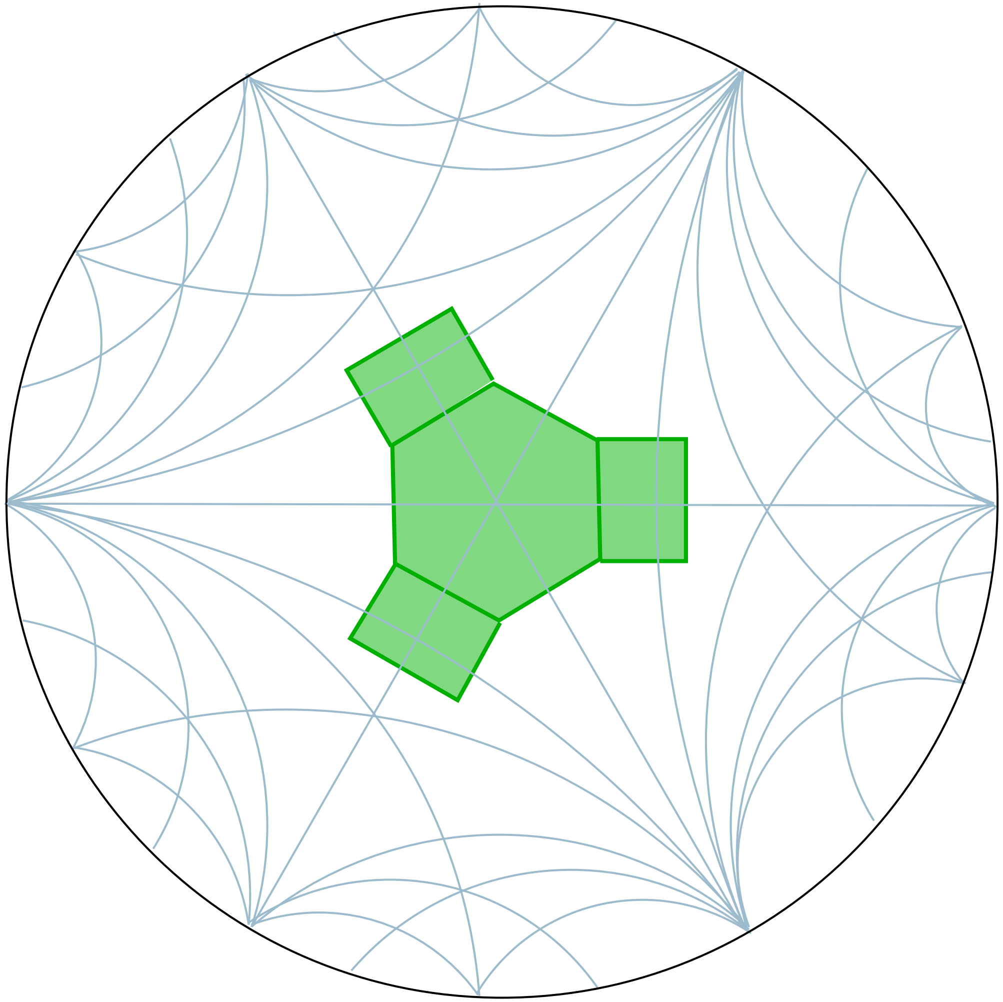
$U = \bigcup \{ \ (g, X_J) \mid g \in G, \ G_J \subseteq G$ finite standard $\}$

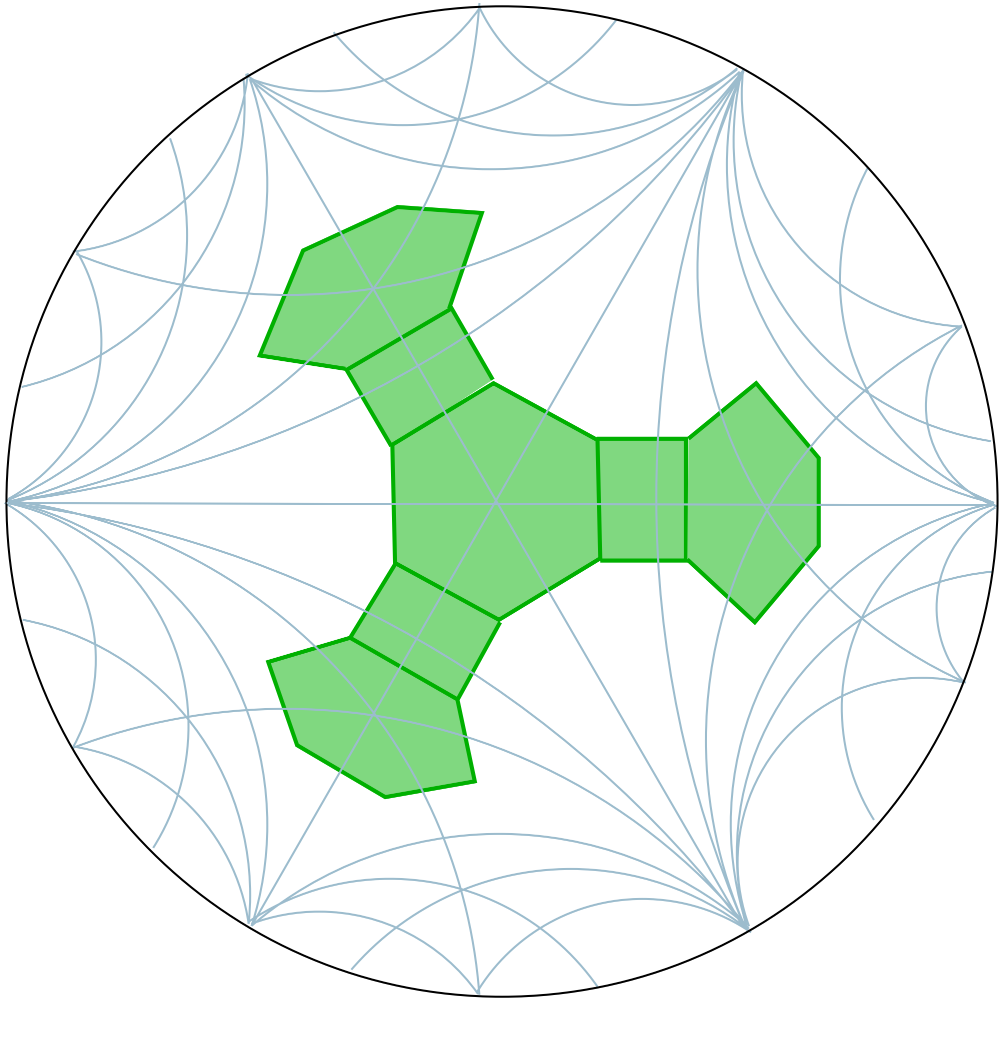
$(gh, x) \sim (h, g^{-1}x)$, whenever $g \in G$, $h \in G_J$, $x \in X_J$.
Cells (g, X_K) and (g, X_L) are glued along the face $J = K \cap L$.

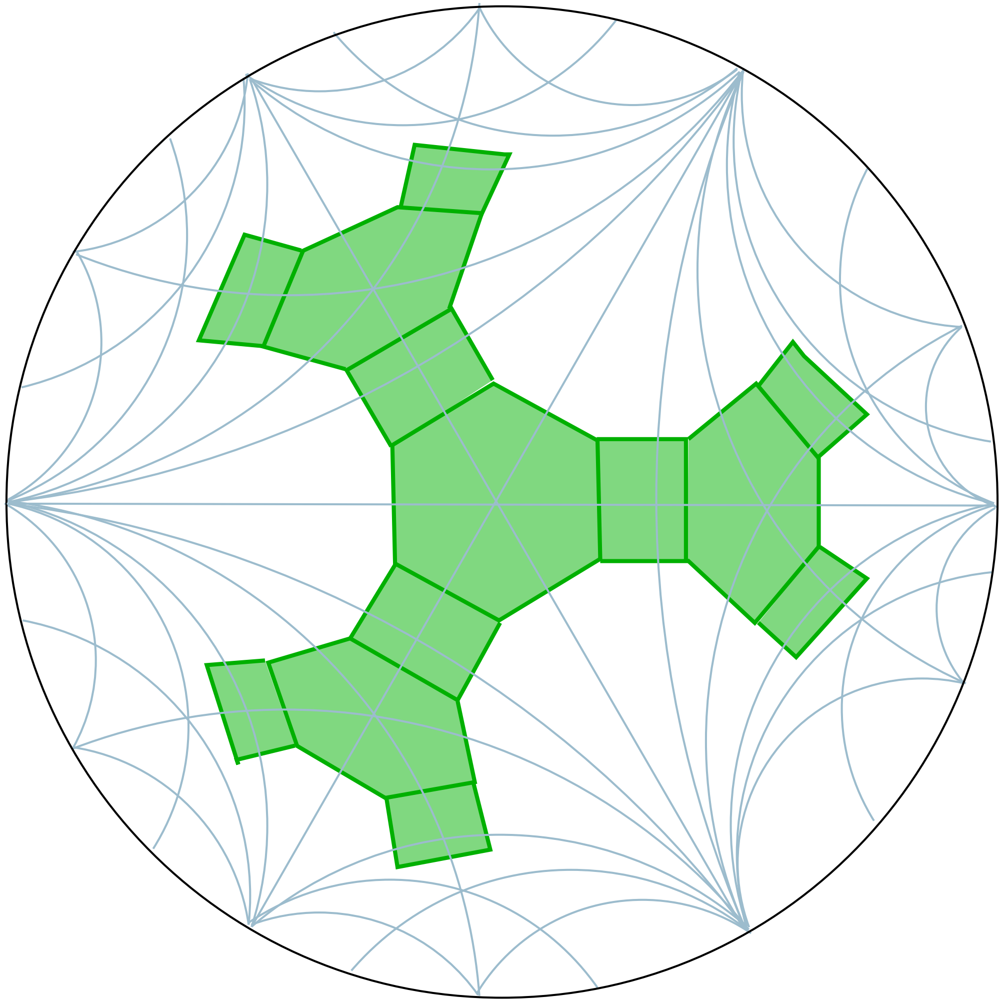
$\Sigma = U/ \sim$.

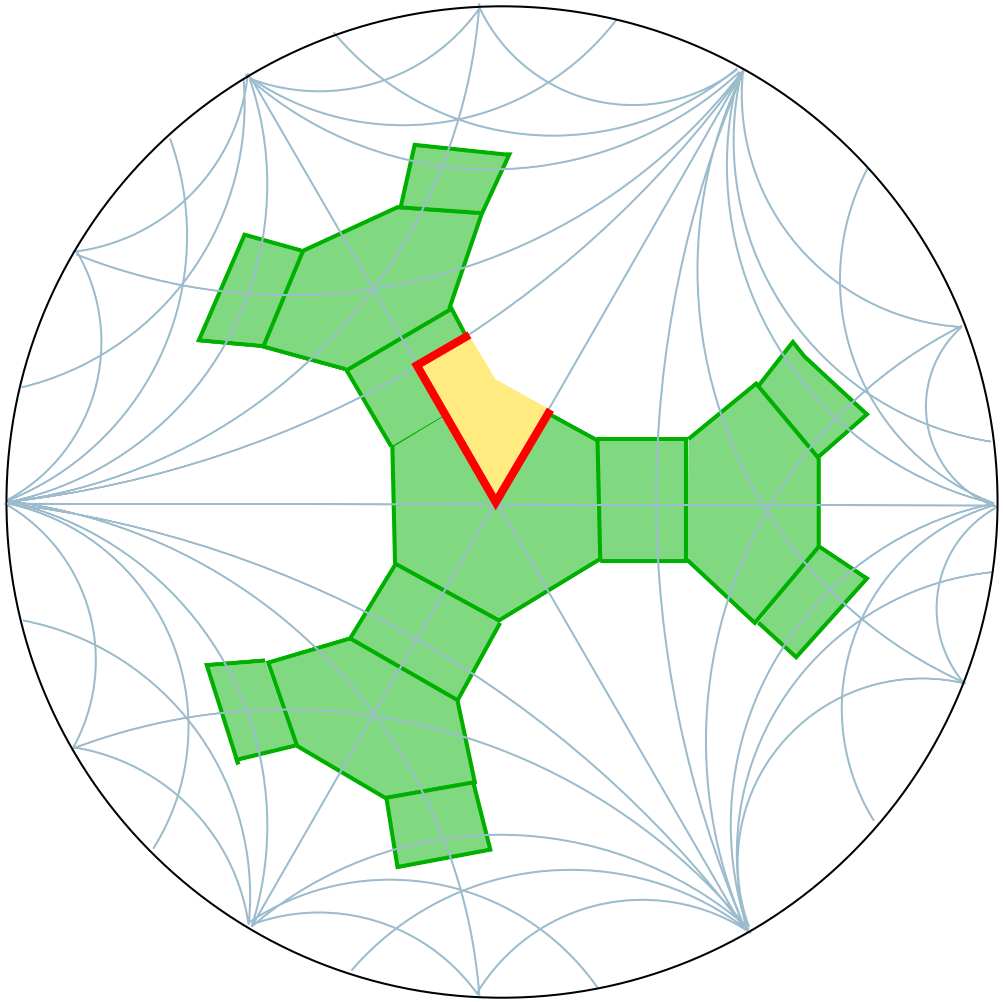












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Let $P \subset \Sigma$ be an acute-angled polytope, let a and b be facets of P and α and β be the walls containing a and b respectively.

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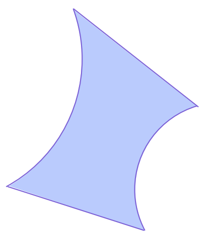

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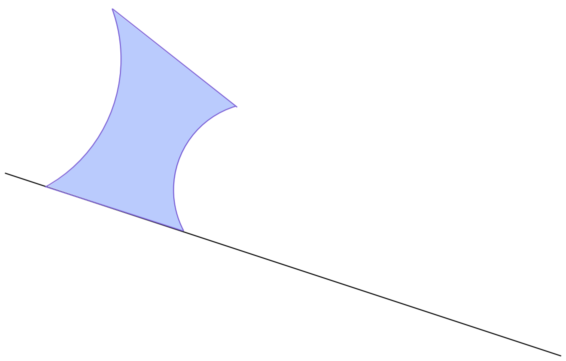

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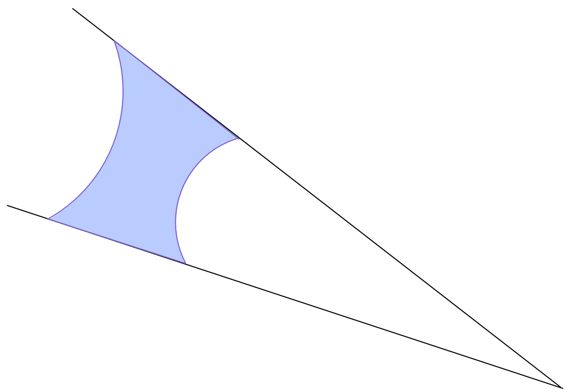

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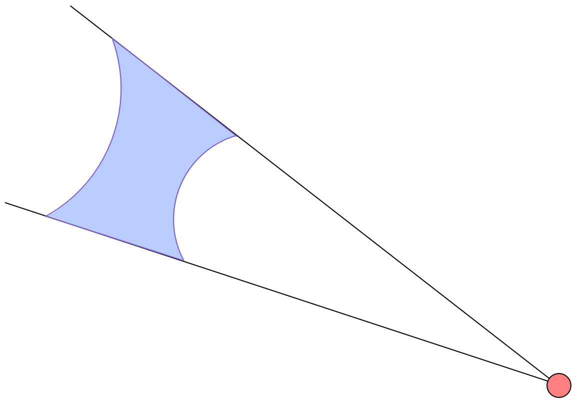

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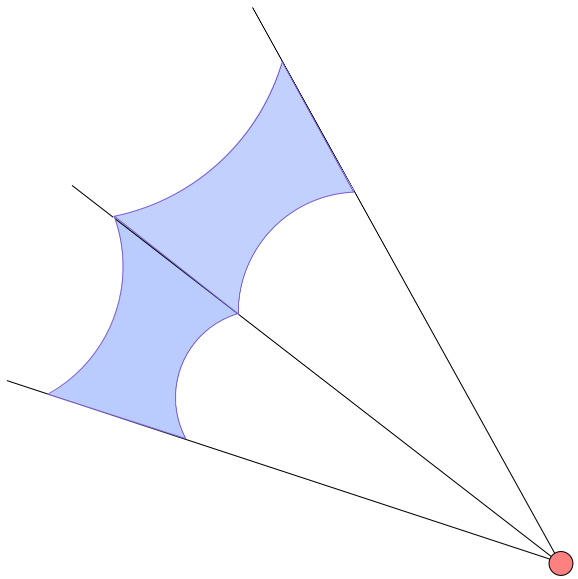

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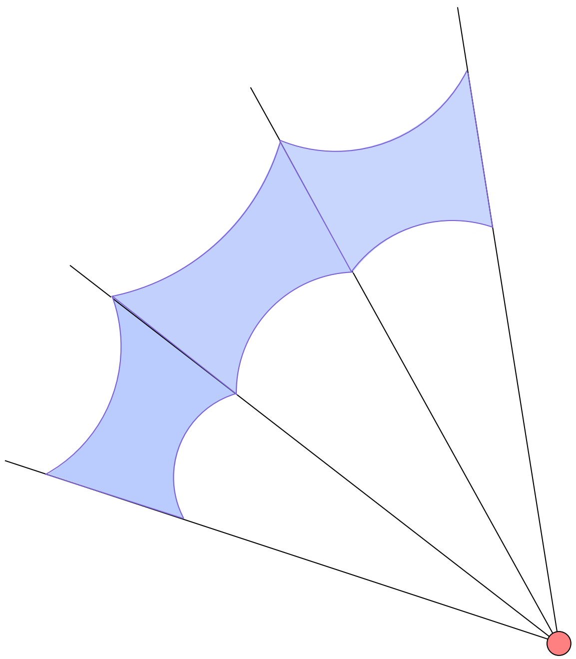

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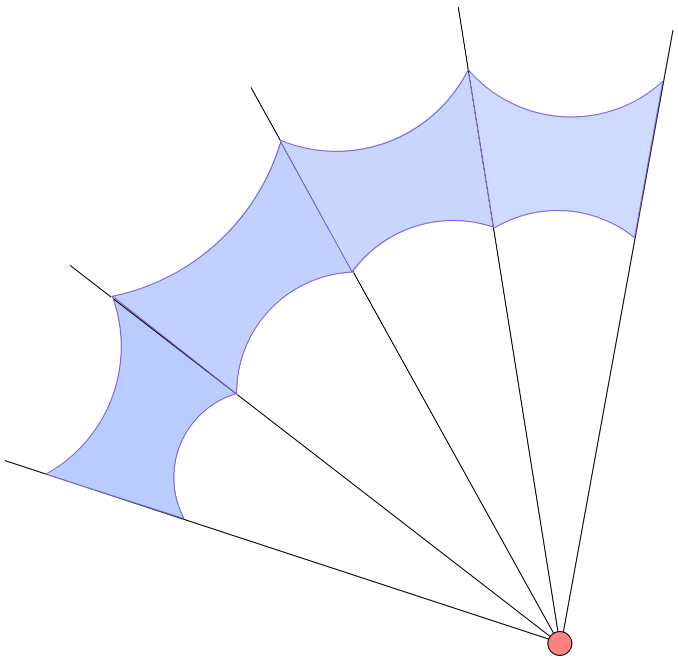

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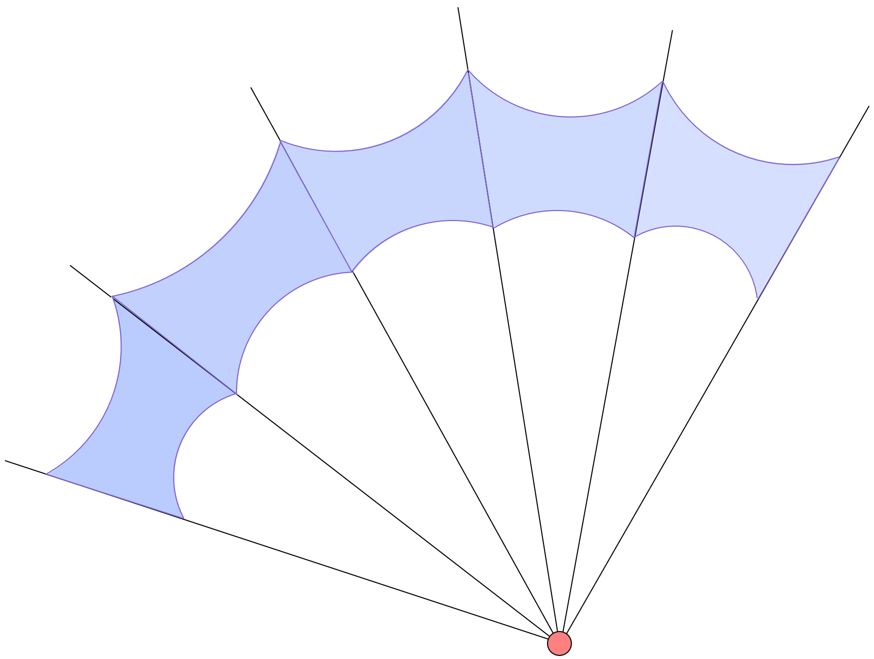

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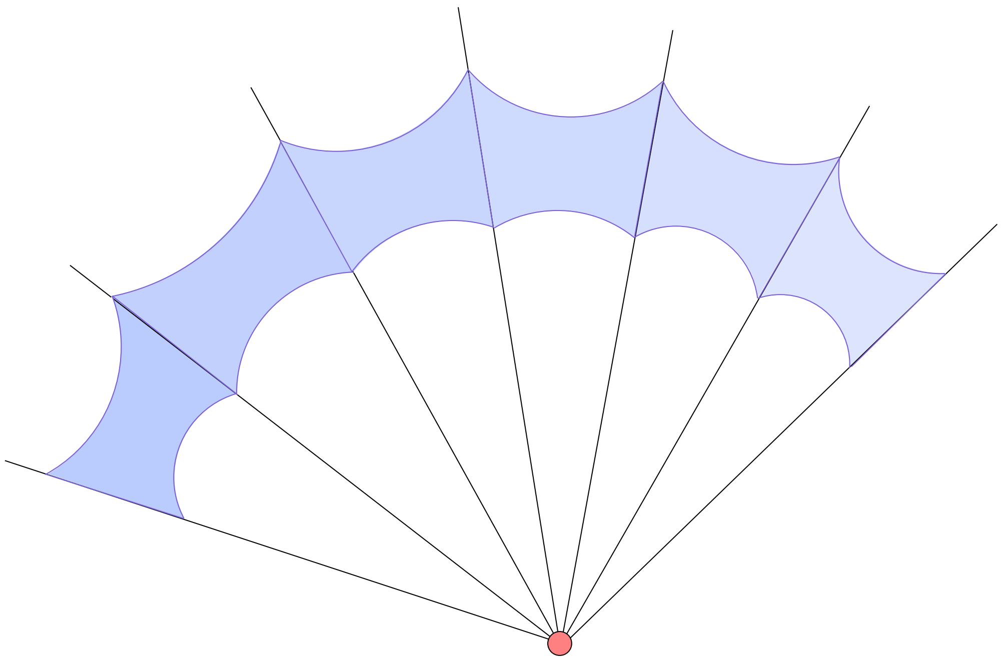

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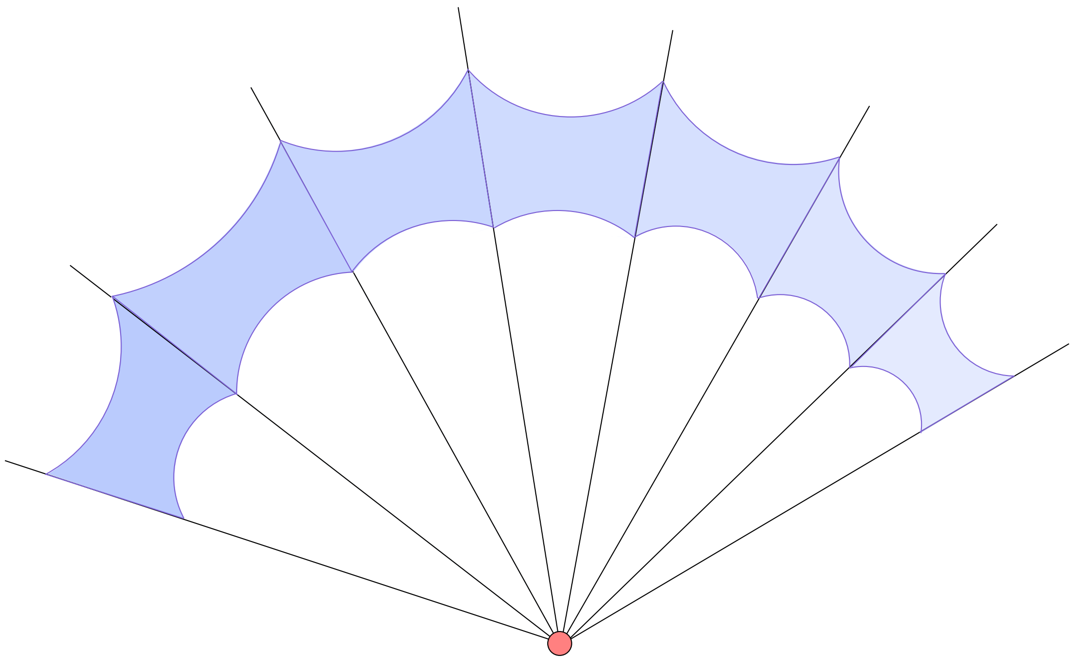

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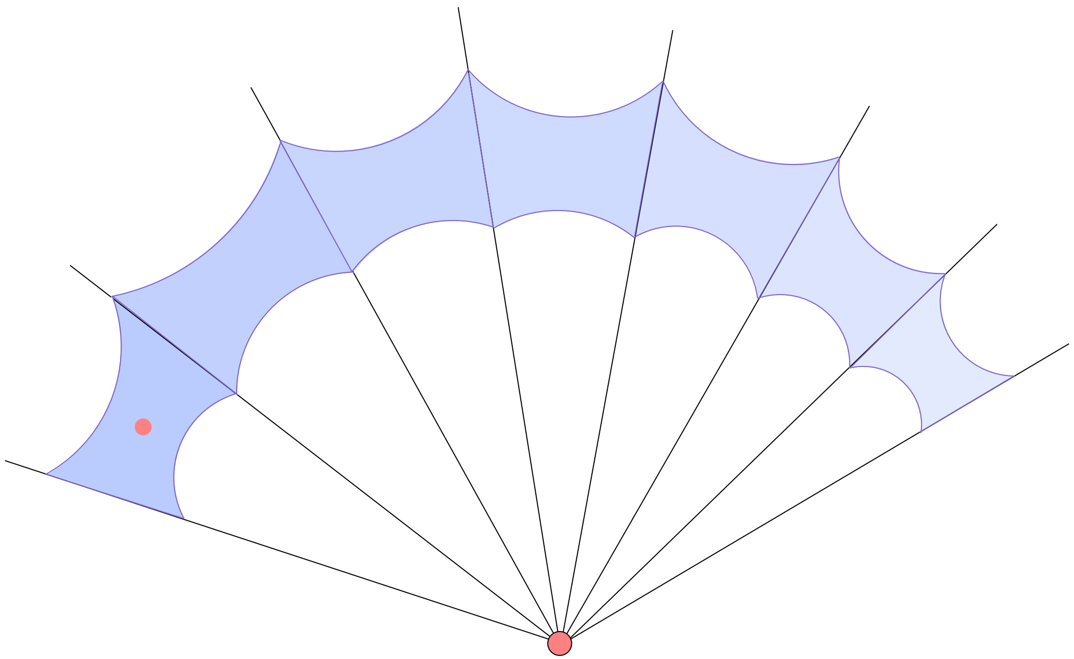

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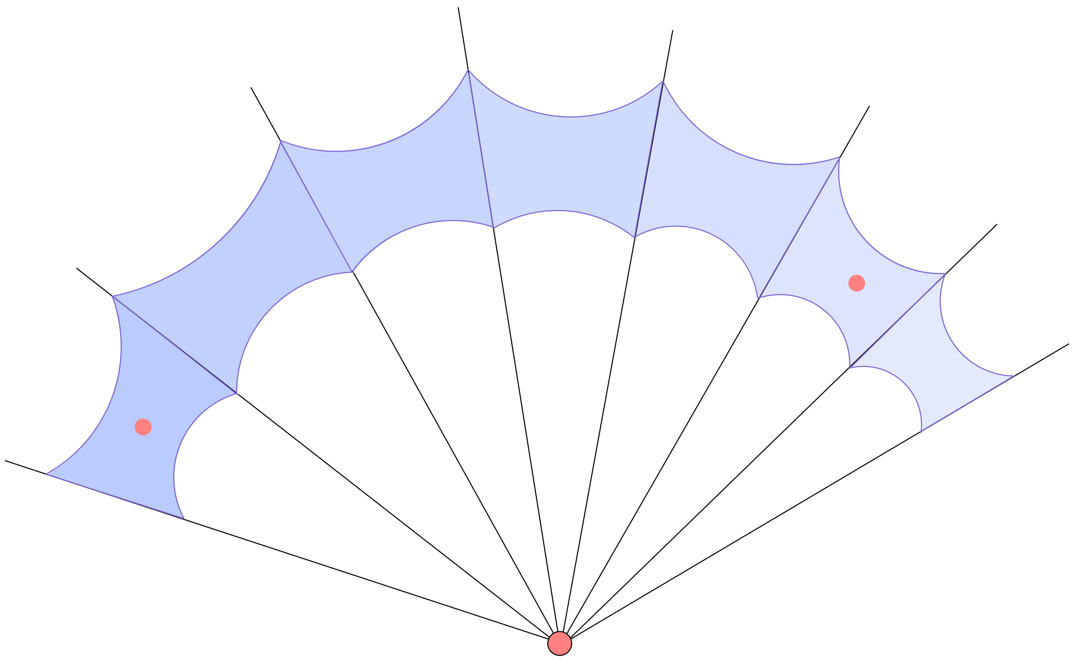

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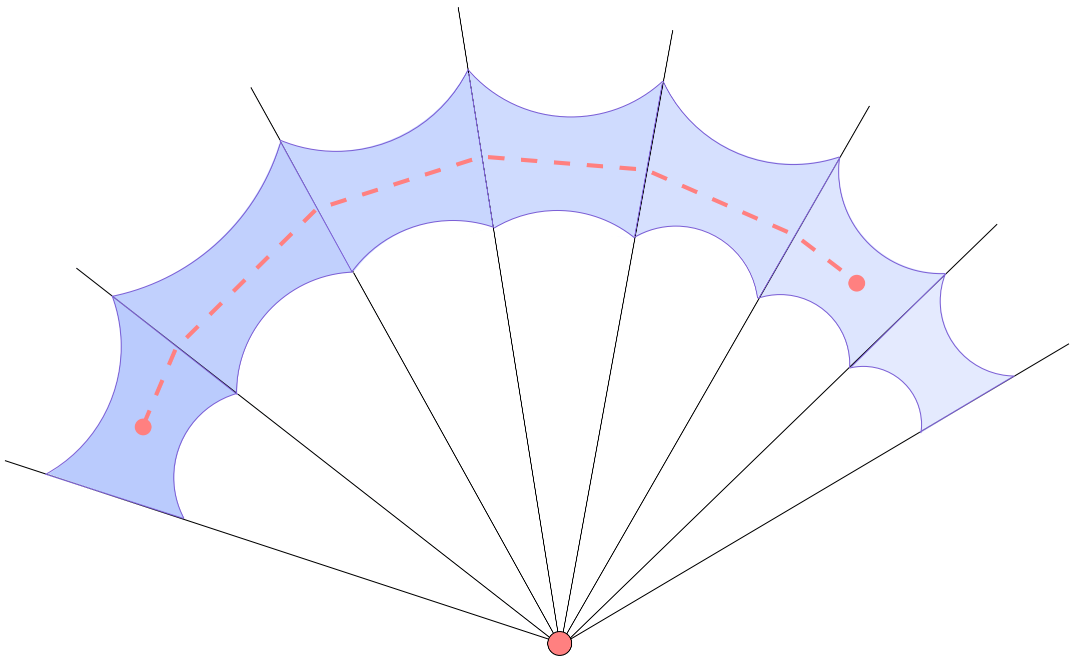

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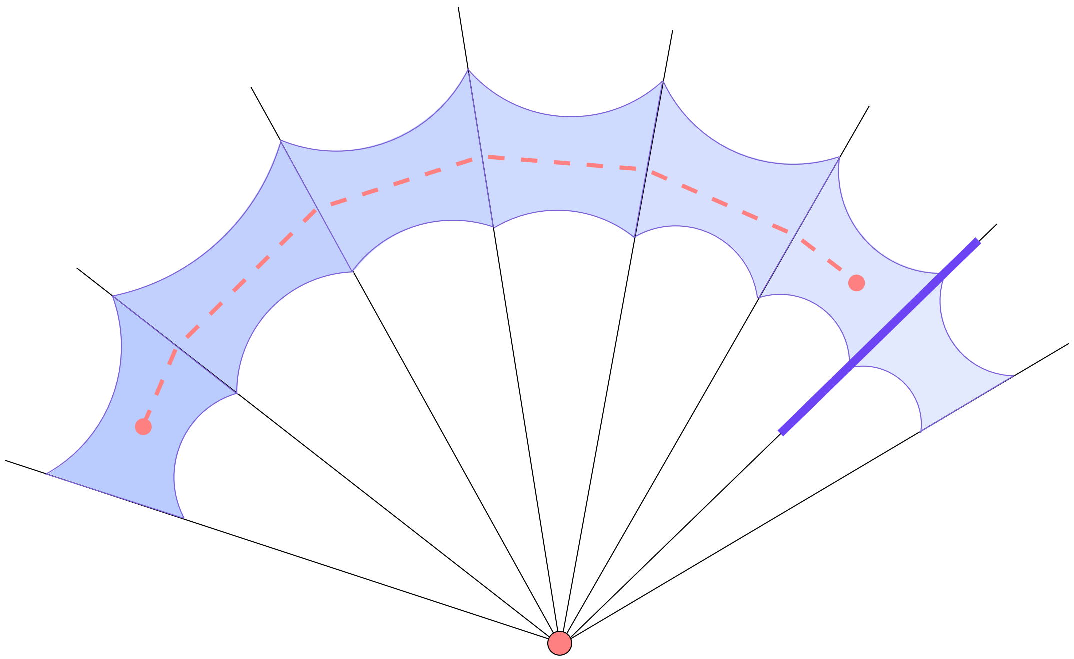

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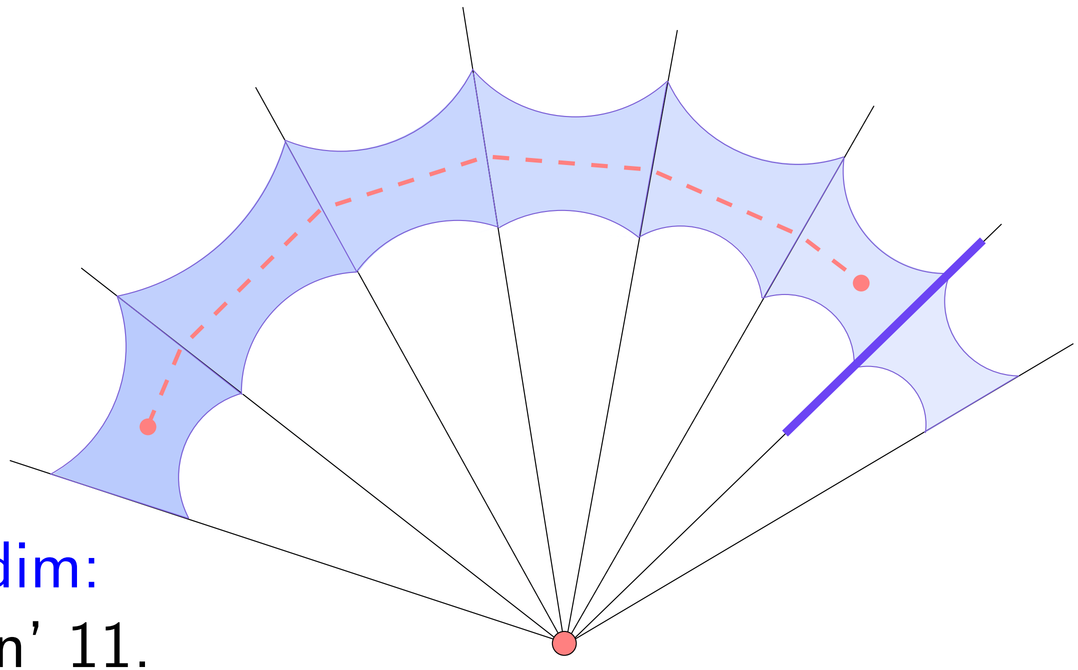

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Jessica Fintzen' 11.

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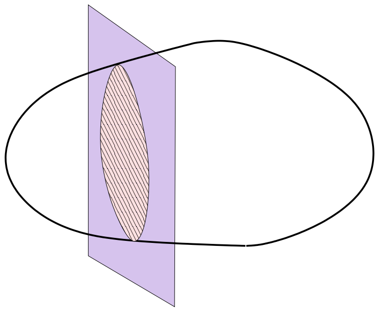
Proof is the same as in the geometric case:

- an induction based on polytopes;
- Deodhar's theorem.

Proof of Deodhar's theorem:

$G = \langle s_0, s_1, \ldots, s_n \rangle$, infinite, indecomposable;
$H = \langle s_1, \ldots, s_n \rangle$. Want: $[G : H] = \infty$.



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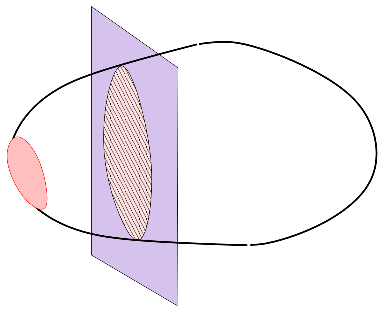

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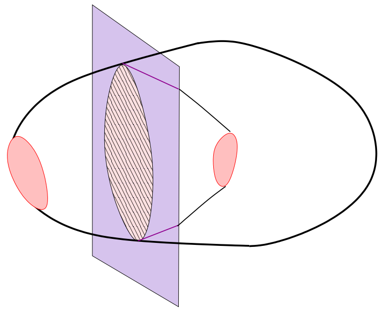

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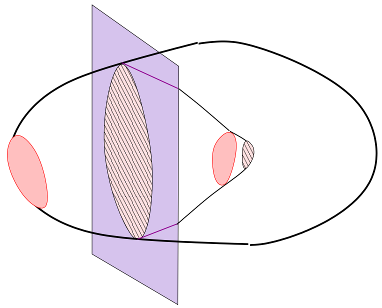

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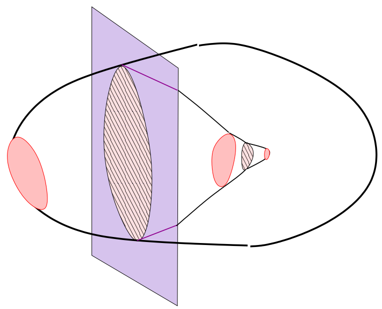

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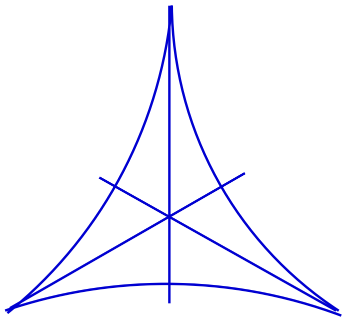

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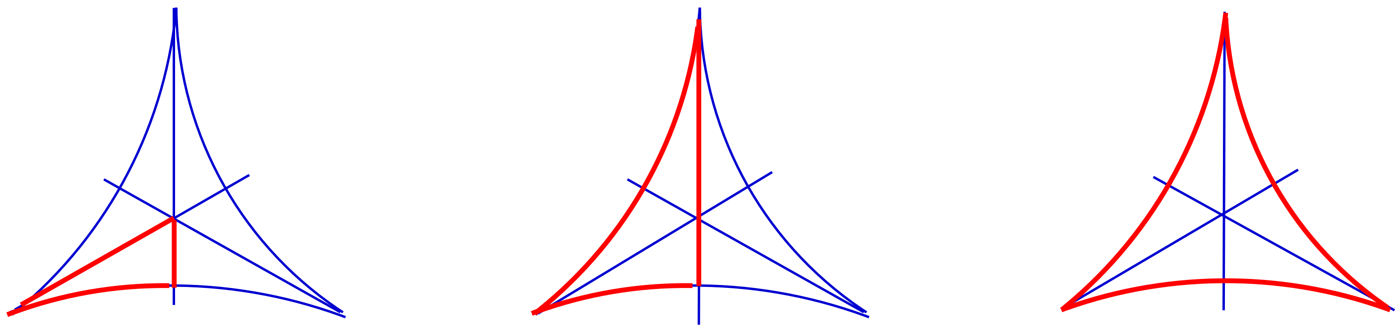

Theorem (F&T, 08). Let (G, S) be a Coxeter system, where G is an infinite indecomposable Coxeter group. If $H \subset G$ is a finite index reflection subgroup then $rank\, H \geq rank\, G$.

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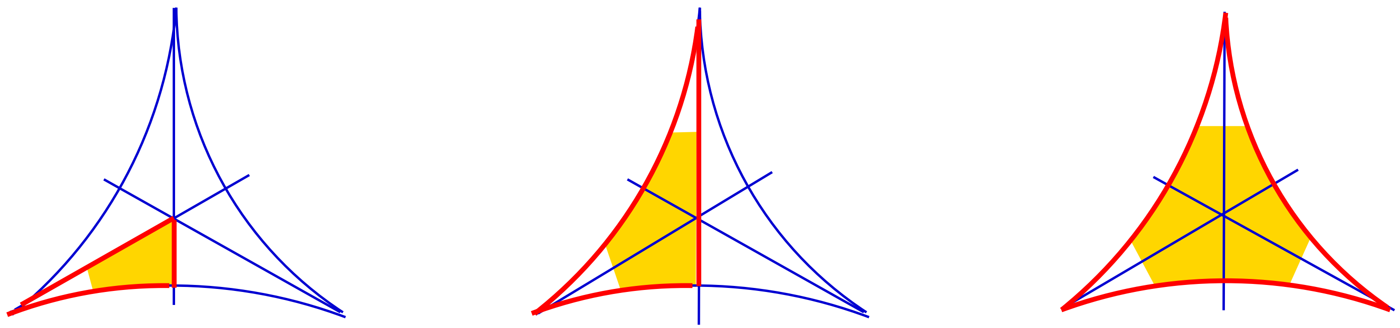

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Thm. If $rank\, H = rank\, G$ then $\exists\ s_0 \in S$ s.t.
either $s_0 s_i = s_i s_0$ for all but one $s_i \in S$;
or the order of $s_0 s_i$ is finite for all $s_i \in S$.

For compact polytopes in $\mathbb{H}^n$:

Thm. If $G : \mathbb{H}^n$ cocompactly, and $rank\, H = rank\, G$, then F is combinatorially equivalent to P.

(F = fundamental chamber of G,
P = fundamental chamber of H)

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Corollary. If $H \subset G$ is a visual subgroup, then $[G : H] = \infty$.

Existance of finite index reflection subgroups ?

Prop. Let $G = G_1 * G_2$, where $G_i \subset G$ is a reflection subgroup.
Then G has a finite index refl. subgr. iff
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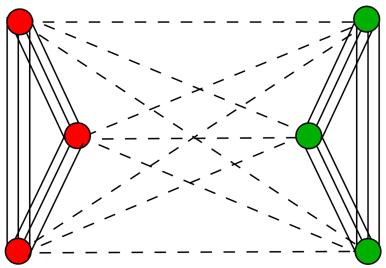

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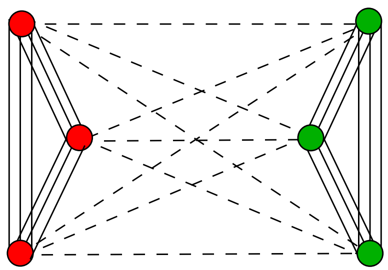

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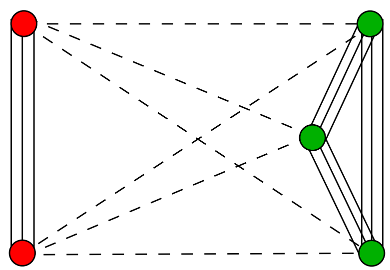

index 10
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Odd-angled groups:

$$G = \langle s_i \in S \mid (s_i s_j)^{m_{ij}} \rangle, \qquad \text{where } \; m_{ij} \notin 2\mathbb{Z} \quad \text{for all} \quad i, j.$$

Why to study odd-angled gps?

Example: tilings of triangles by hyperbolic Coxeter triangles

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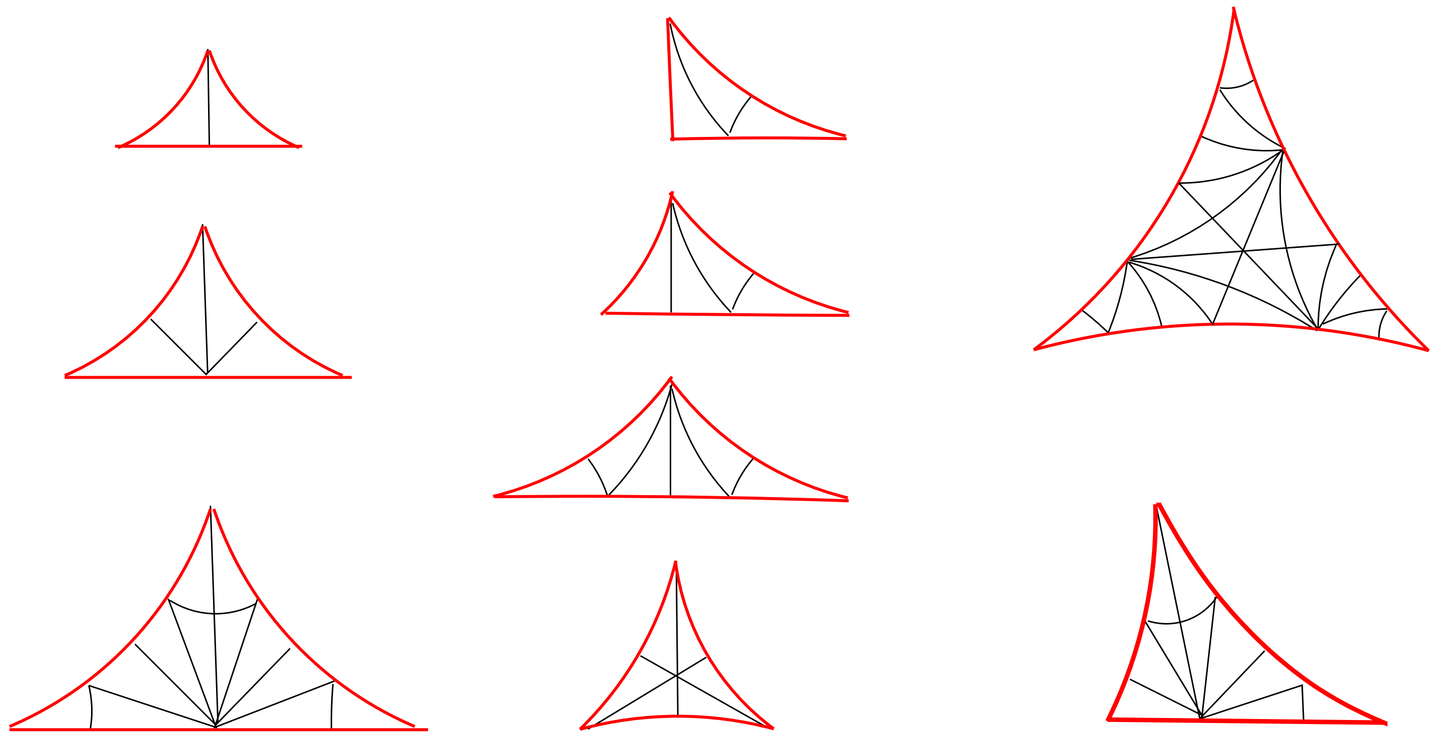

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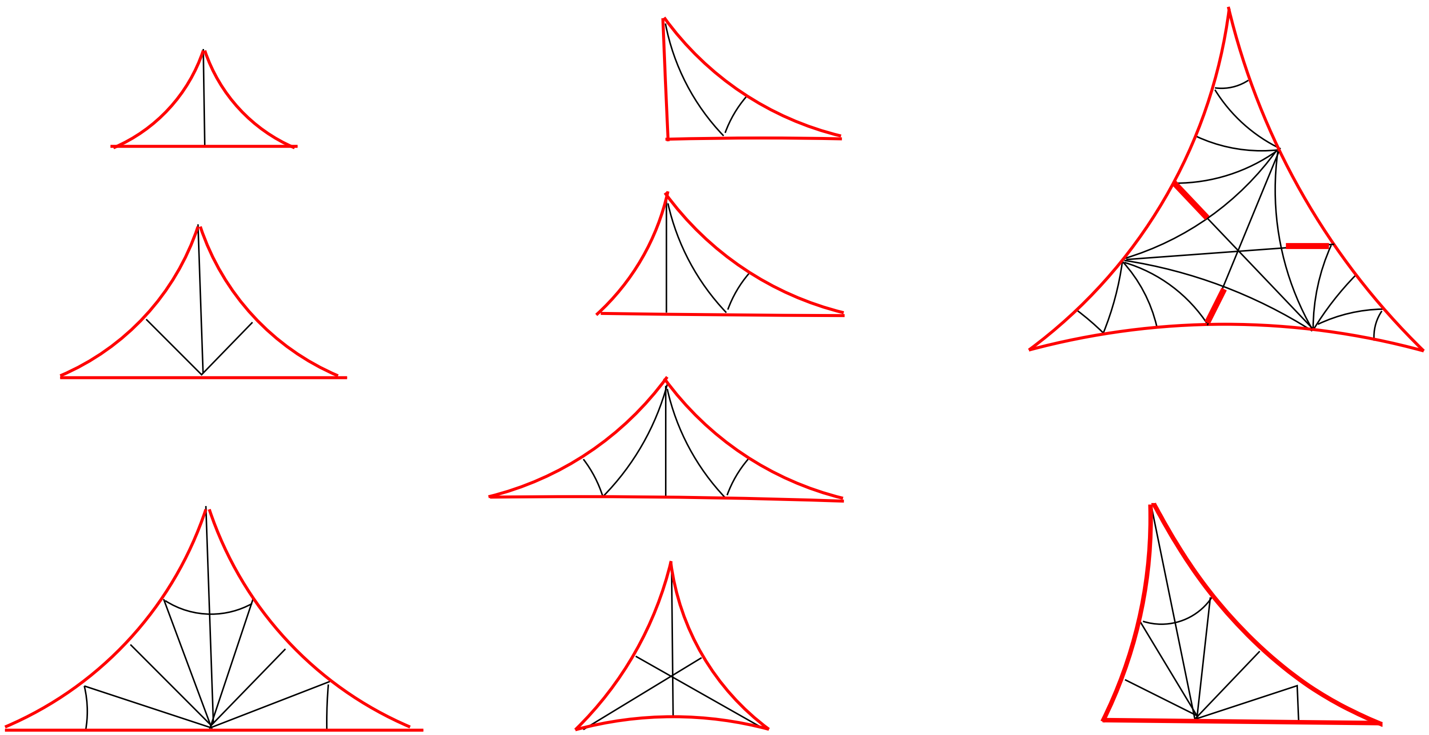

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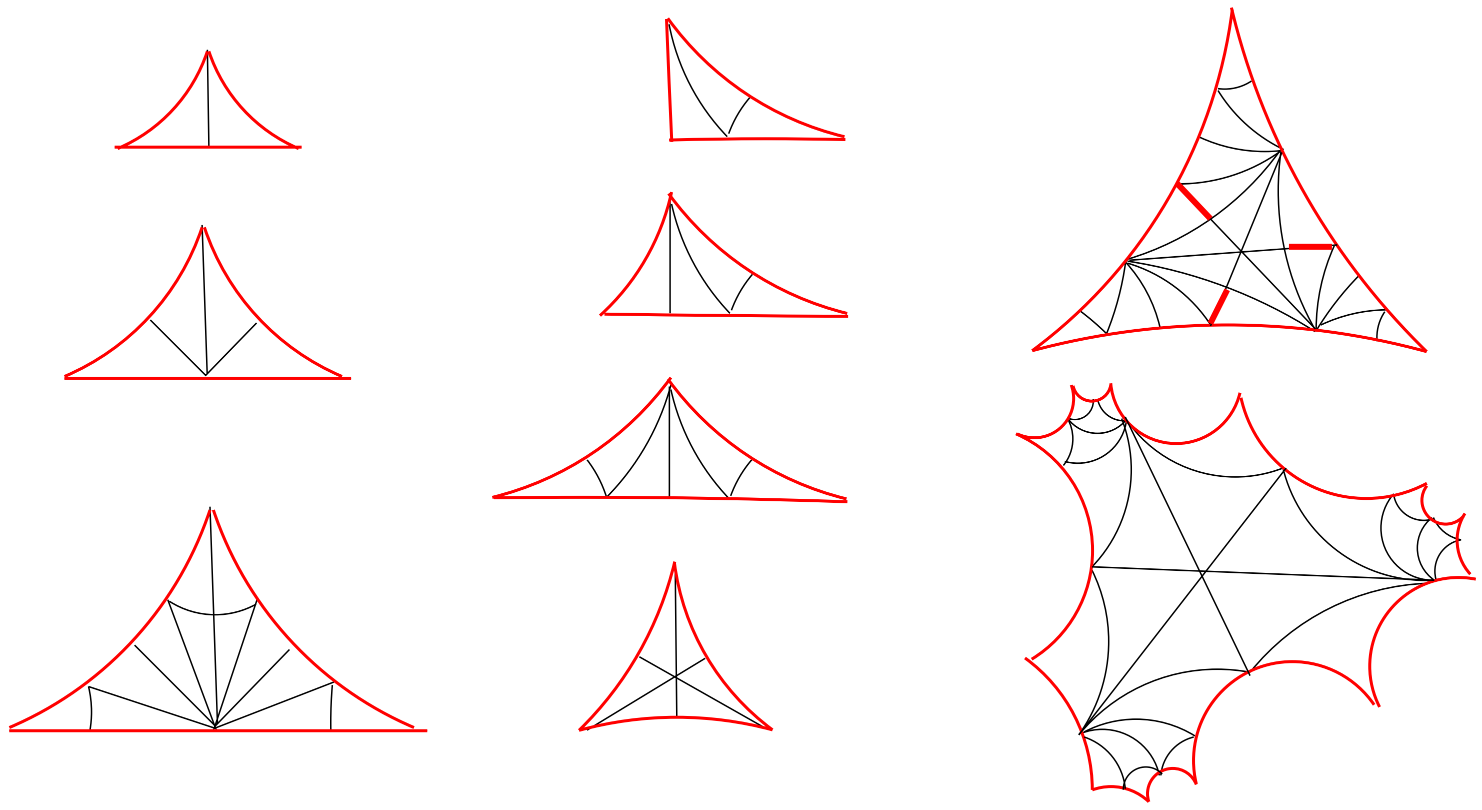

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How to study odd-angled gps?

Divisability Coxeter diagram $Cox_{div}(G)$:

- $\bullet\!\overset{k_{ij}}{\rule[0.5ex]{2em}{0.4pt}}\!\bullet$, where k_{ij} is a minimal nontrivial divisor of m_{ij}.
- no edge for $m_{ij} = \infty$.

Thm. (with P.T. and Jessica Fintzen, 2012)
An odd-angled Coxeter group G contains a finite index reflection subgroup iff $Cox_{div}(G)$ contains at least one connected component C of one of the three types:

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An odd-angled Coxeter group G contains a finite index reflection subgroup iff $Cox_{div}(G)$ contains at least one connected component C of one of the three types:

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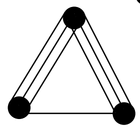
2. C contains at most one non-simple edge;
3. C contains a subdiagram $D =$ 

   and all other edges of C are simple.

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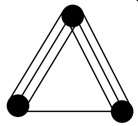
3. C contains a subdiagram $D =$ 

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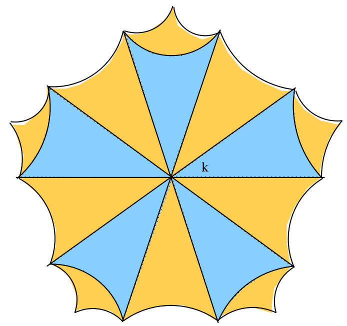
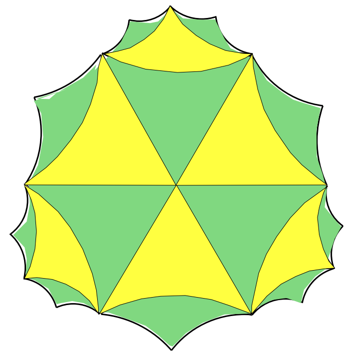
Rotation subgp





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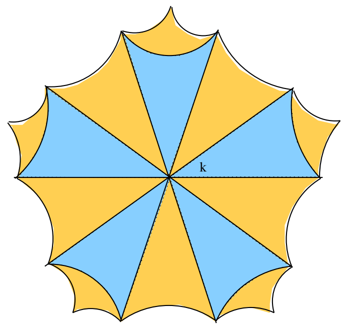

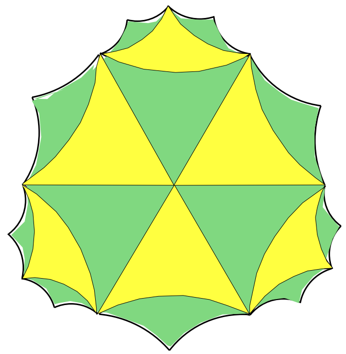

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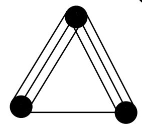
3. C contains a subdiagram $D =$ 

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THANKS !