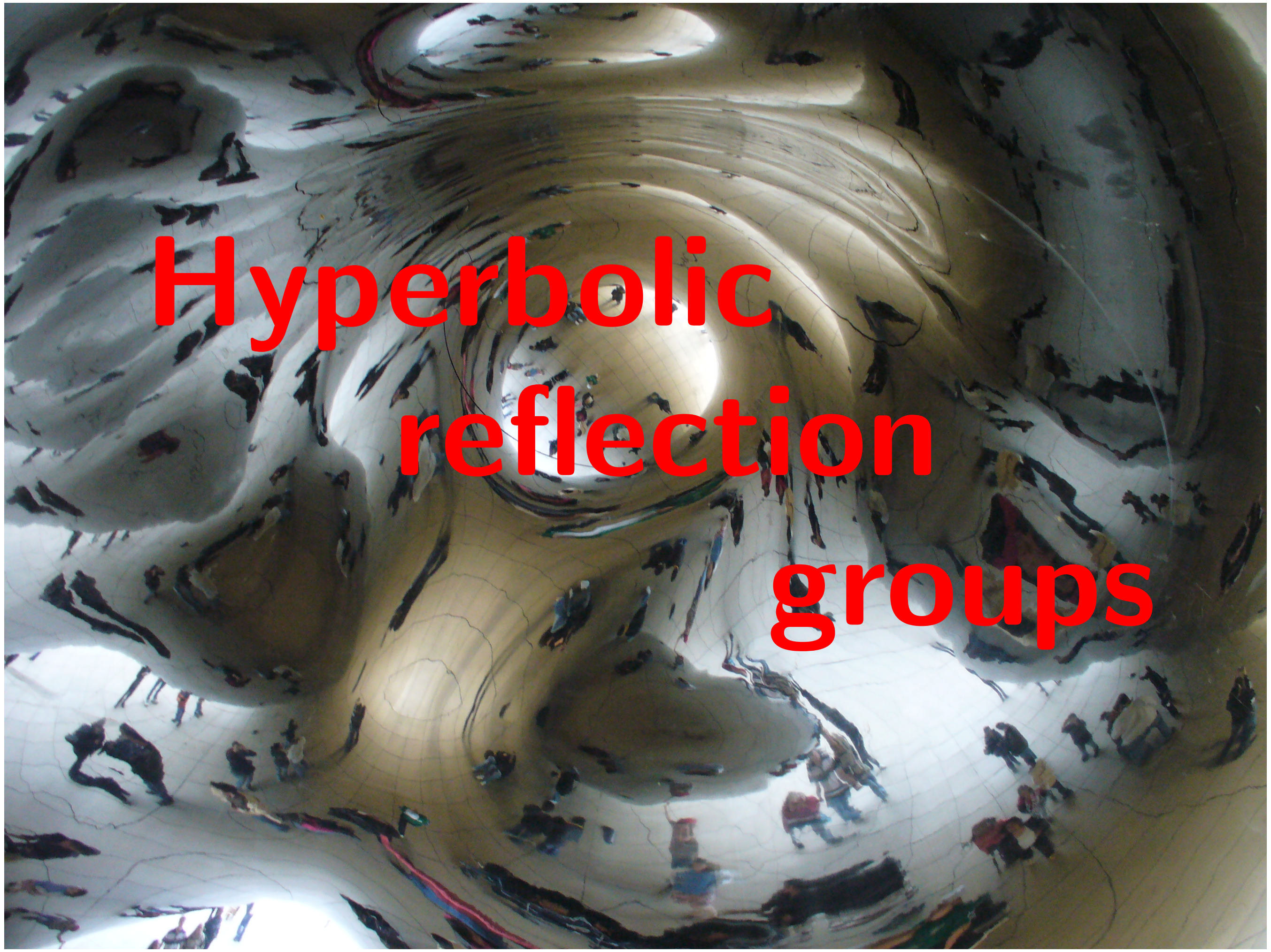
Hyperbolic
reflection
groups

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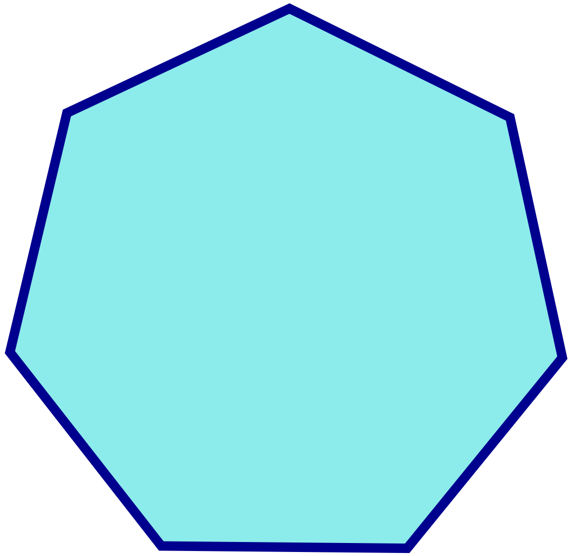

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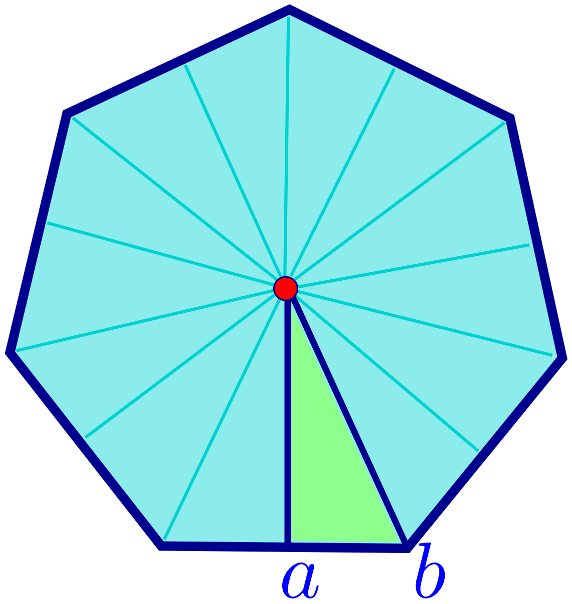

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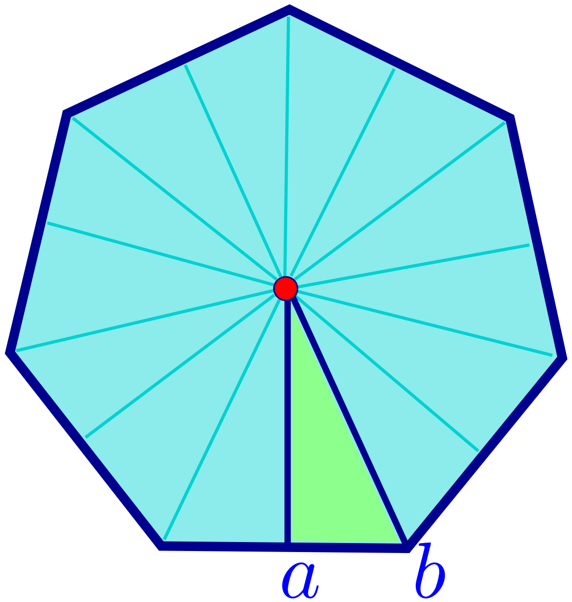

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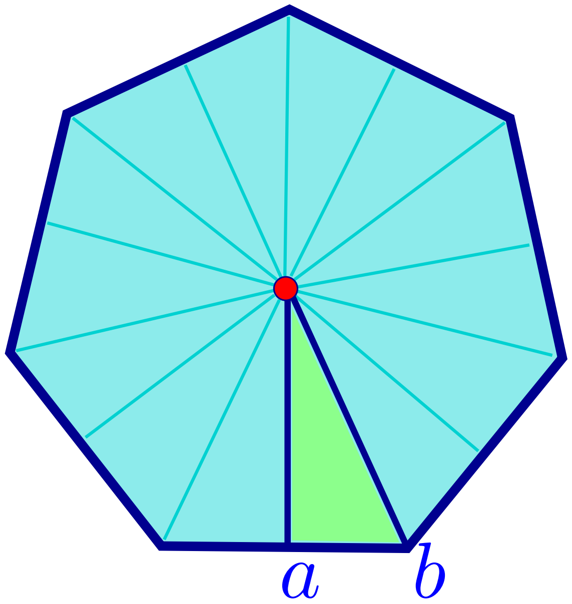

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- generated by reflections r_a and r_b
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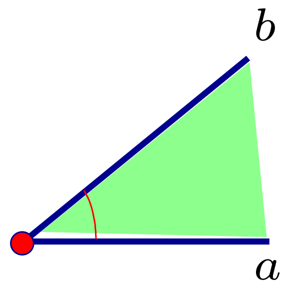

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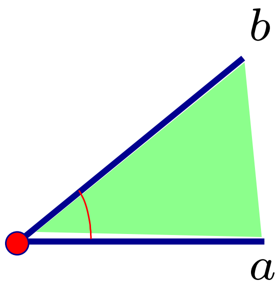

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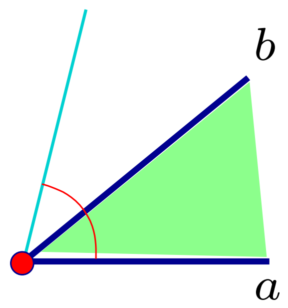

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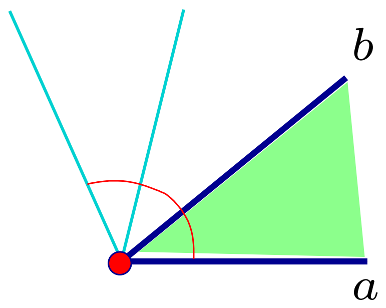

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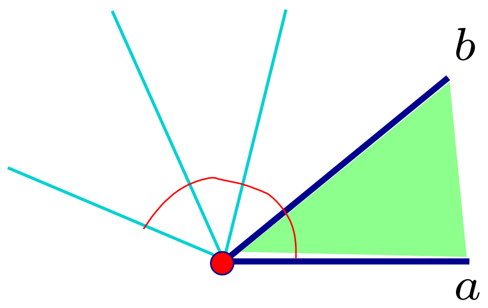

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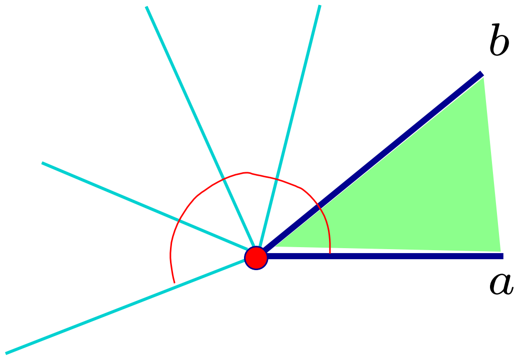

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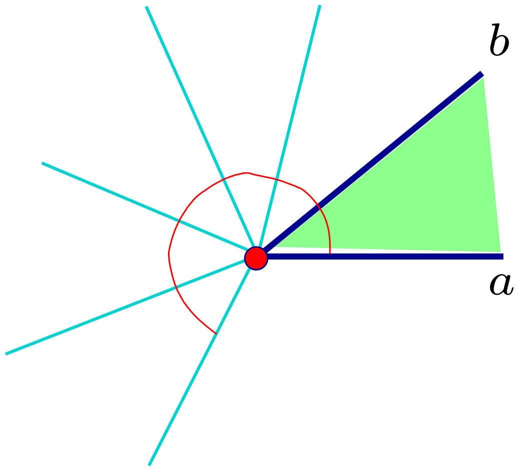

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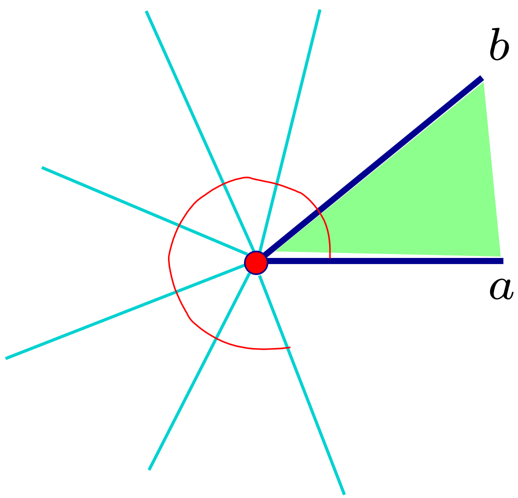

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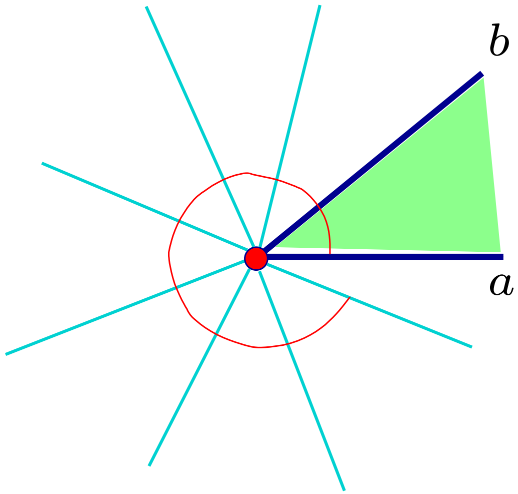

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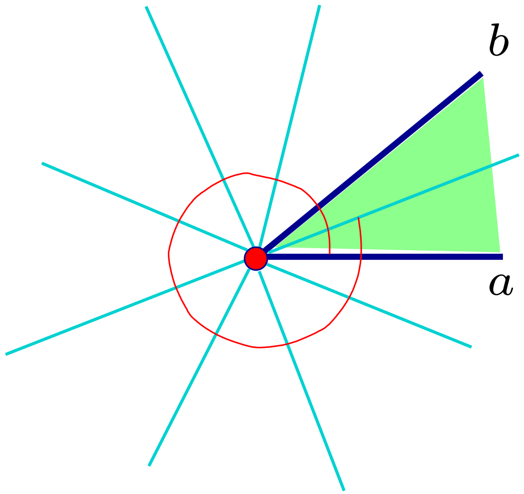

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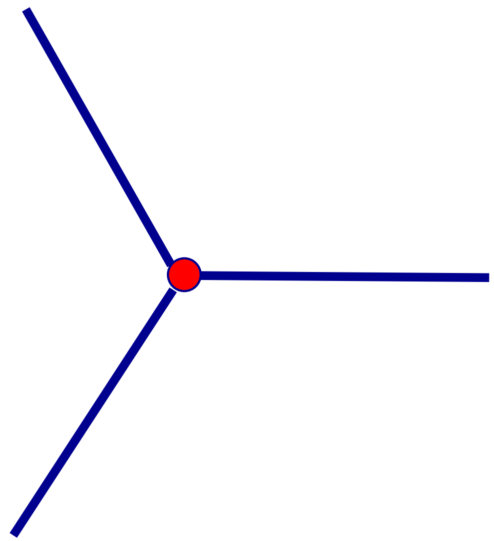

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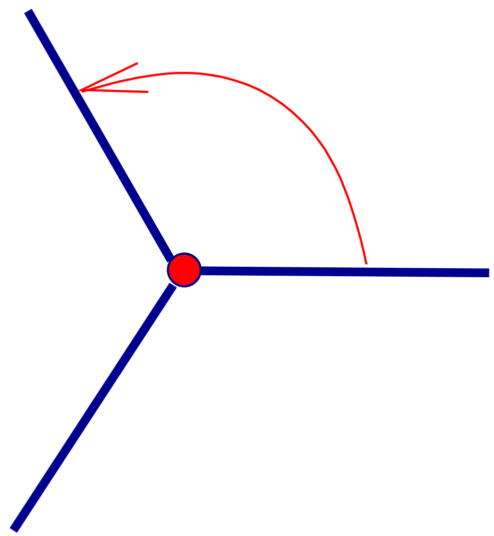

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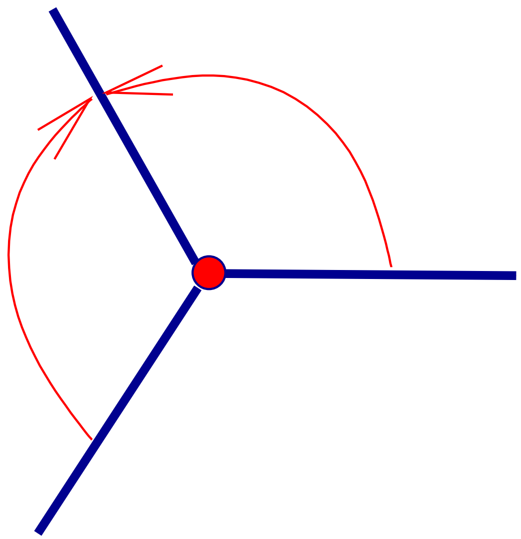

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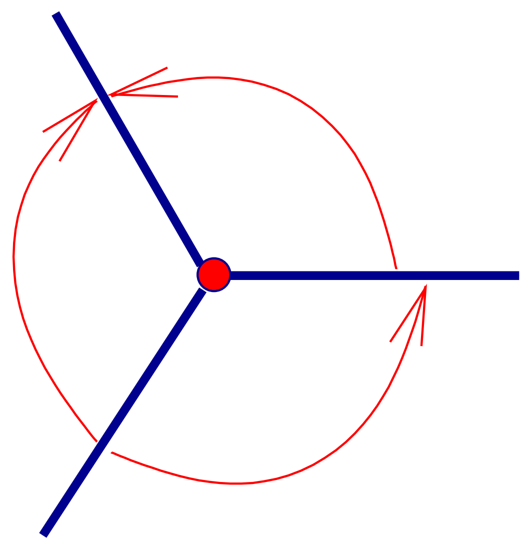

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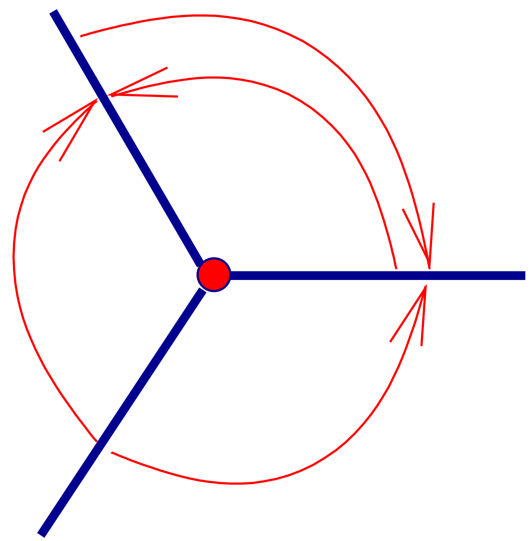

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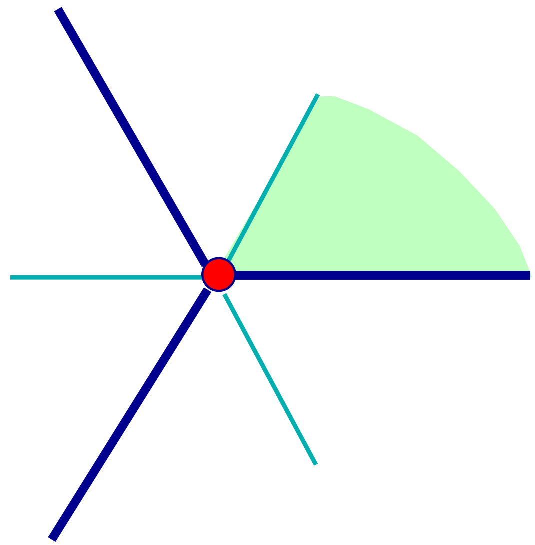
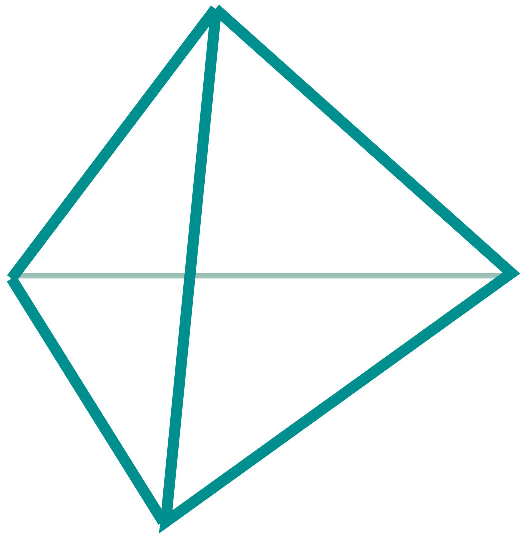


Example: regular tetrahedron T



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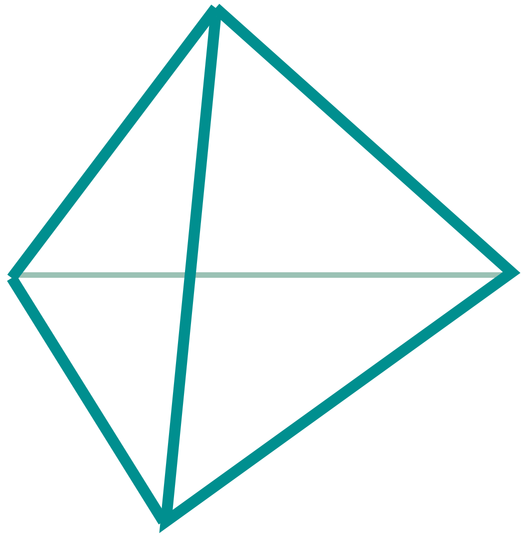
group of symmetries $Sym(T)$?



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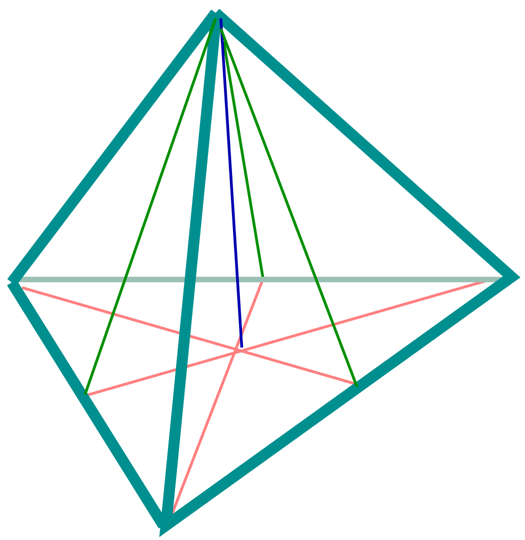

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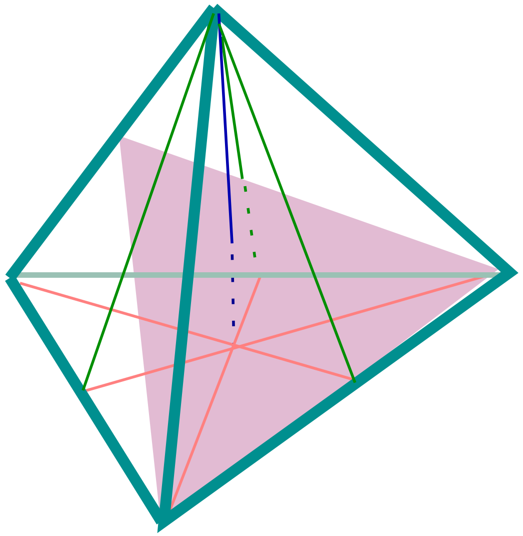

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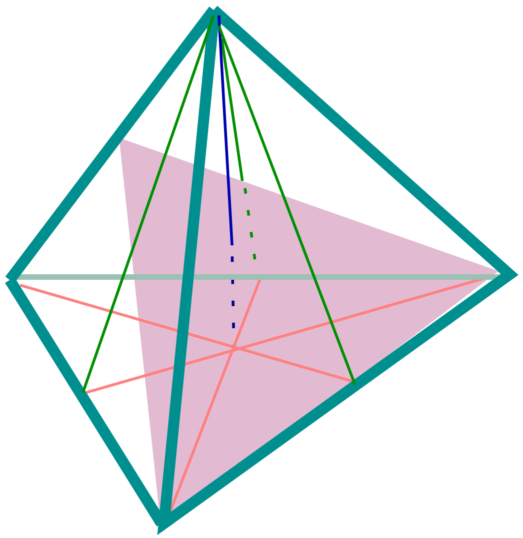

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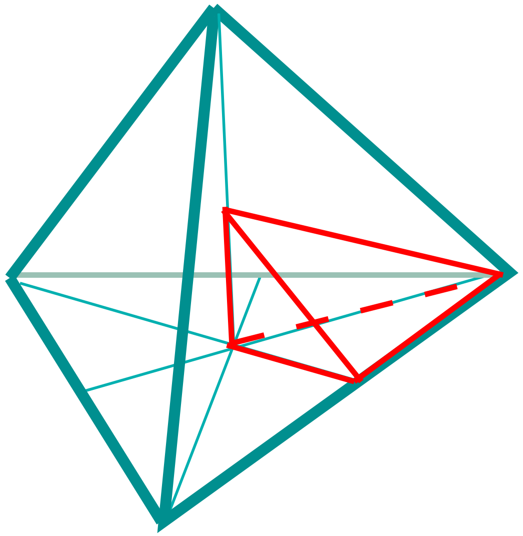

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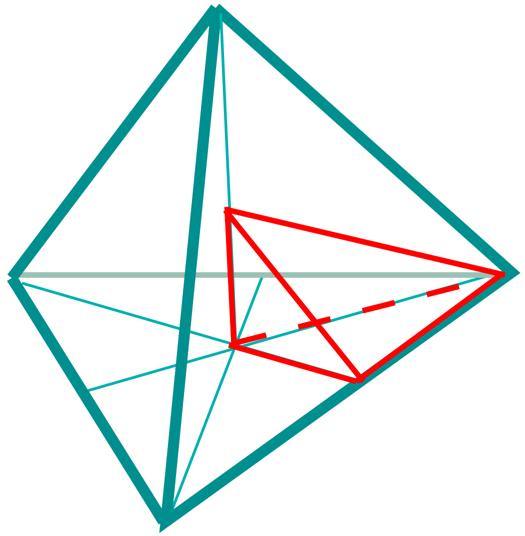

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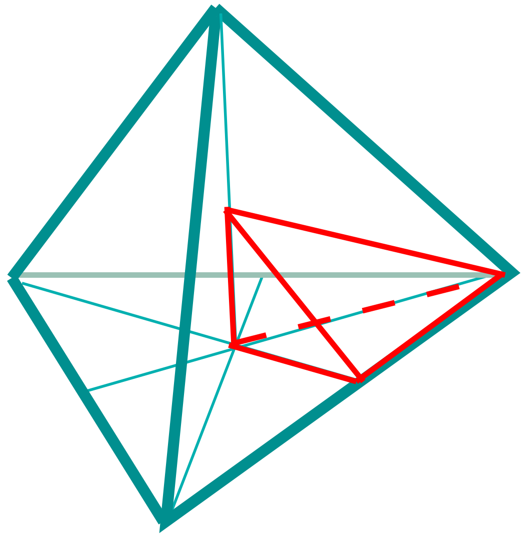

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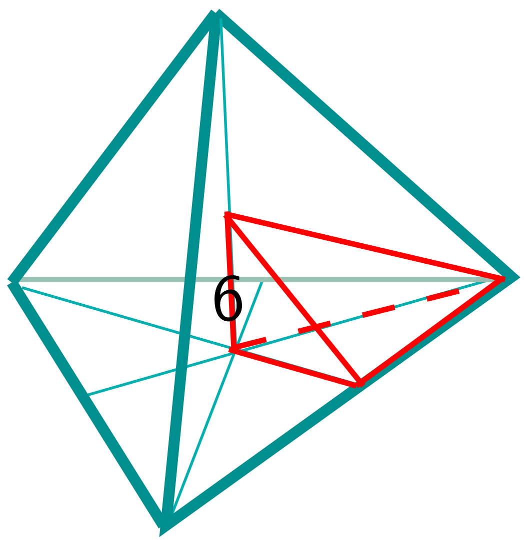

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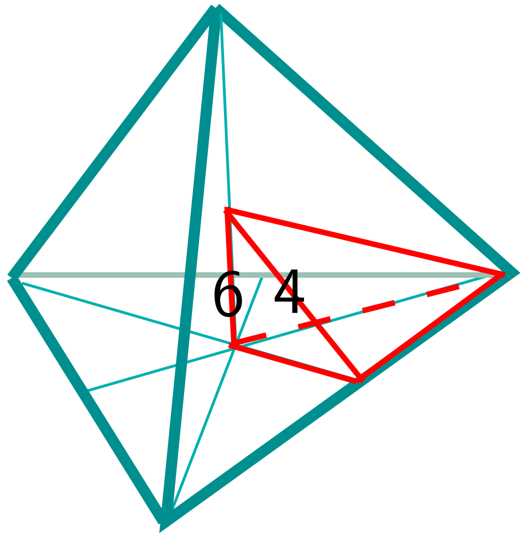

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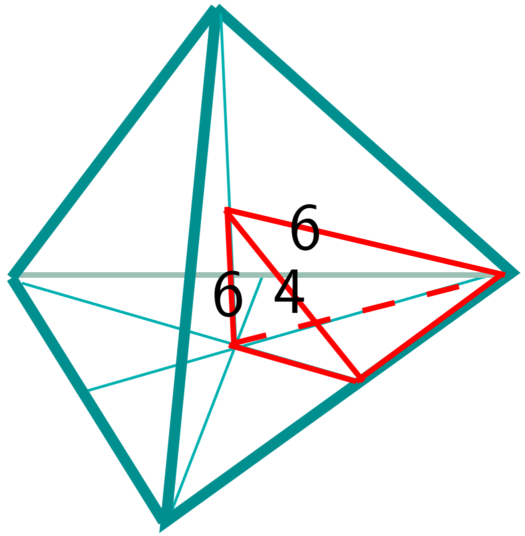

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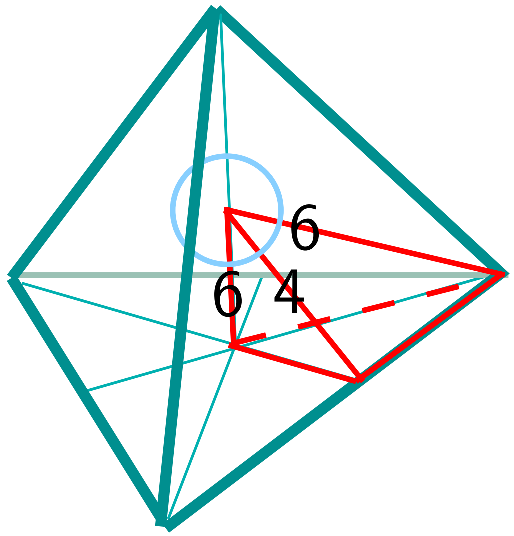

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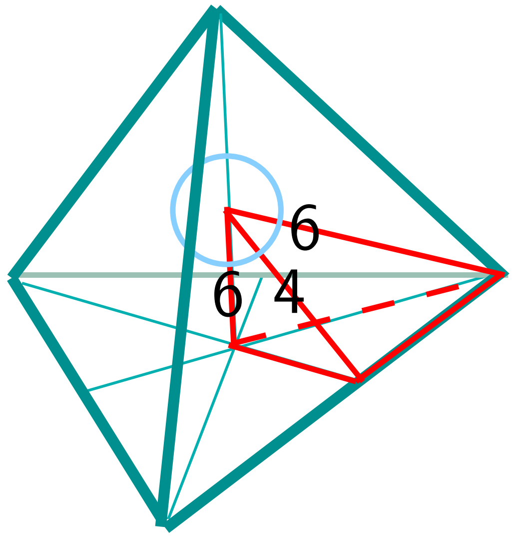

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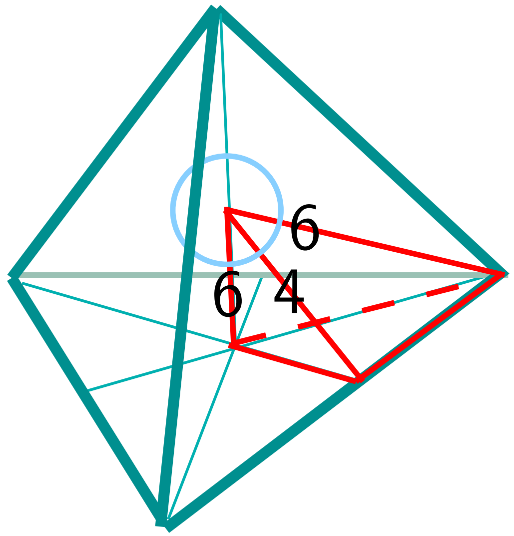

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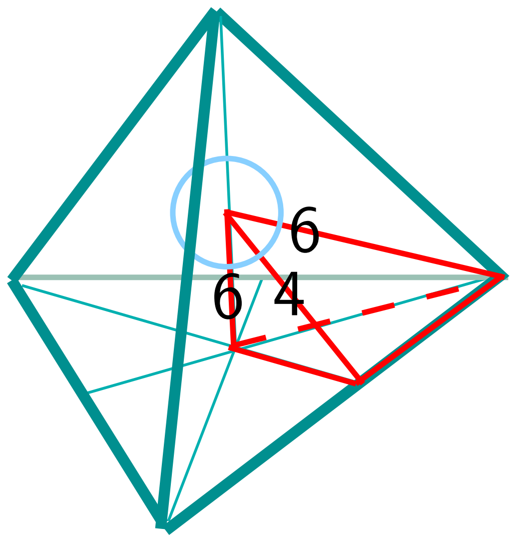

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- $Sym(T) = S_4$

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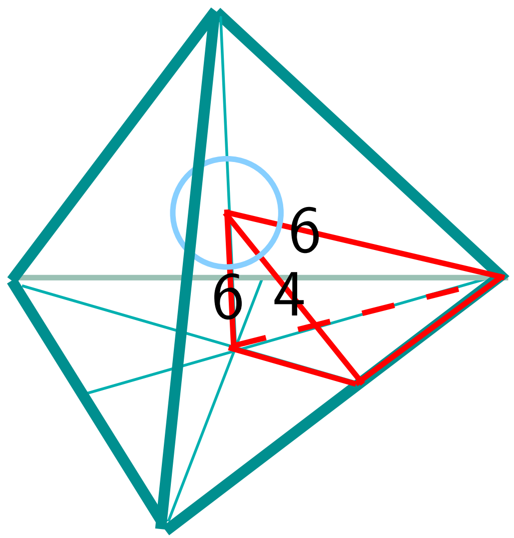

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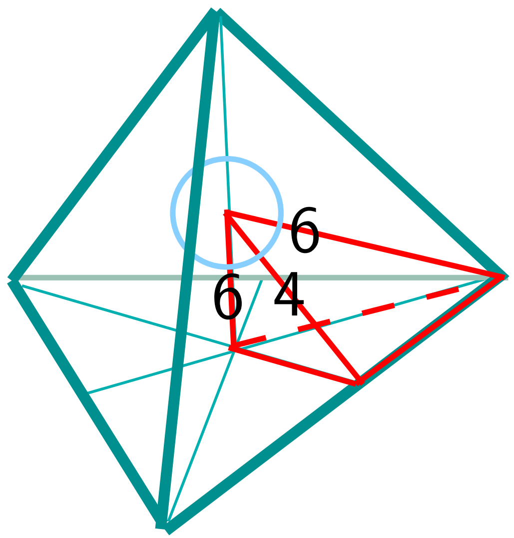

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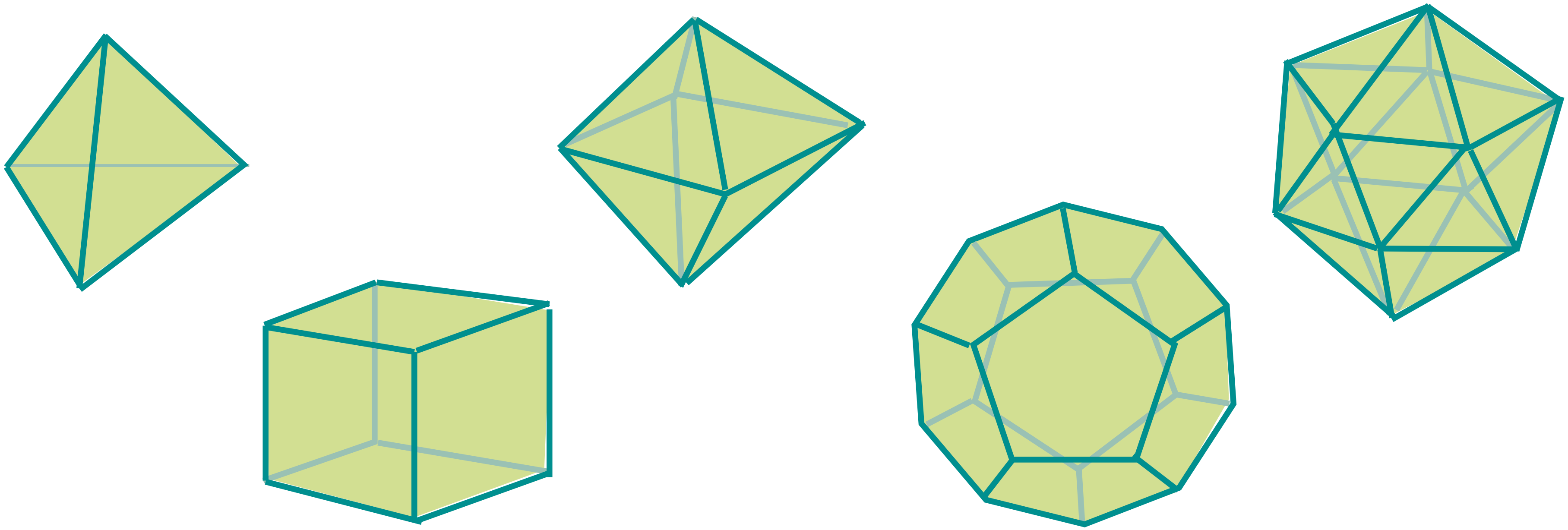
- S_4 is gen. by transpositions $\leftrightarrow$ $Sym(T)$ is gen. by reflections

Example: regular 3-polytopes



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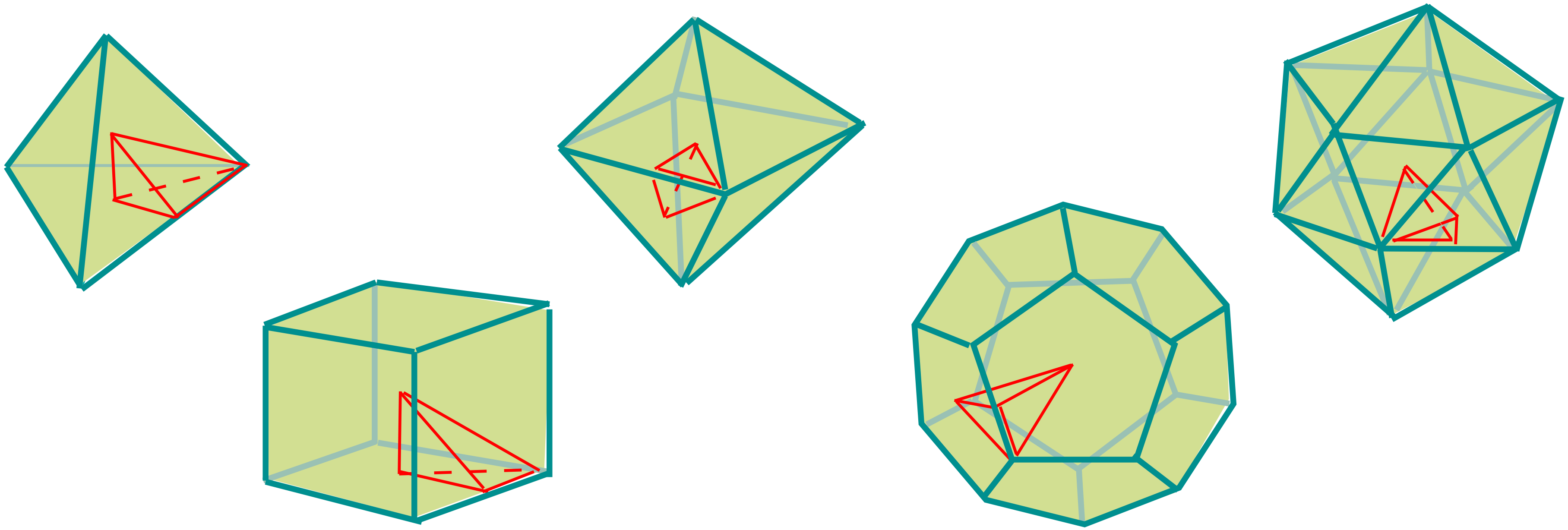

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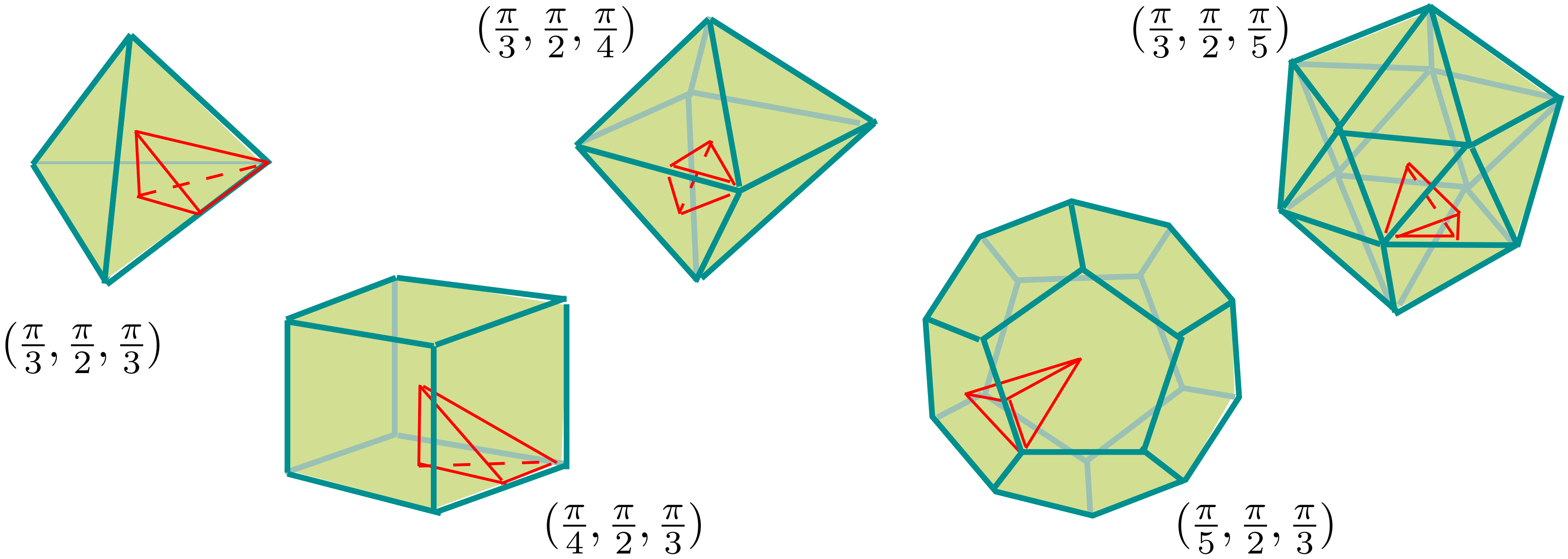

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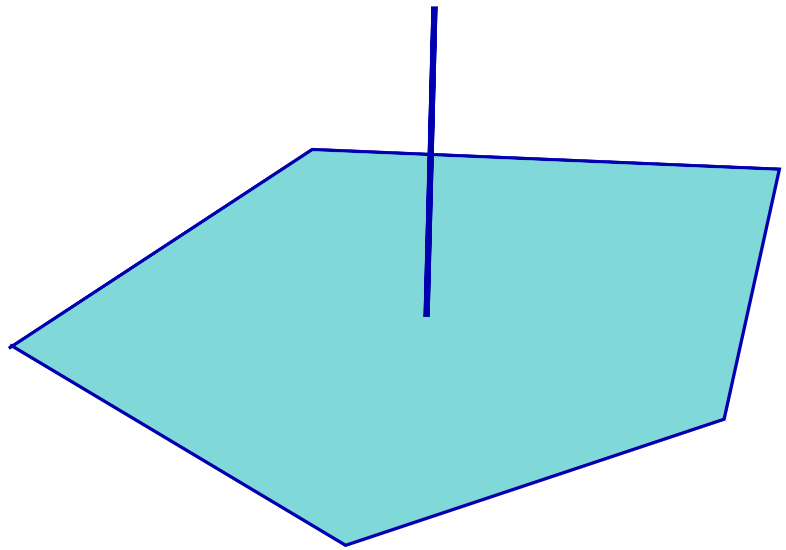

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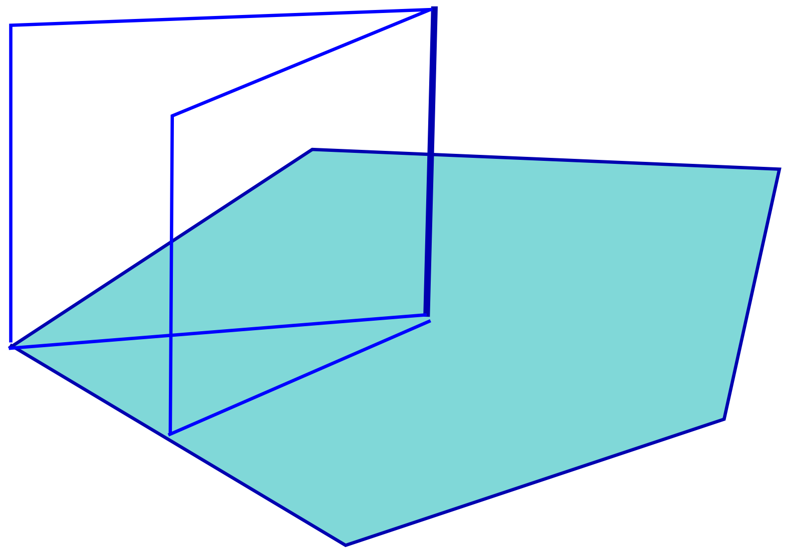

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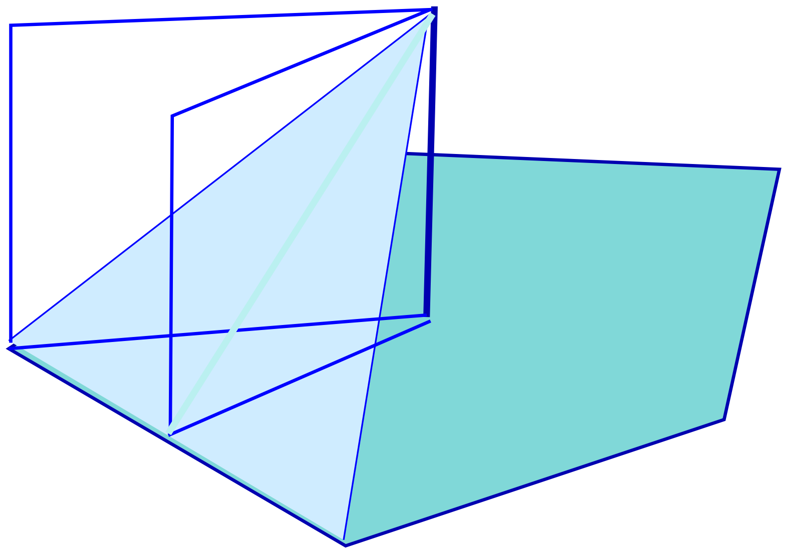

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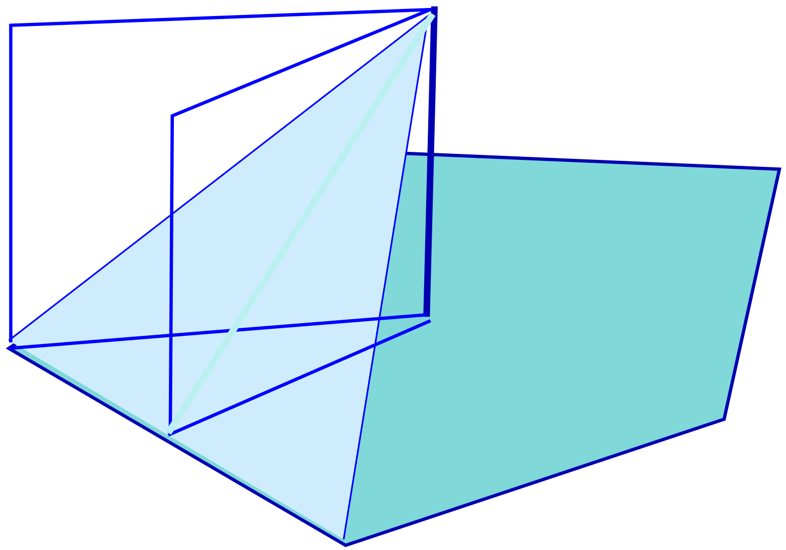

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- Chambers have good dihedral angles (otherwise G is not discrete).

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  - $G_P = \langle r_{f_i},\ f_i \in \{\text{facets of } P\} \ \mid \ r_{f_i}^2 = e,\ (r_{f_i} r_{f_j})^{m_{ij}} = e \rangle$

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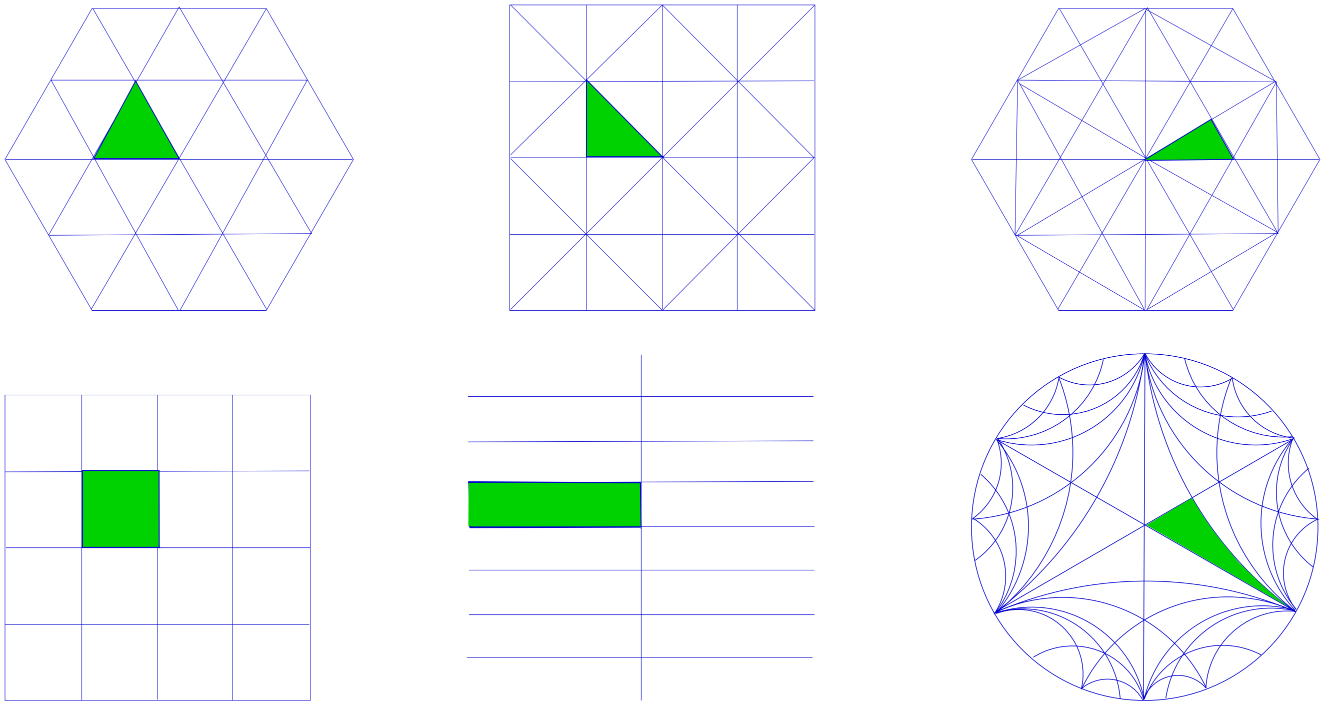
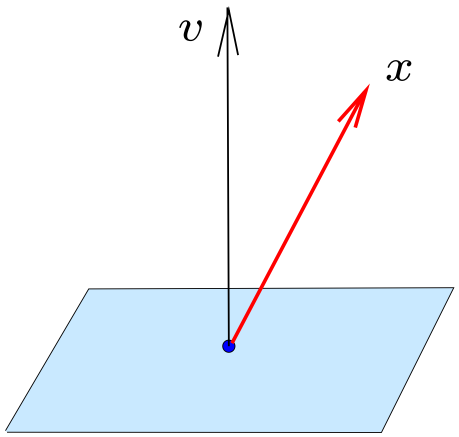
  i.e. G_P is a Coxeter group.

Examples of infinite groups:



- we are mainly interested in finite volume groups, where $vol(\mathbb{X}/G) < \infty$.

How to describe a reflection?



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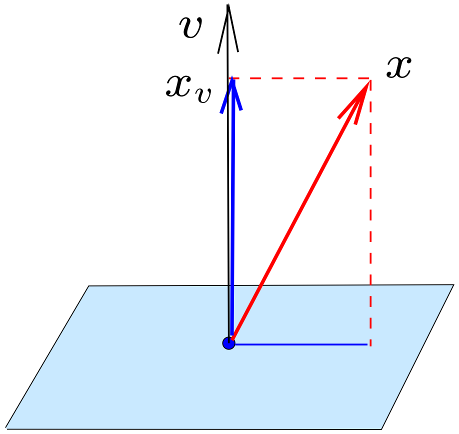

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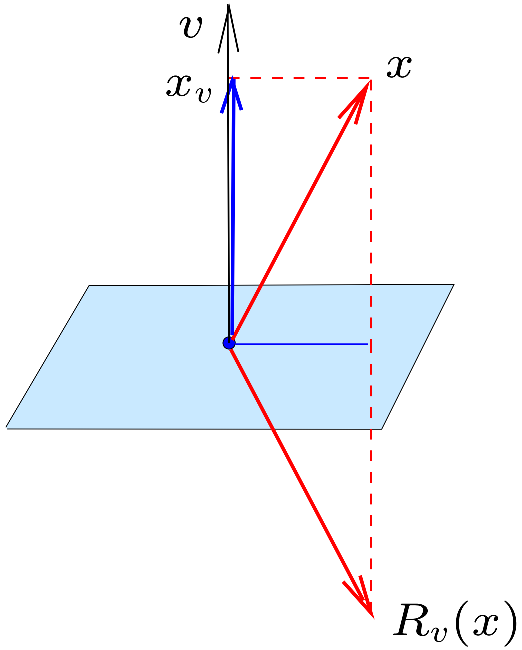

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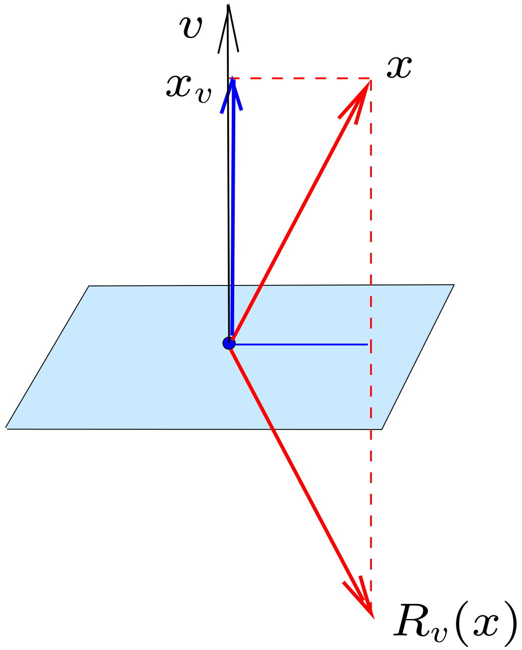

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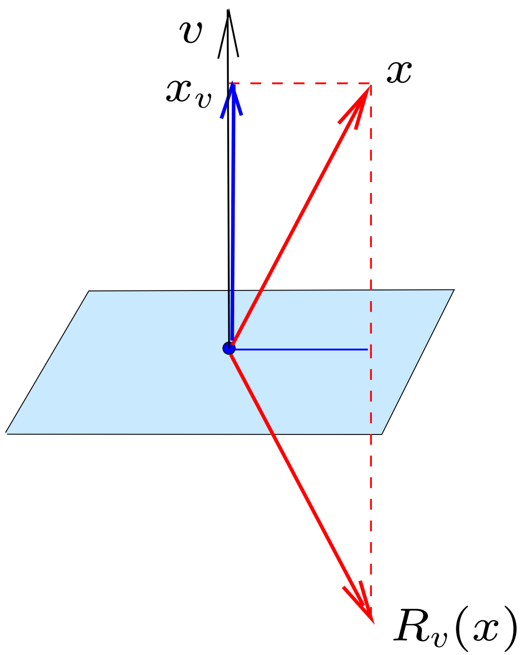

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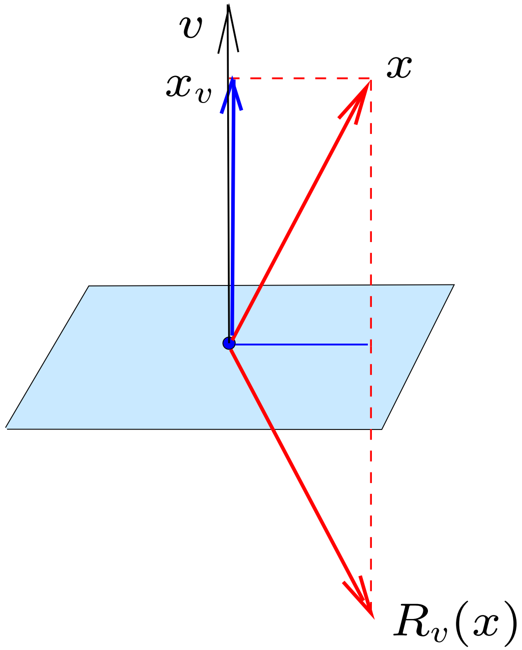

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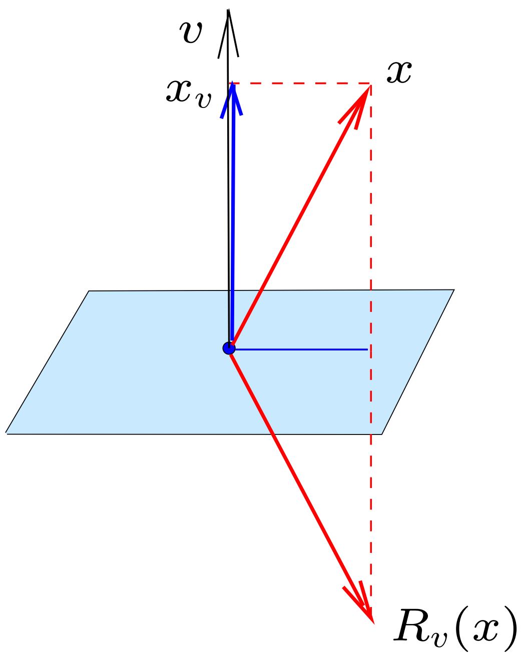

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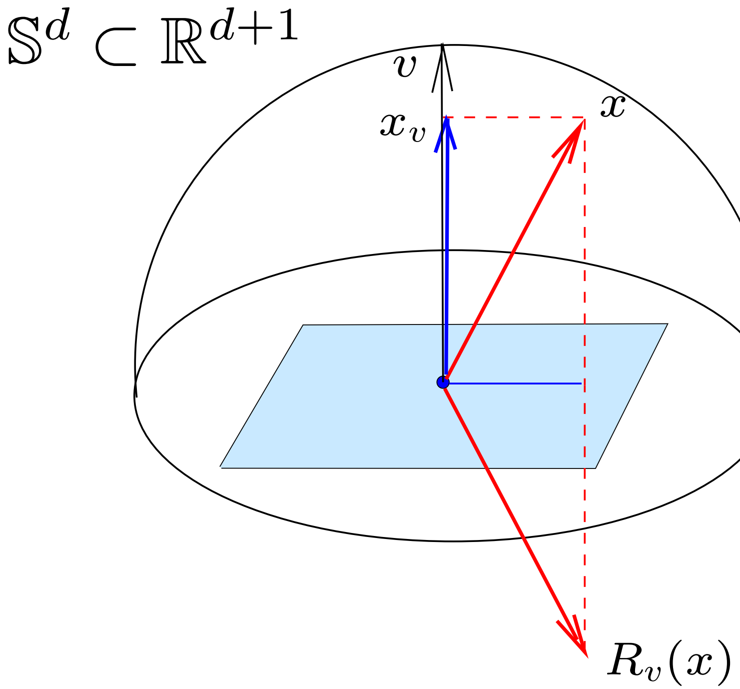

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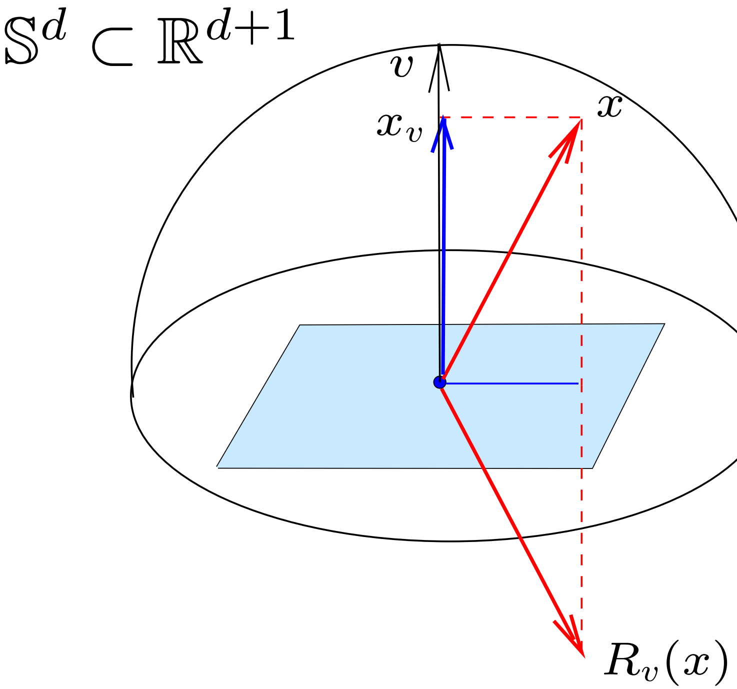

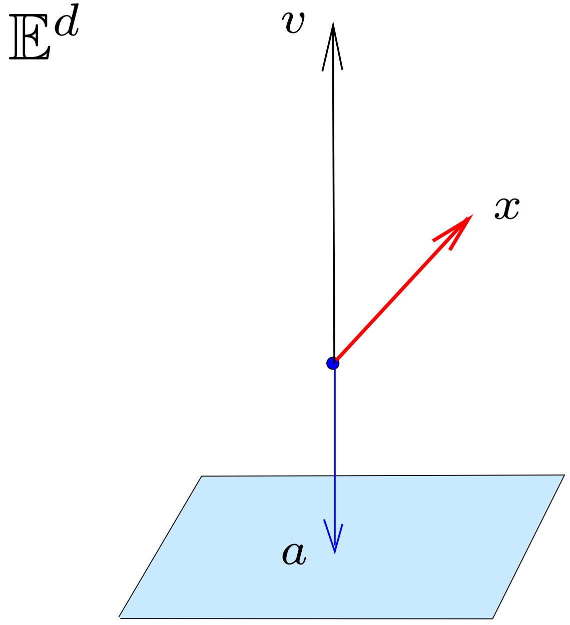

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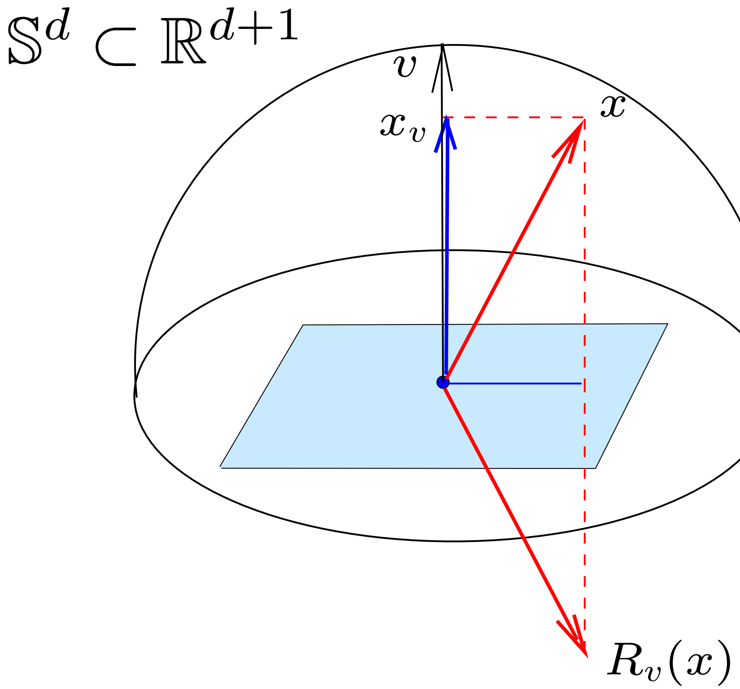

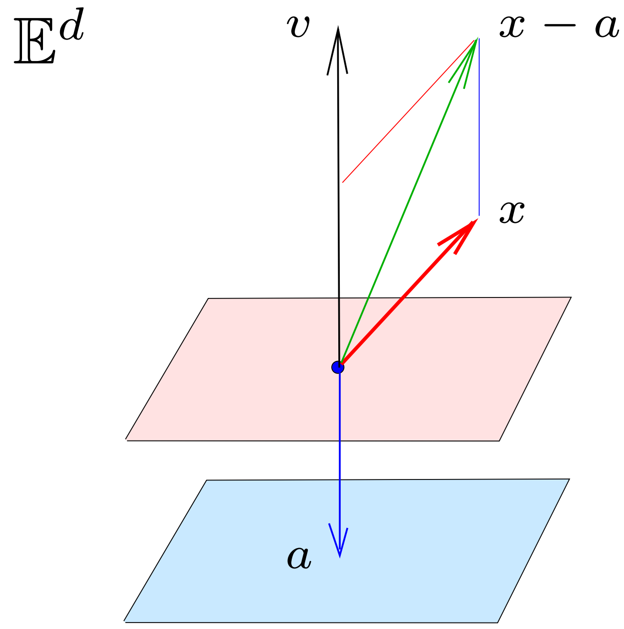

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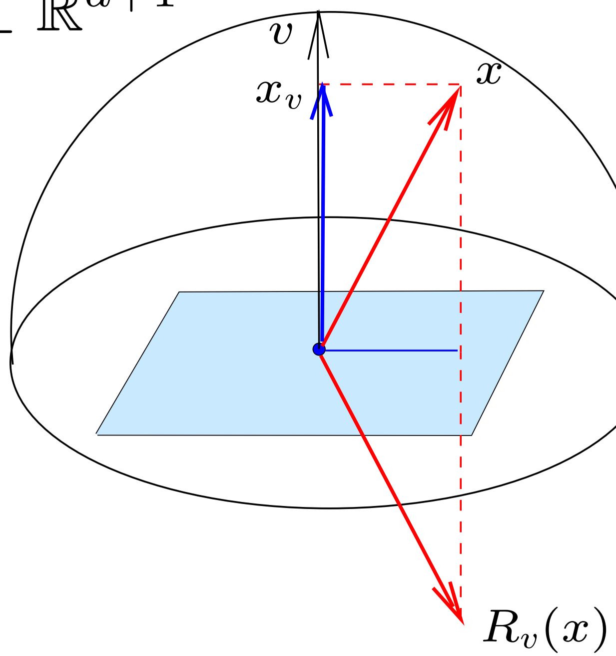
$\mathbb{S}^d \subset \mathbb{R}^{d+1}$



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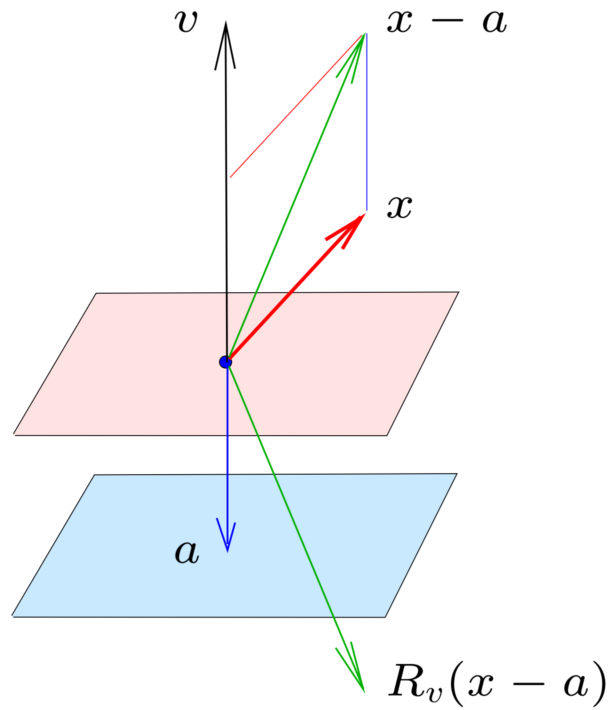

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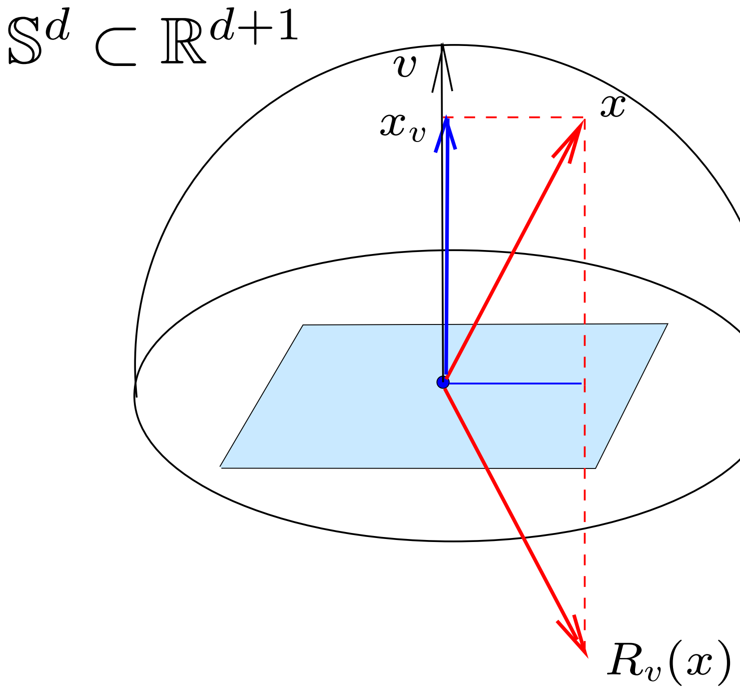

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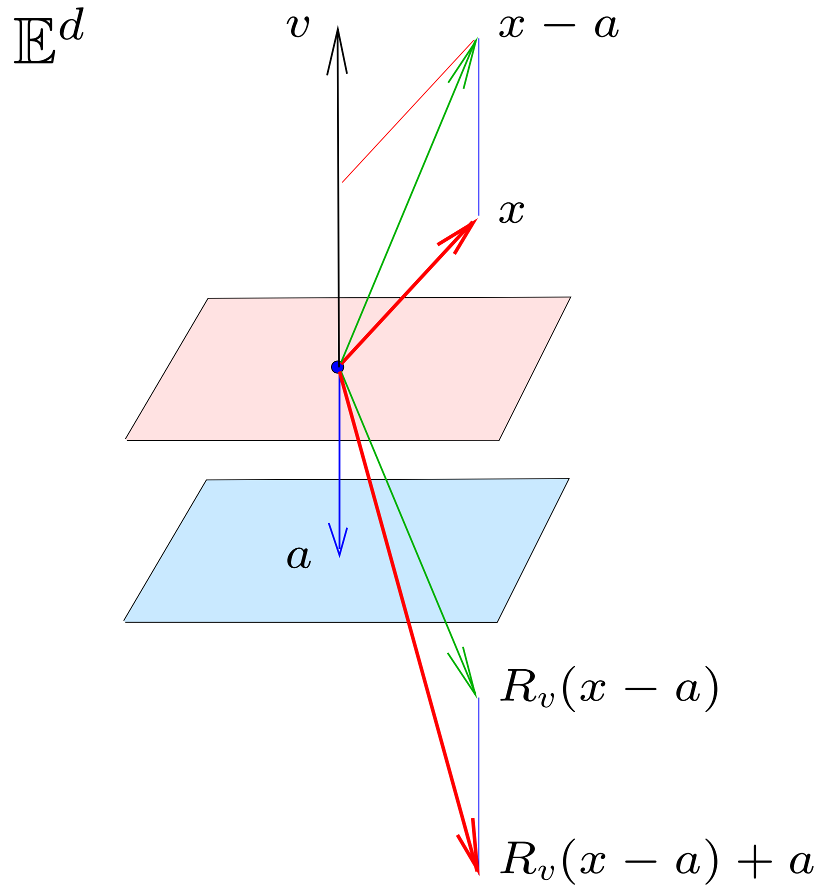

Linear model of $\mathbb{H}^d$

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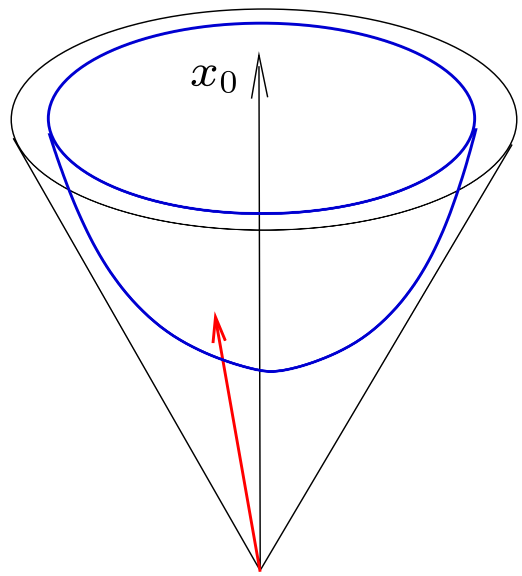

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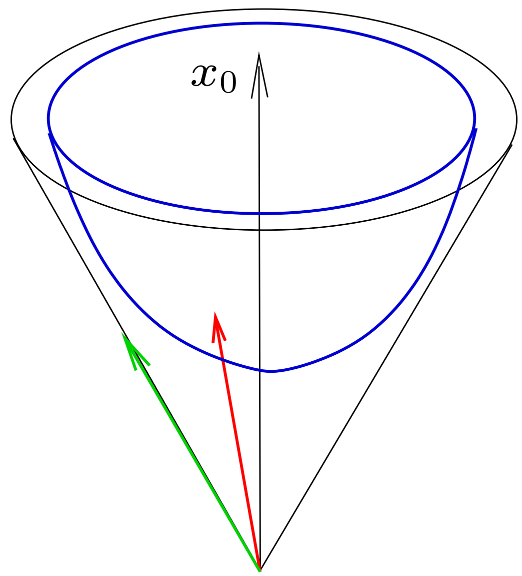

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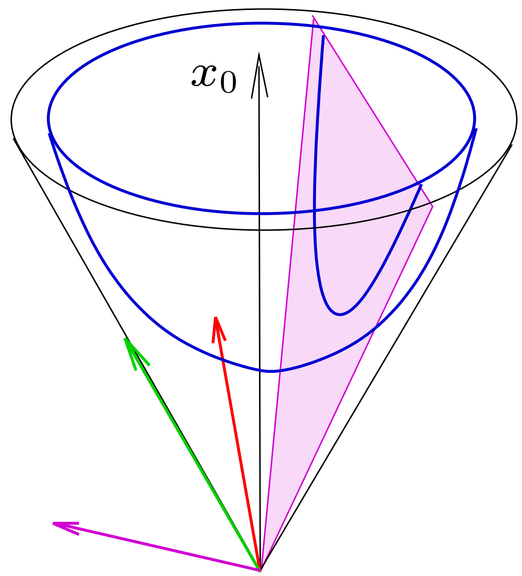

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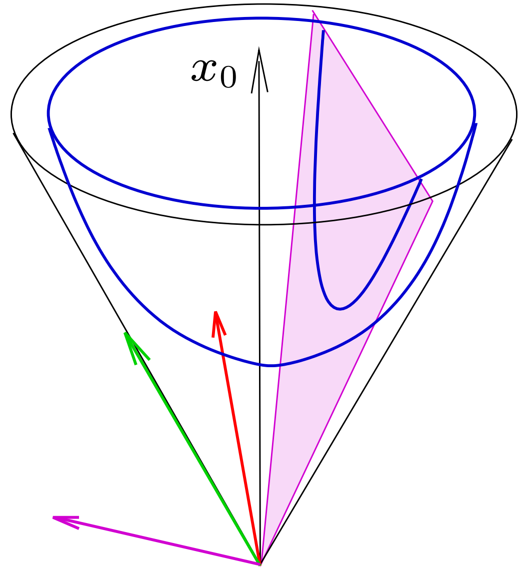

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- Pair of hyperplanes:

$$
\begin{aligned}
(u,v) &= -\cos(\angle(H_u,H_v)) &&\leftrightarrow\quad H_u \cap H_v \neq \emptyset\\
(u,v) &= -1 &&\leftrightarrow\quad H_u \cap H_v \subset \partial\mathbb{H}^d\\
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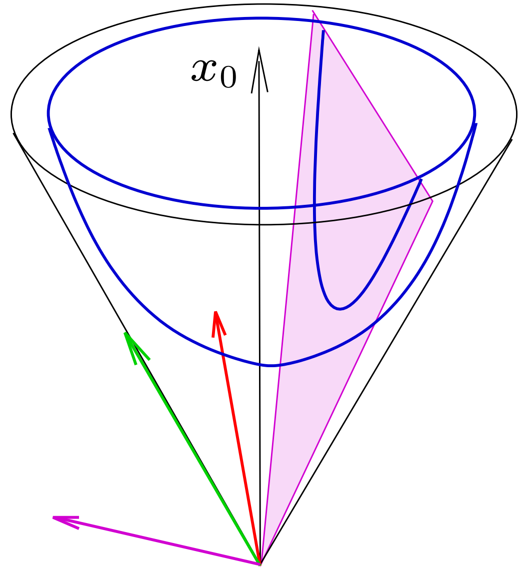

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- Reflection across H_v: $\quad R_v(x) = x - 2\frac{(x,v)}{(v,v)}v$.

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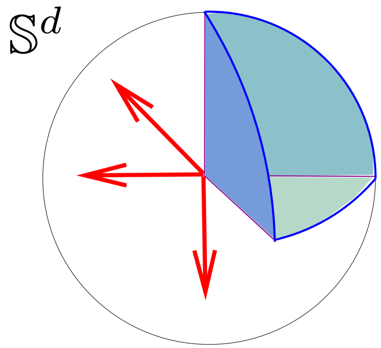

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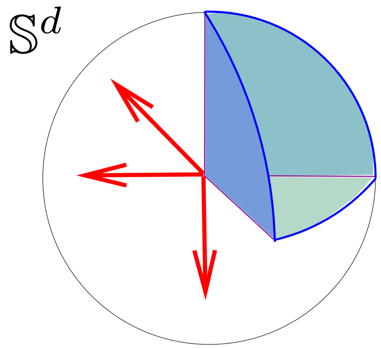

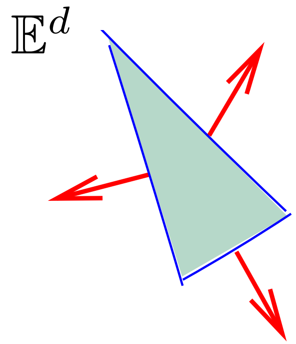

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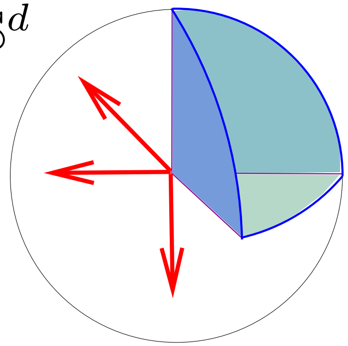
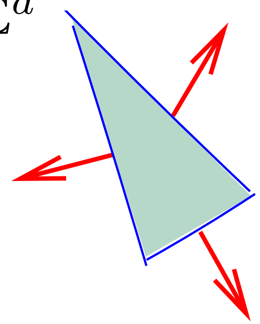
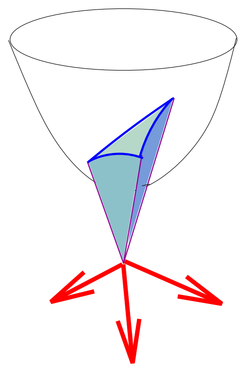


$\mathbb{E}^d$



$\mathbb{H}^d$



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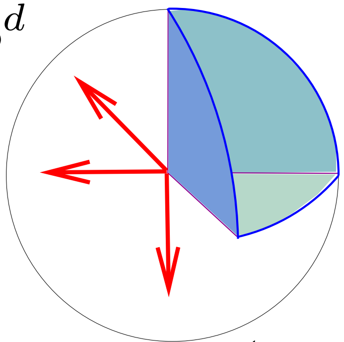

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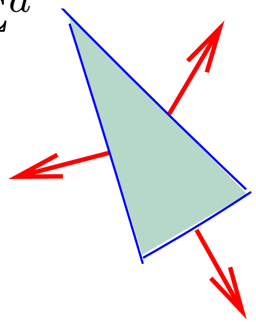

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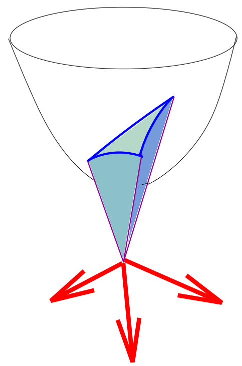
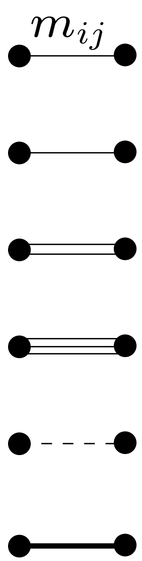


$sgn\ (G(P)):$ $\quad (d+1, 0) \quad (d, 0, 1) \quad (d, 1)$

Coxeter diagram $\Sigma(P)$

- Nodes $\longleftrightarrow$ facets f_i of P
- Edges:



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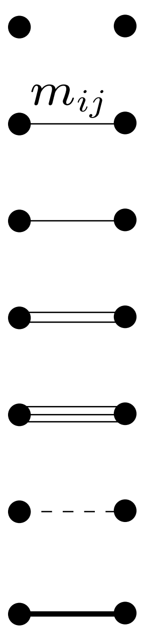

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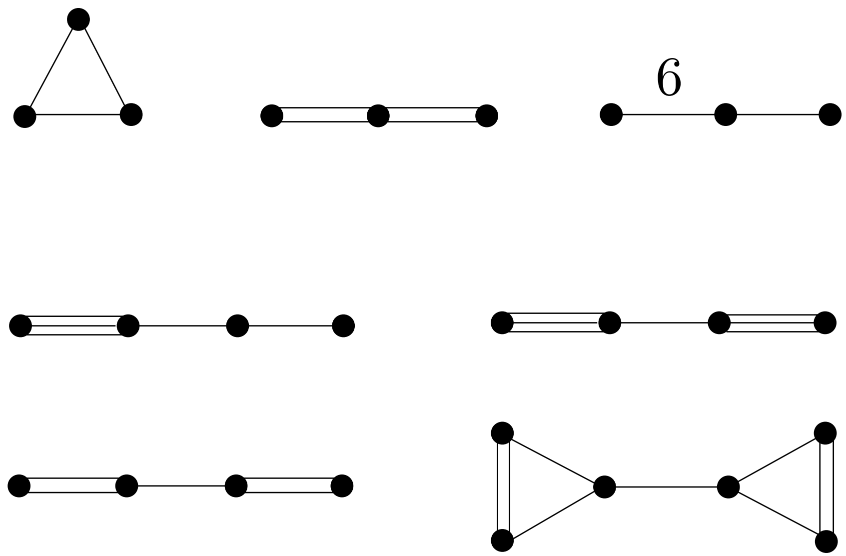
Examples:



Coxeter polytopes in $\mathbb{S}^d$, $\mathbb{E}^d$ and $\mathbb{H}^d$:

- $P \subset \mathbb{S}^d$. Finitely many in each dimension, Classified (Coxeter, 1934).
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- $P \subset \mathbb{H}^d$. Infinitely many, No classification.

Spherical Coxeter polytopes

- $P \subset \mathbb{S}^d \Rightarrow P$ is a simplex.
- $G(P) > 0 \quad \Rightarrow$ Any connected component of $\Sigma(P)$ has
  - no $\bullet\!\overset{k}{\text{———}}\!\bullet$ for $k > 5$,
  - no cycles,
  - at most one multiple edge,
  - no nodes of valency ≥ 4,
  - at most one node of valency 3.

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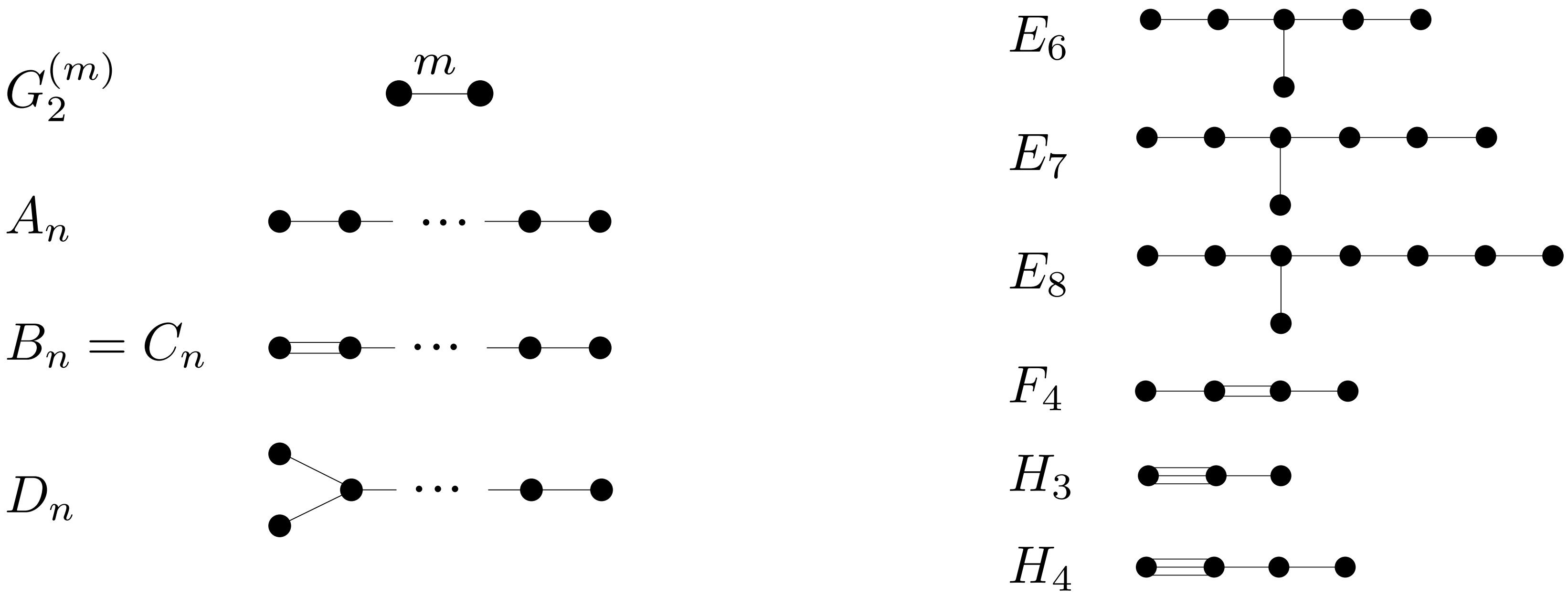
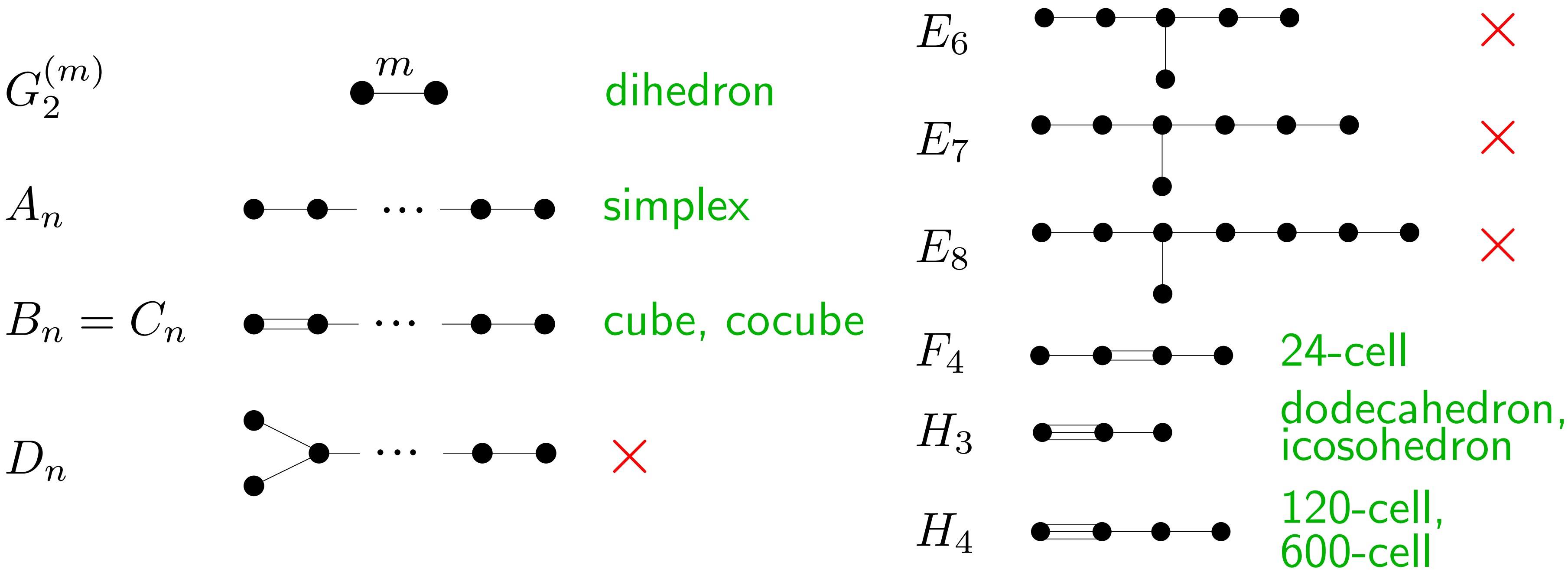
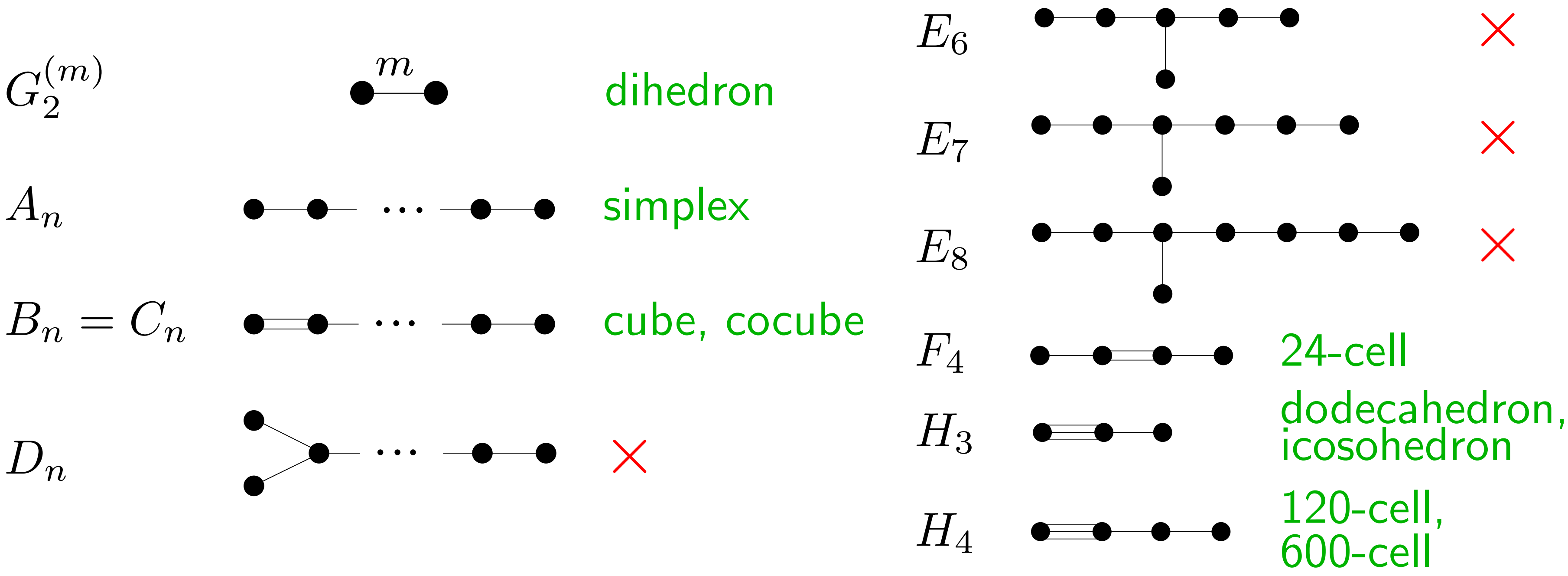


Regular polytopes: classification



Regular polytopes: classification

- Regular polytopes correspond to linear elliptic diagrams:



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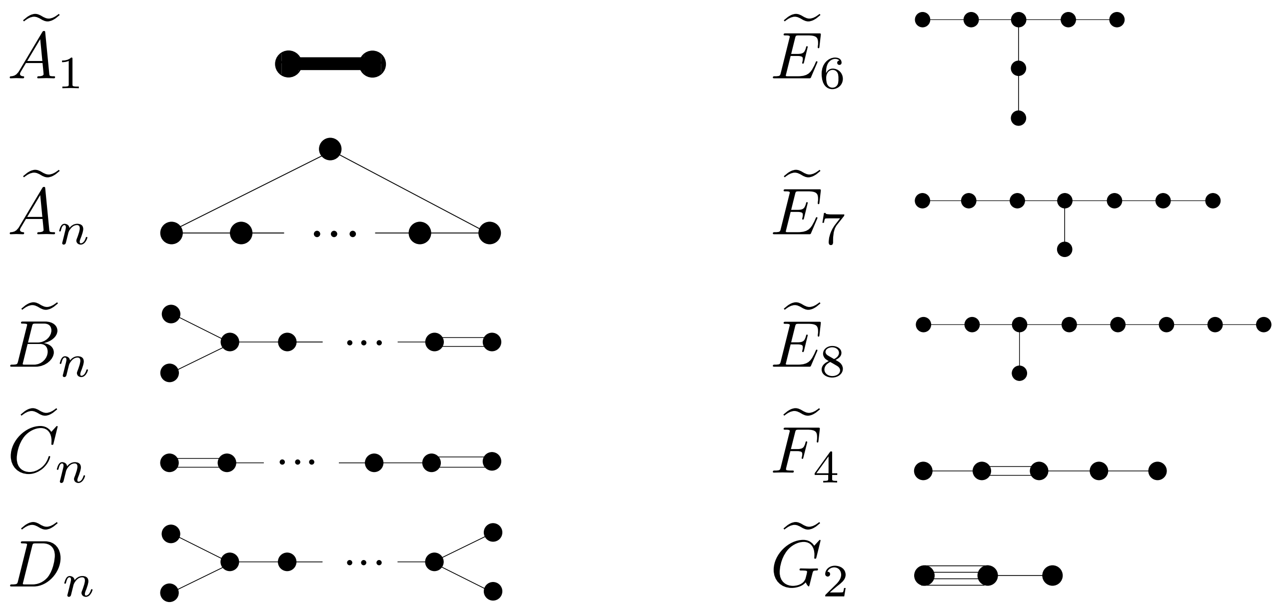

Acute-angled polytopes in $\mathbb{E}^d, \mathbb{S}^d$ and $\mathbb{H}^d$

A polytope is acute-angled if all its dihedral angles are acute (α is called acute if $\alpha \le \pi/2$).

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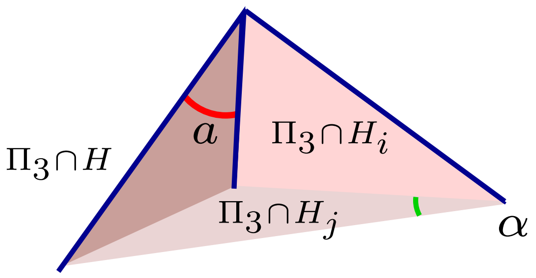

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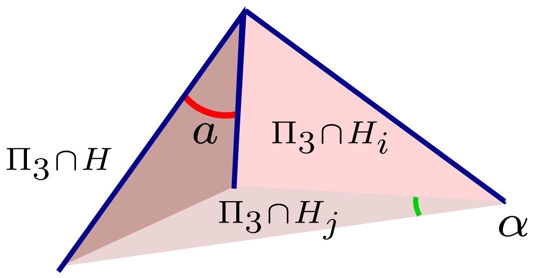

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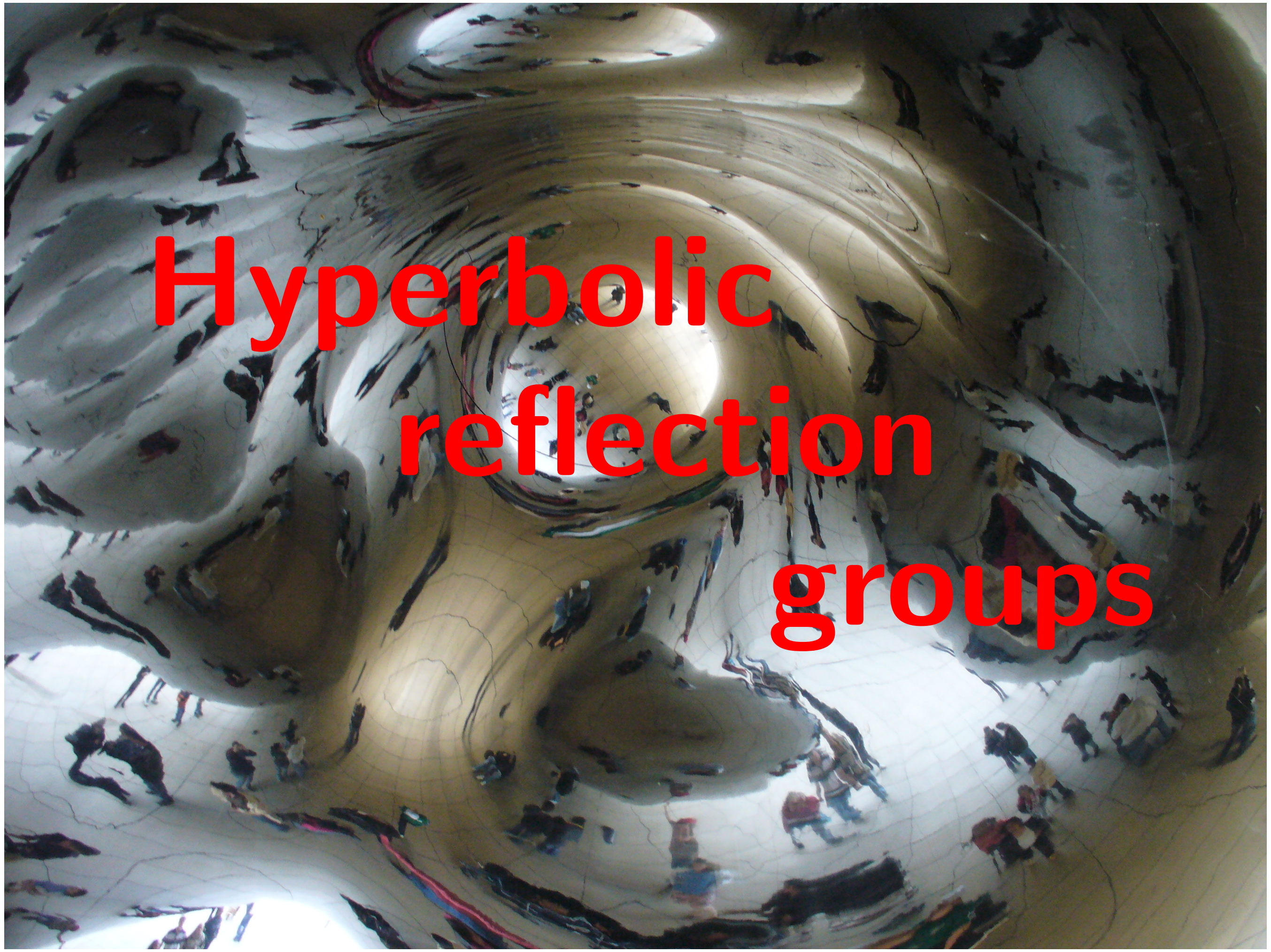
Cor. Any compact Coxeter polytope in $\mathbb{E}^d$ and $\mathbb{H}^d$ is simple.

Thm. Any acute-angled polytope in $\mathbb{E}^d$ is a direct product of several simplices and a simplicial cone.

Lemma. $L = \{e_1, ..., e_s\}$ indecomposable system of vectors in $\mathbb{E}^d$, $(e_i, e_j) \leq 0$, $i \neq j$. Then L is either linearly independent or there is a unique linear dependence with positive coefficients.


Hyperbolic
reflection
groups

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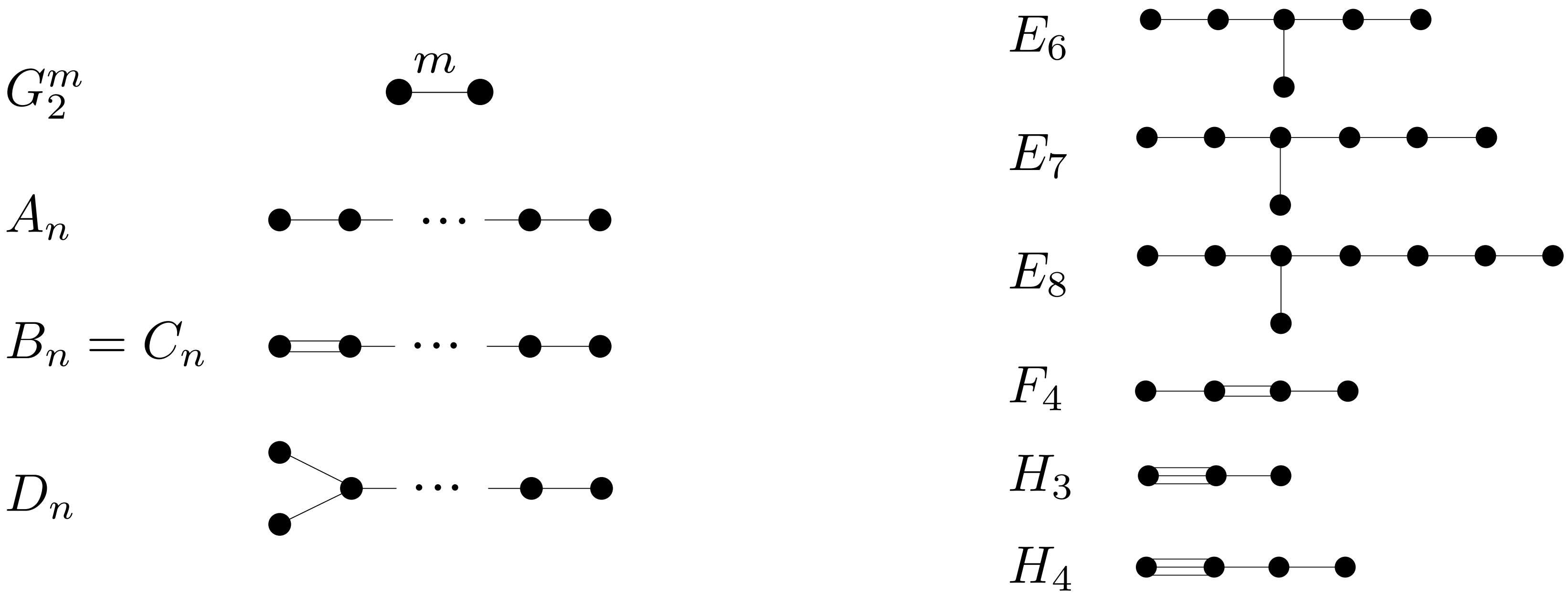

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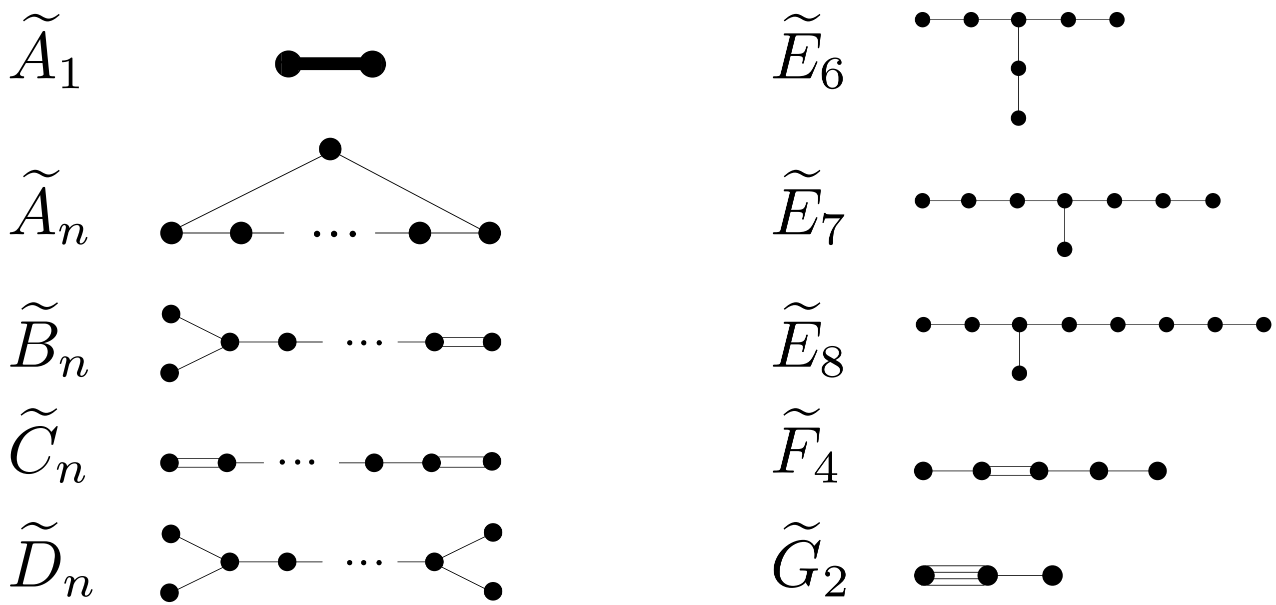

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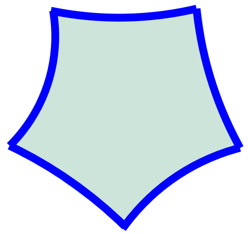

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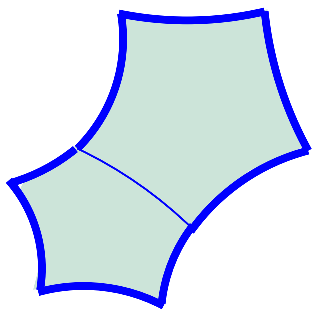

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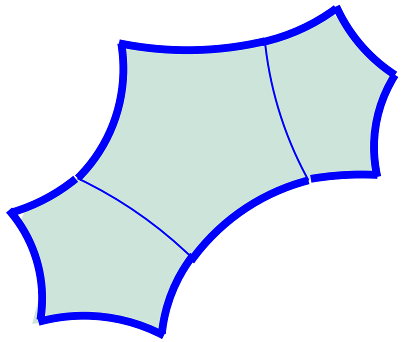

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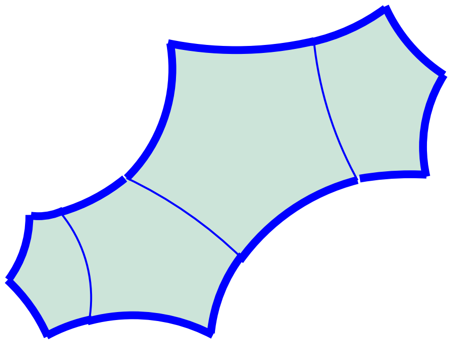

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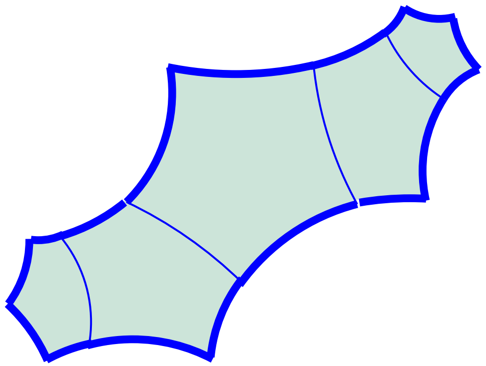

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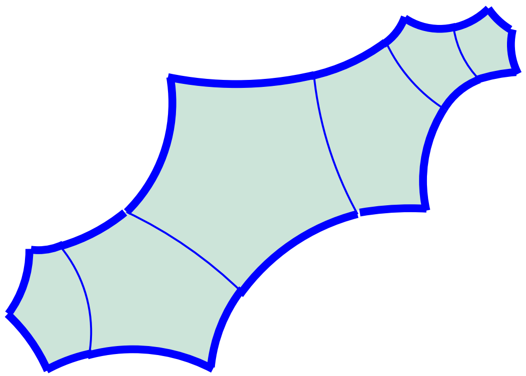

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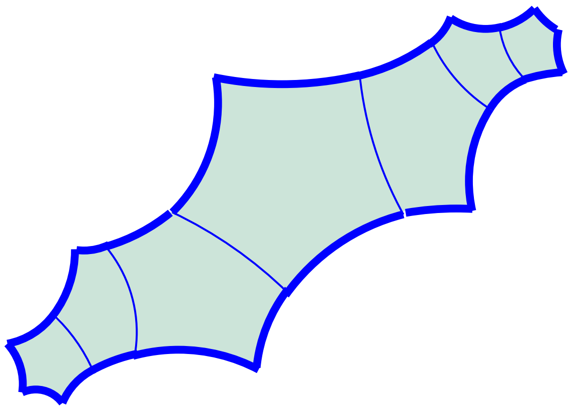

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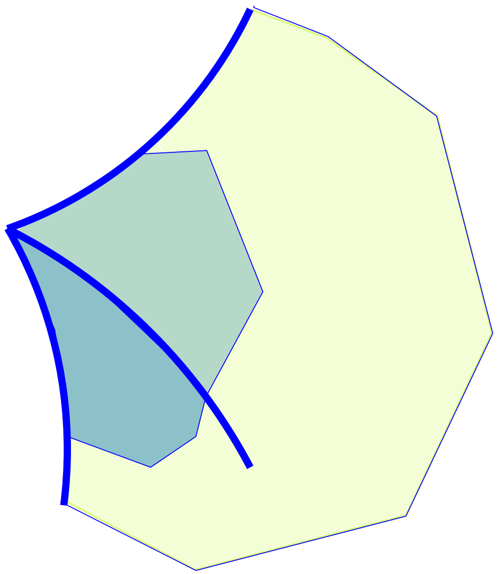
- **Thm.** (Allcock' 05) *There are infinitely many finite-volume Coxeter polytopes in* $\mathbb{H}^d$, *for every* $d \leq 19$.

  There are infinitely many compact Coxeter polytopes in $\mathbb{H}^d$, *for every* $d \leq 6$.

- If P is compact then P is simple.

  (i.e. d facets through each vertex)



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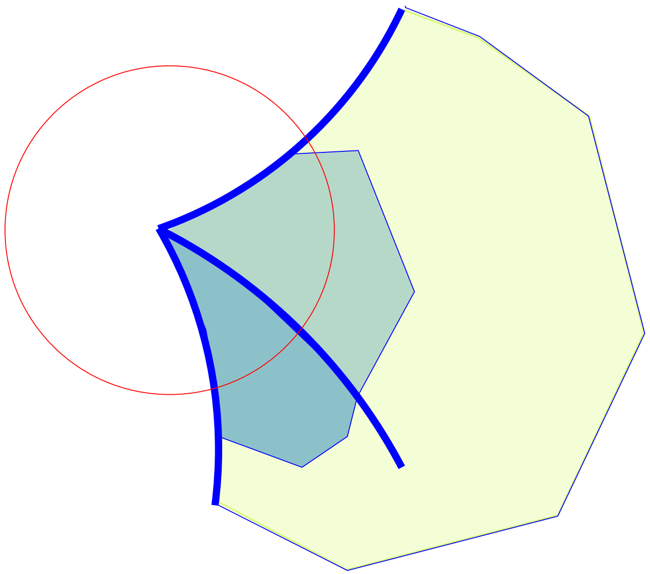

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$$G(P) = \begin{pmatrix} \boxed{} & 0 \\ 0 & \boxed{} \end{pmatrix}$$

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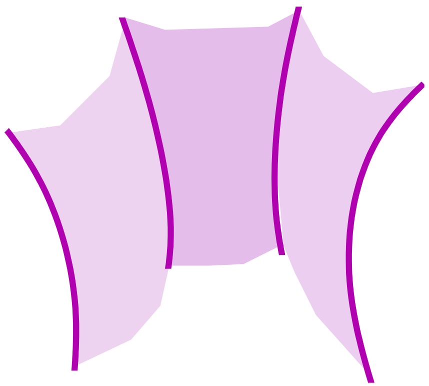

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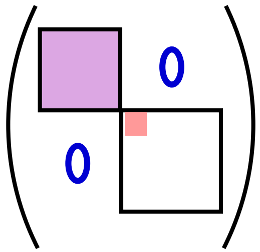
$G(P) =$ 

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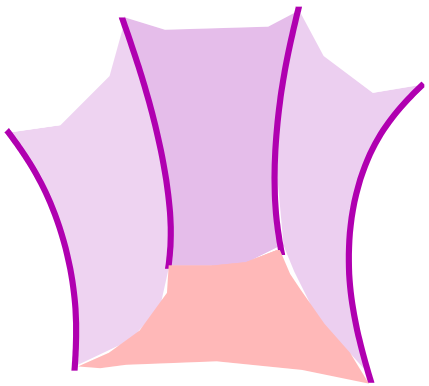

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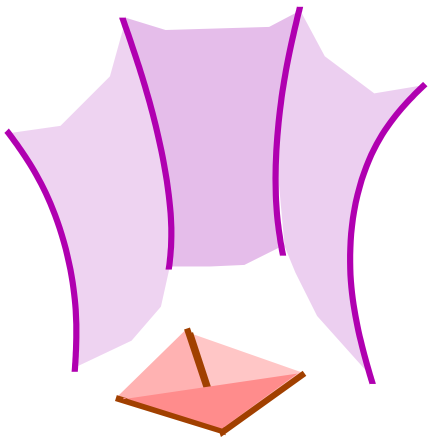

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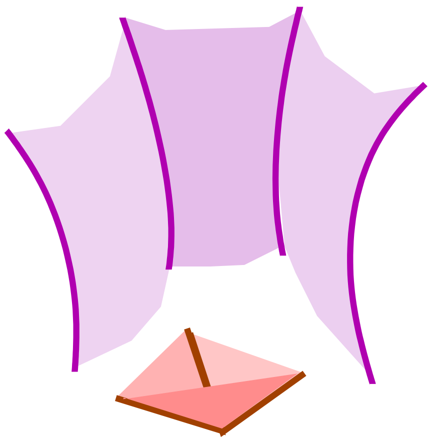

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$$F_H = \bigcup_{g \in H} gF$$

$P, \Sigma(P)$

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Indecomposable, symmetric matrix G, $sgn(G) = (d, 1)$,

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$g_{ij} \leq 0$.

Then there exists a convex polytope $P \subset \mathbb{H}^d$, such that $G = G(P)$ (unique up to an isometry of $\mathbb{H}^d$).

- P is compact (finite volume) $\Rightarrow$ P combinatorially equivalent to some Euclidean polytope.
  1. P has at least one vertex (ideal vertex).
  2. each edge has two ends.
- 1, 2 $\Rightarrow$ P is compact (finite volume).

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3. Thm. (Nikulin, 81):
   For any simple, compact, convex polytope $P \subset \mathbb{E}^d$
   and any $i < k \leq [d/2]$ holds

$$\alpha_k^i < \binom{d-i}{d-k} \frac{\binom{[d/2]}{i} + \binom{[(d+1)/2]}{i}}{\binom{[d/2]}{k} + \binom{[(d+1)/2]}{k}}$$

where α_k^i = average number of i-faces of a k-face of P

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$\Rightarrow$ a lots of triangular and quadrilateral 2-faces

More precisely:

Plane angles $\longrightarrow$ weights

vertex $A \longrightarrow \sigma(A) = \sum$ of weights of plane angles at A

2-face $F \longrightarrow \sigma(F) = \sum$ of weights of plane angles of F

L. If for all A, F $\quad \sigma(A) \leq cd$ and $\sigma(F) \geq 5 - n_F$ then $d < 8c + 6$.
($n_F = \#$ of sides of F)

plane angle $\leftrightarrow$ diagram Σ_A of a vertex A with two "black" nodes a and b (corresp. to facets not containing F).

$$weight = 1, \text{ if } dist_{\Sigma_A}(a, b) \leq 7$$
$$weight = 0, \text{ otherwise.}$$

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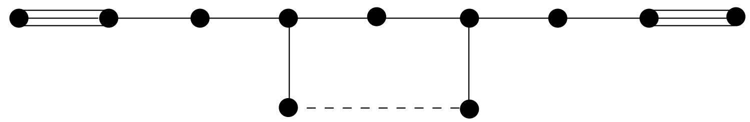
  Examples known for $d \leq 8$.
  Unique Ex. for $d = 8$ (Bugaenko '92):



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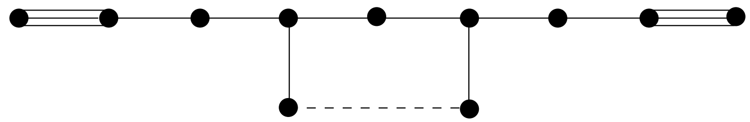

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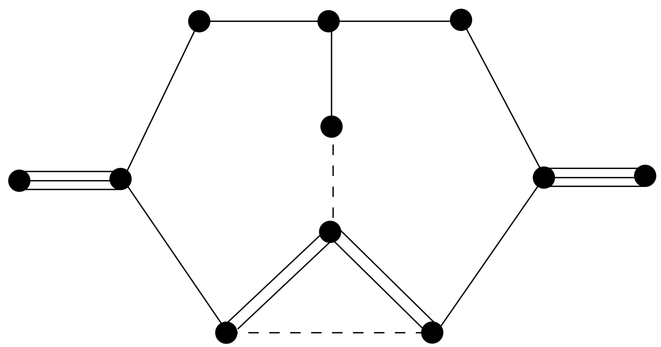

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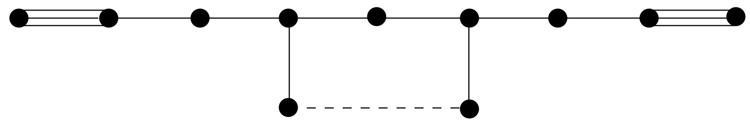

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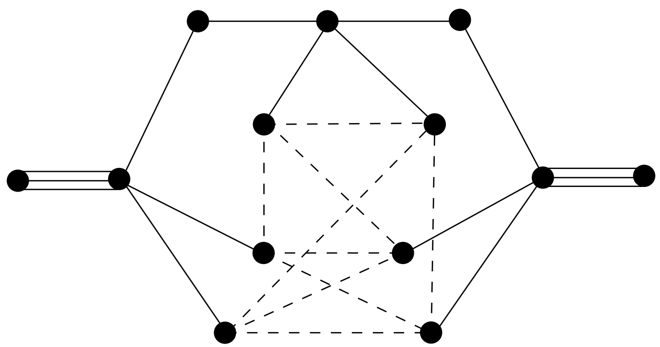

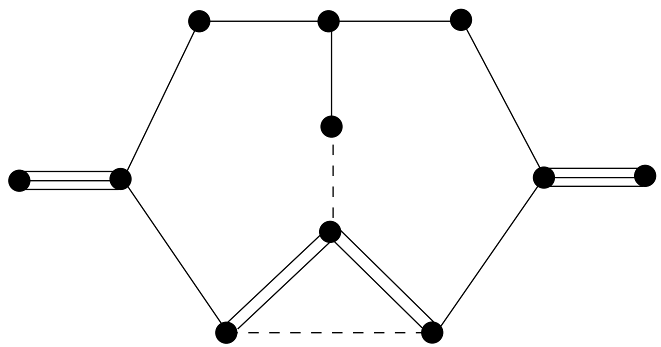

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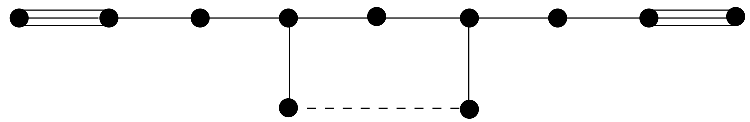

- If $P \subset \mathbb{H}^d$ is of finite volume then $d < 996$.
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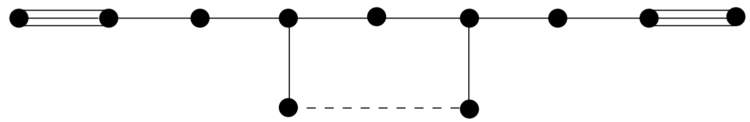

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  Examples known for $d \leq 19$ (Vinberg, Kaplinskaya '78)
  $d = 21$ (Borcherds '87).

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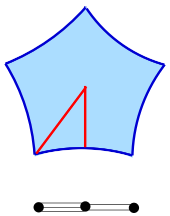	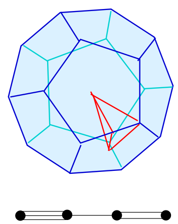	regular 4-polytope with 120 dodecahedral facets 

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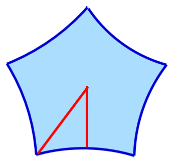
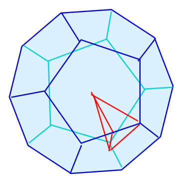
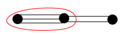
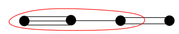
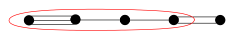
		regular 4-polytope with 120 dodecahedral facets
		

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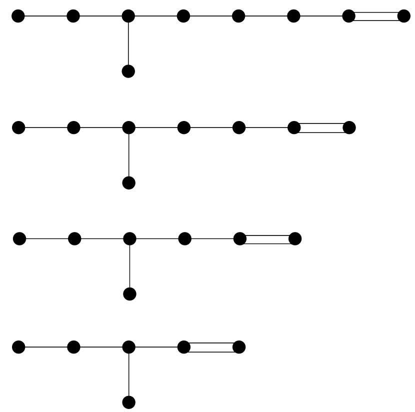

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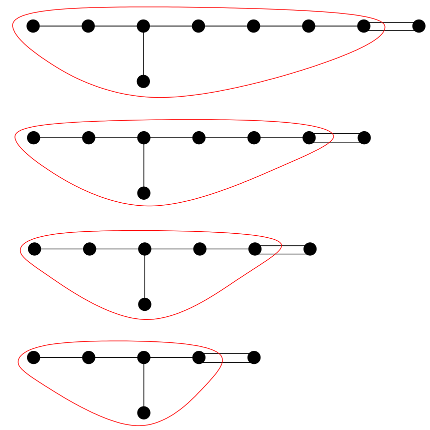

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Thm. (Andreev '70):
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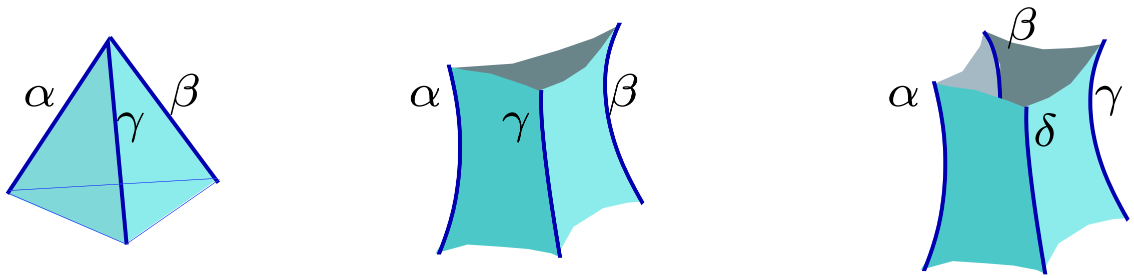
- 

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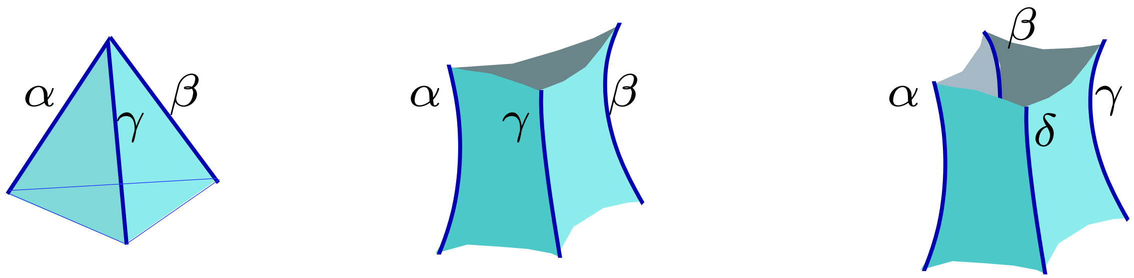
- 

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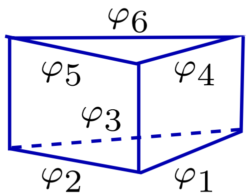

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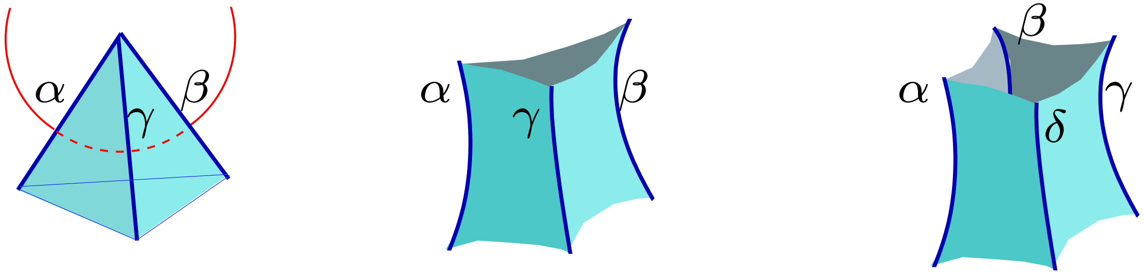
- 

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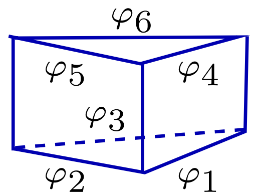

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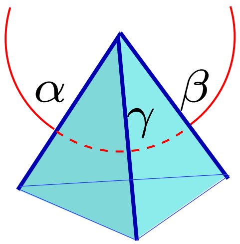
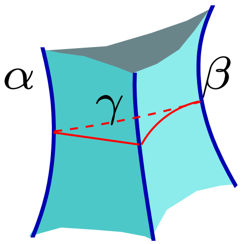
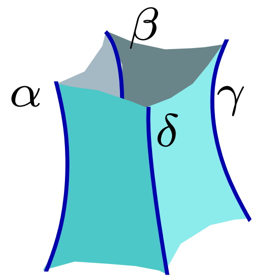
- 





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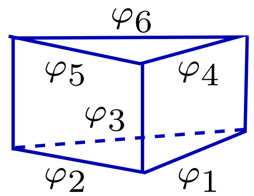

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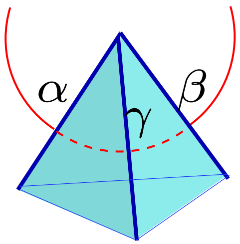
- 

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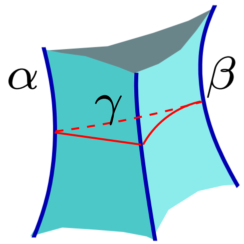

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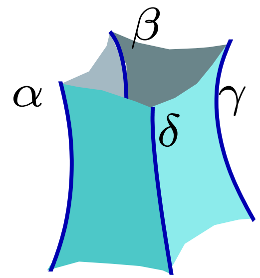

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$$\begin{pmatrix} 1 & a & 0 & 0 \\ a & 1 & 0 & 0 \\ 0 & 0 & 1 & b \\ 0 & 0 & b & 1 \end{pmatrix}$$

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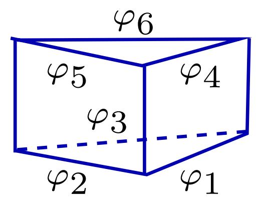
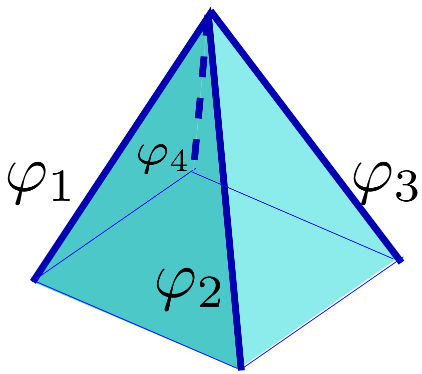
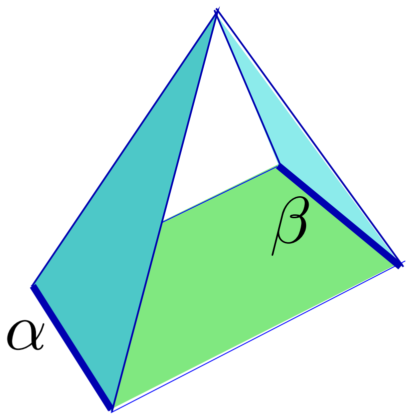


Additional conditions for finite volume polytopes:

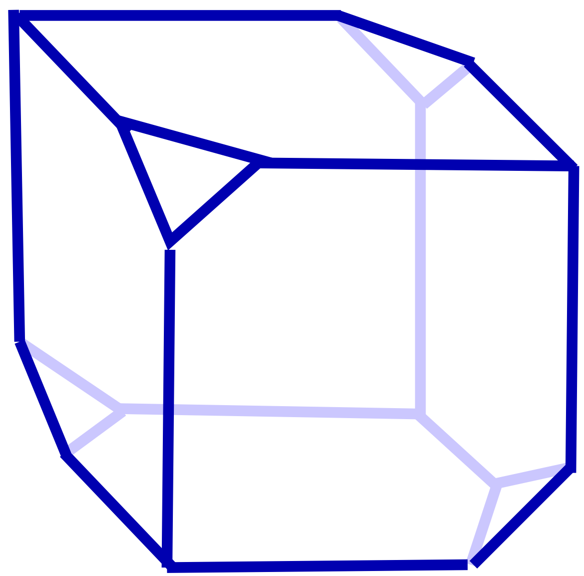
- 

$\Rightarrow \quad \varphi_i = \frac{\pi}{2},\ i = 1, \ldots, 4.$

- 

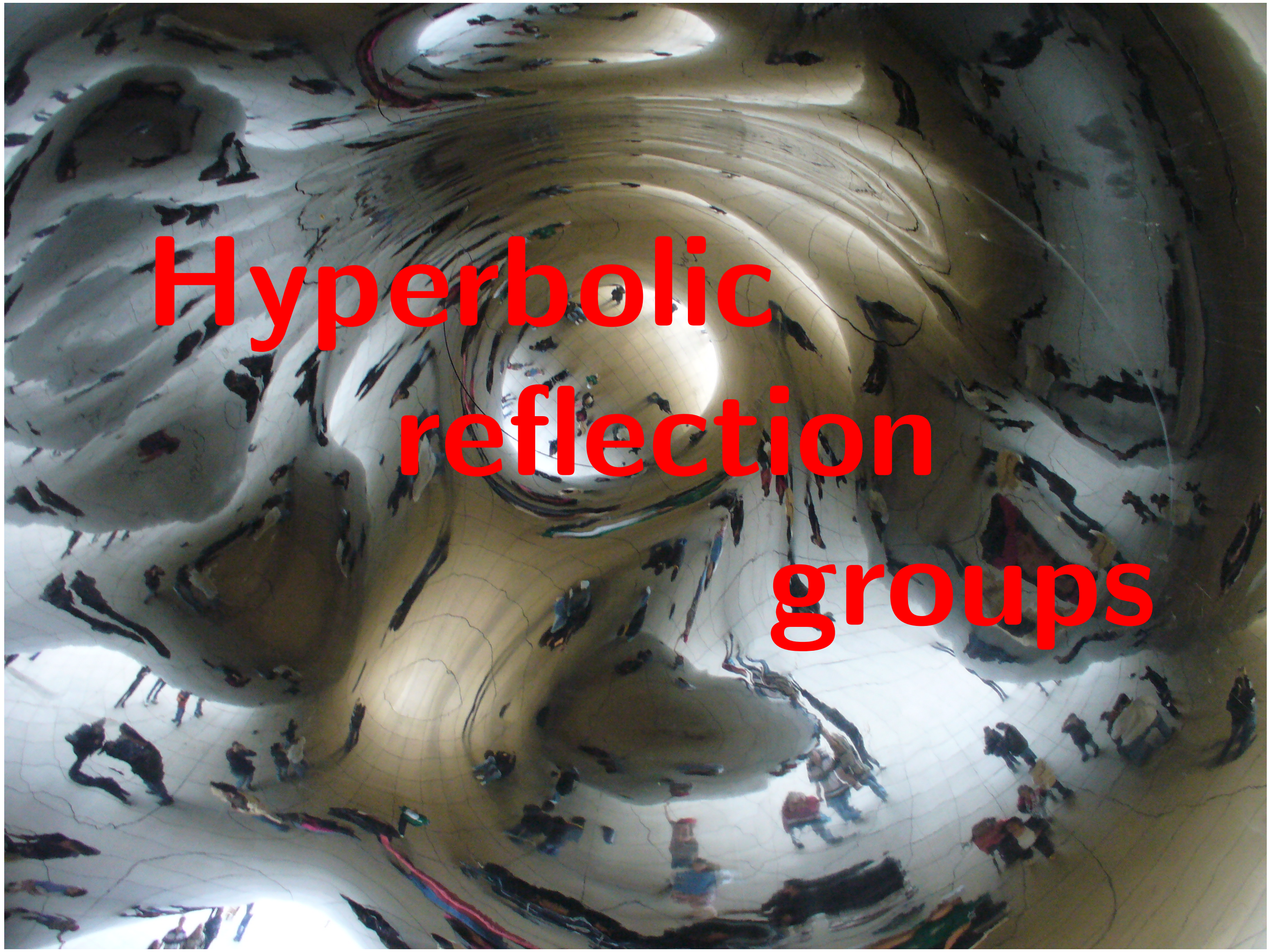
$\Rightarrow$ either $\alpha \neq \frac{\pi}{2}$ or $\beta \neq \frac{\pi}{2}$.

Example: no angles of this polytope would satisfy the conditions of the theorem!



Thm. (Andreev '70) Let P be an acute-angled polytope in $\mathbb{H}^d$, a, b be its faces, and $\bar{a}$, $\bar{b}$ be planes spanned by a and b.

If $a \cap b = \emptyset$ then $\bar{a} \cap \bar{b} = \emptyset$.


Hyperbolic
reflection
groups

Compact hyperbolic Coxeter polytopes

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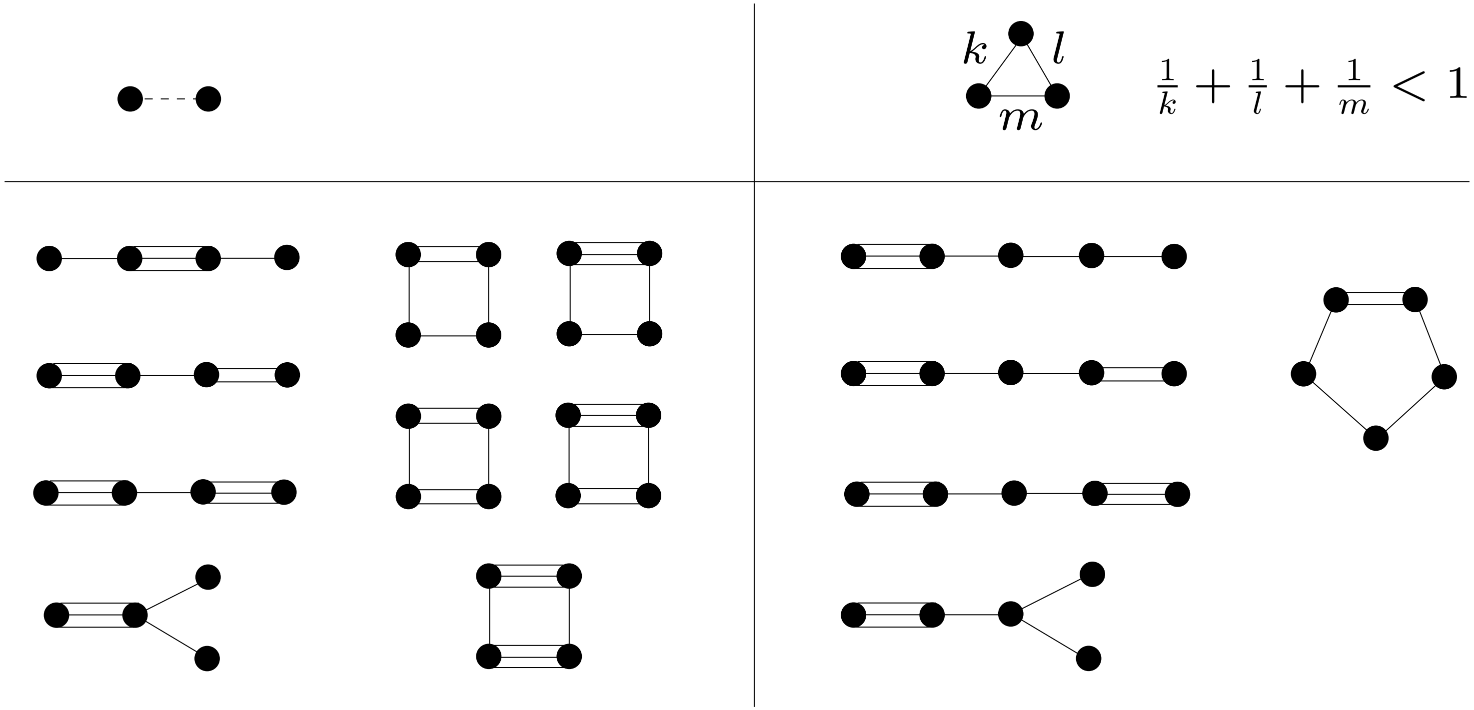

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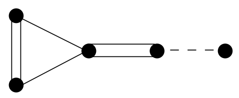
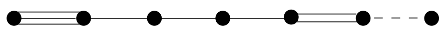
Examples of prisms:





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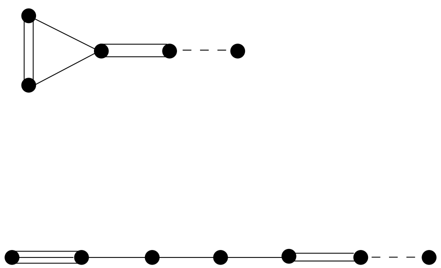
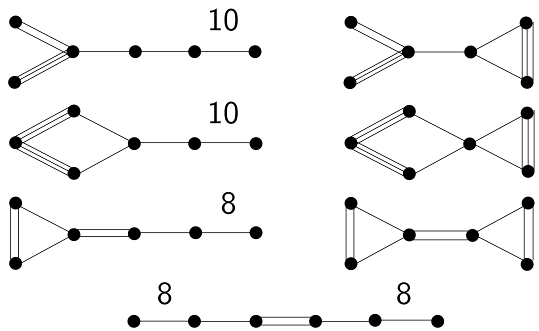


Esselmann's polytopes:



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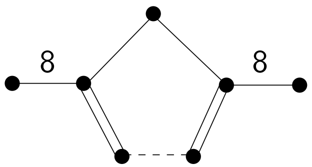

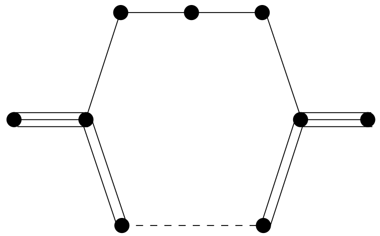

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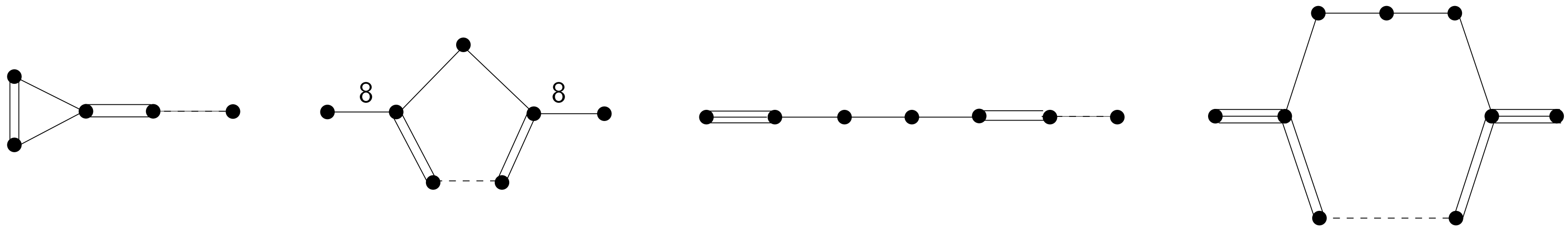

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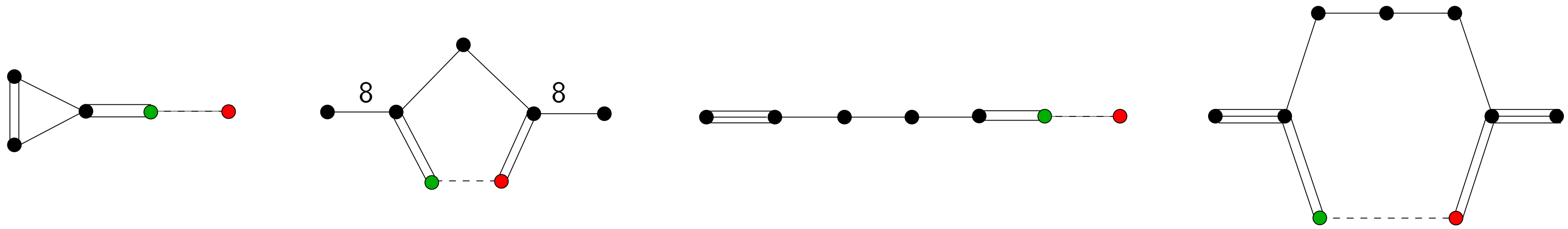

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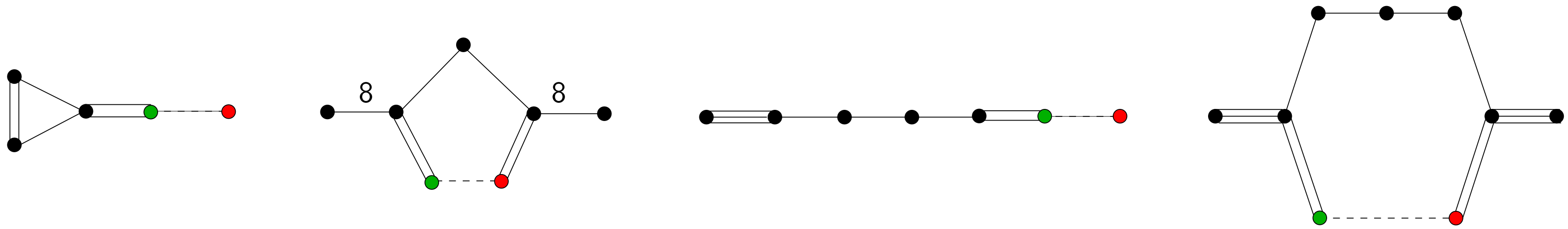

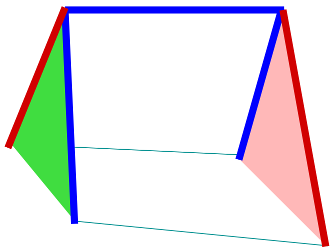

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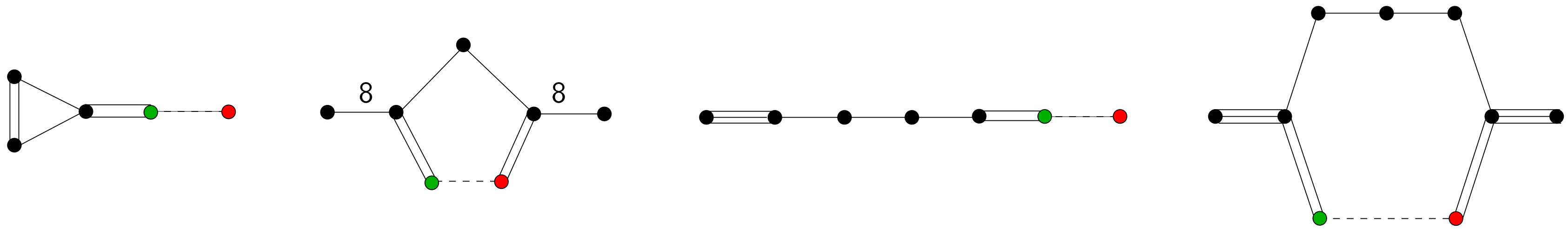

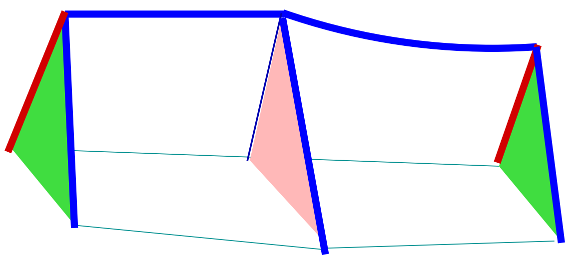

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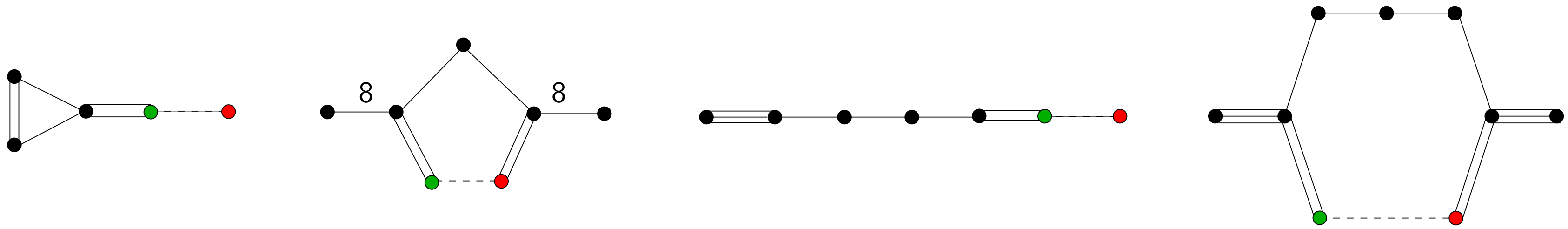

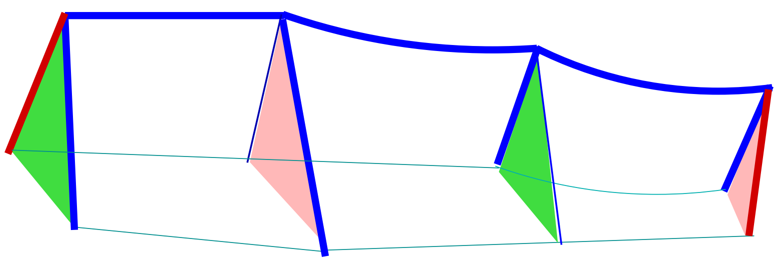

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How to proceed for a given combinatorial type ?

How to list all appropriate combinatorial types ?

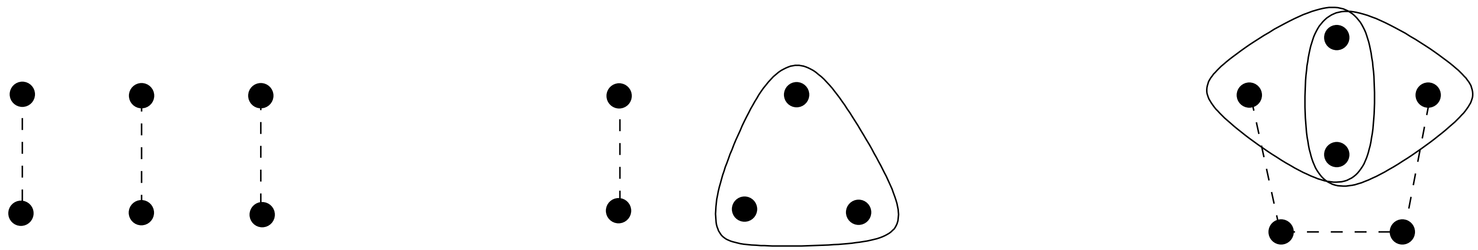
Tools

- Given a combinatorial type, may try to “reconstruct” the polytope (i.e. to find its dihedral angles).

Combinatorics:	Dihedral angles:
Diagram of missing faces	Coxeter diagram

Diagram of missing faces

- Nodes $\longleftrightarrow$ facets of P
- Missing face is a minimal set of facets $f_1, \ldots, f_k$, such that $\bigcap_{i=1}^{k} f_i = \emptyset$.
- Missing faces are encircled.
- Ex:



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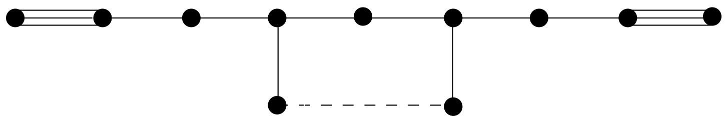

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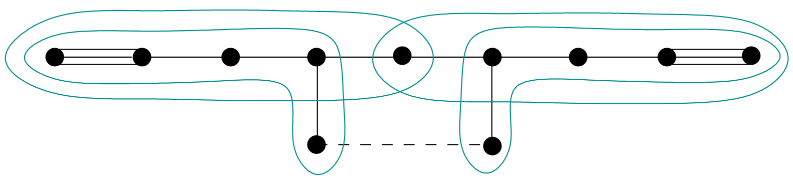

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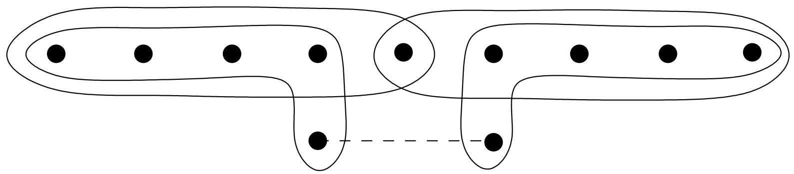

Lannér subdiagrams $\longleftrightarrow$ Missing faces

- If L is a Lannér diagram then $|L| \leq 5$.
- $\#$ of Lannér diagrams of order $4, 5$ is finite.
- For any two Lannér subdiagrams s.t. $L_1 \cap L_2 = \emptyset$, there exists an edge joining these subdiagrams.

Given a combinatorial type may try to check

if there is a Coxeter polytope of this type.

Tools

- Combinatorial type $\longrightarrow$ "reconstruction" of Coxeter polytope
- Coxeter faces.
  - Borcherds '87: $\begin{array}{c}\text{Elliptic subdiagram}\\ \text{without } A_n \text{ and } D_5\end{array} \longrightarrow$ Coxeter face

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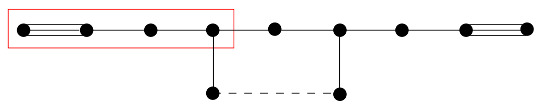

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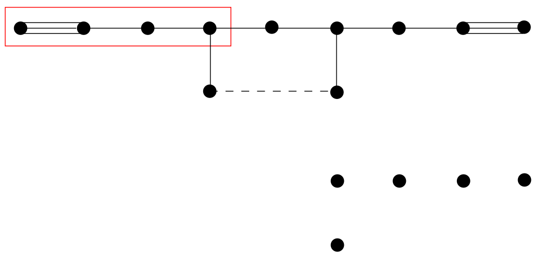

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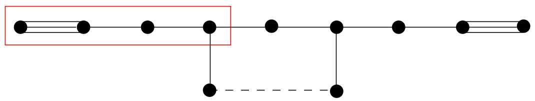

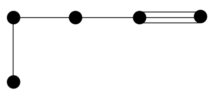

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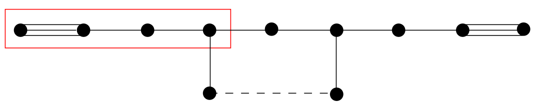

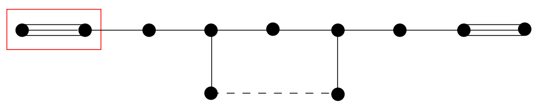

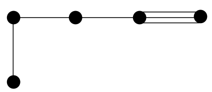

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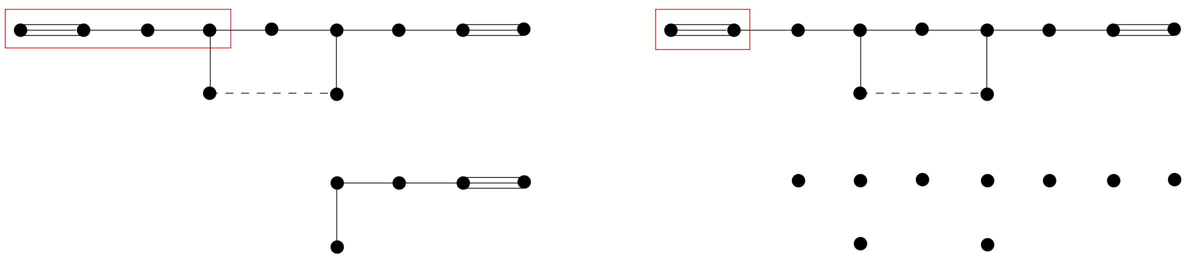

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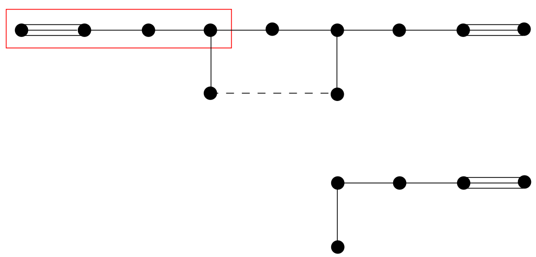

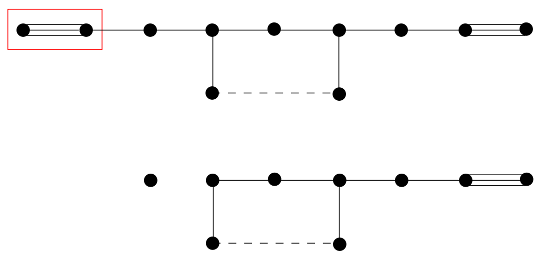

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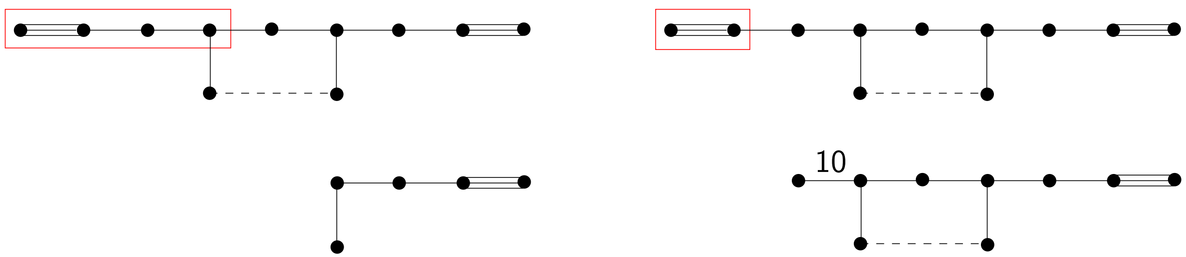

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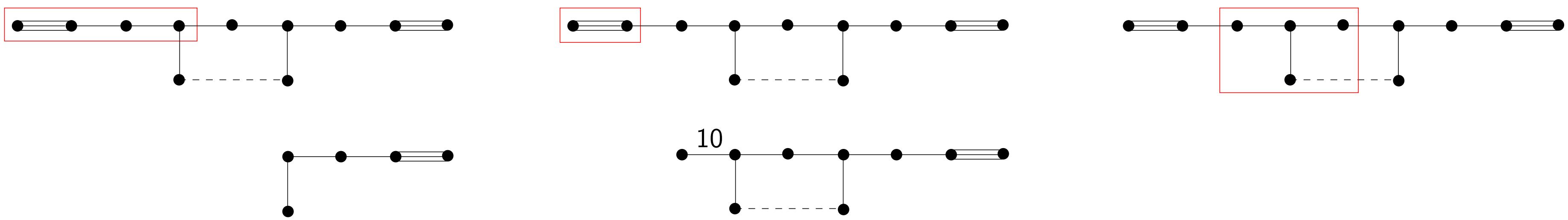

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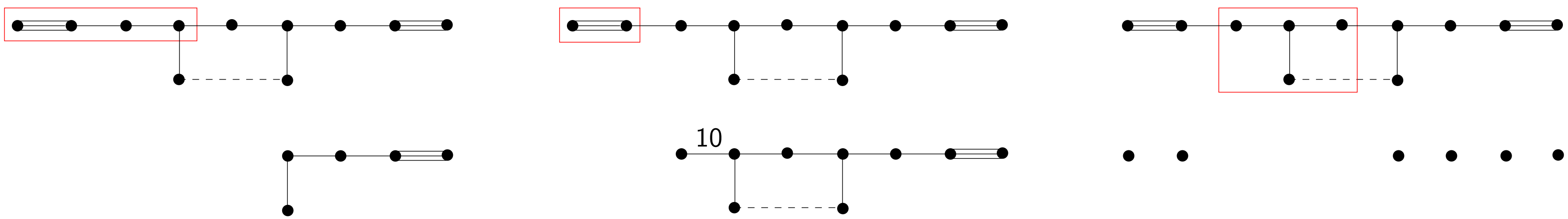

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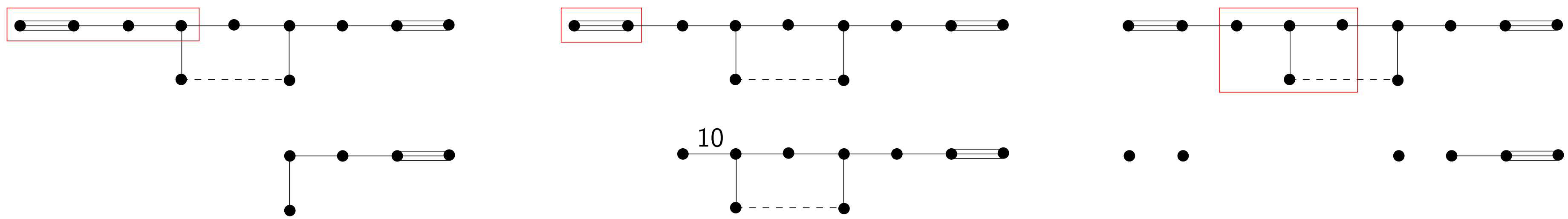

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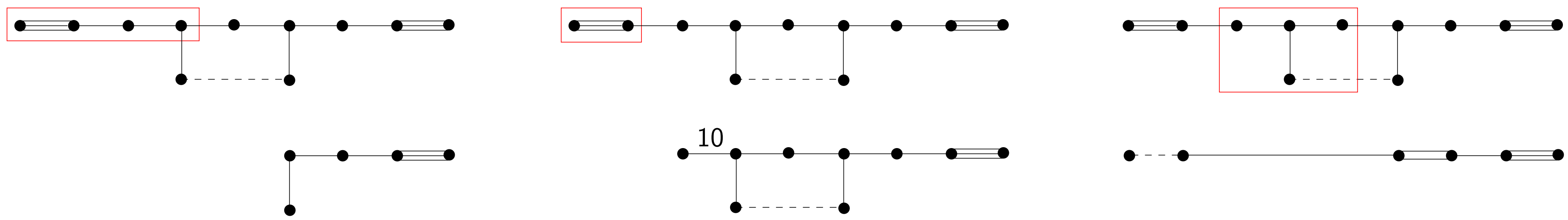

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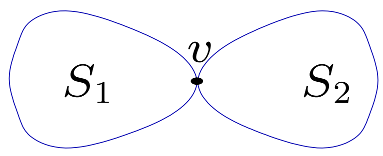
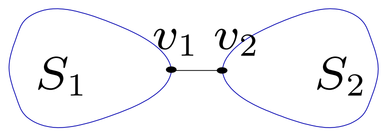
$$det(\Sigma, v) = det(S_1, v) + det(S_2, v) - 1$$



$$det(\Sigma, \langle v_1, v_2 \rangle) = det(S_1, v_1) det(S_2, v_2) - g_{12}^2$$



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To list combinatorial types:

- Gale diagram (works well for $n \le d + 3$ only).

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  - $\forall u \in \Sigma(P)$ $\exists$ Lannér subdiagram L, $u \in L$.
  - $\forall$ Lannér subdiagram L_1 $\exists$ Lan. subd. L_2, $L_1 \cap L_2 = \emptyset$.

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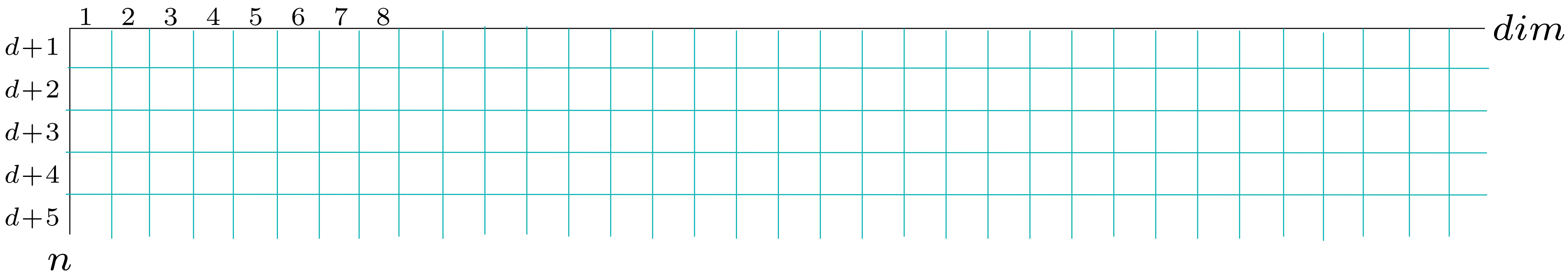
  (T,F '06): $d \le 7$, unique example in $d = 7$.
- $n = d+5$, (T,F '06): $d \le 8$.


1 2 3 4 5 6 7 8
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d+1
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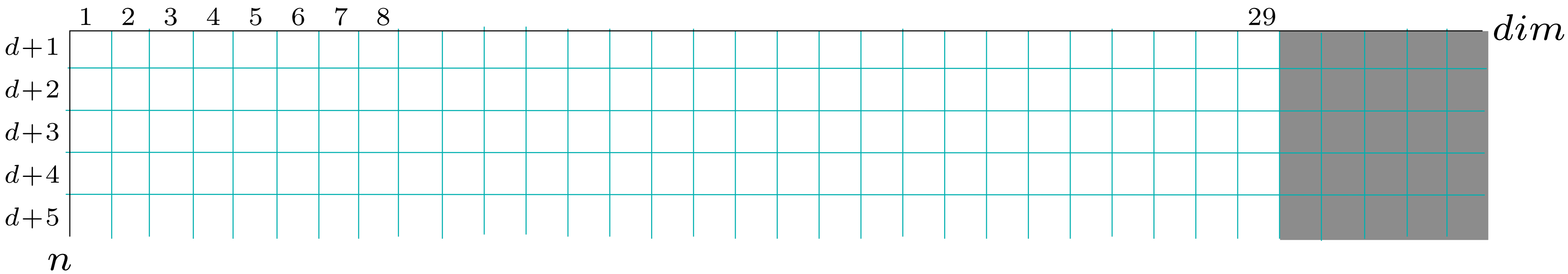
d+4
d+5
n



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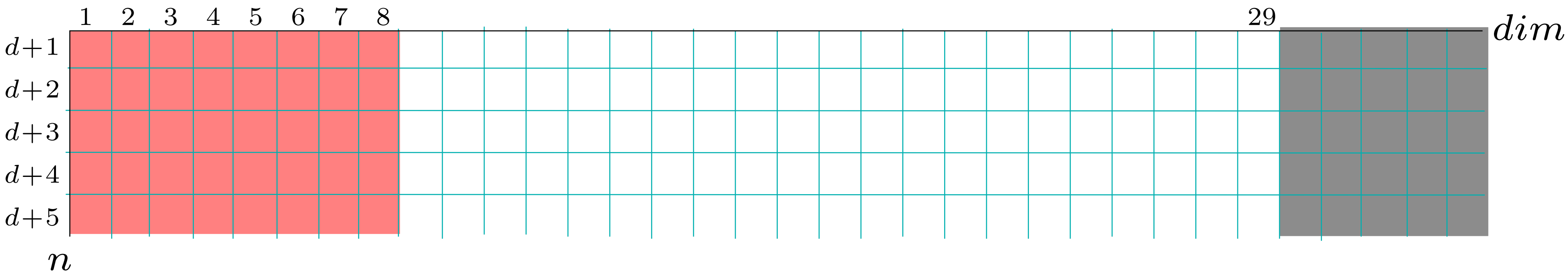
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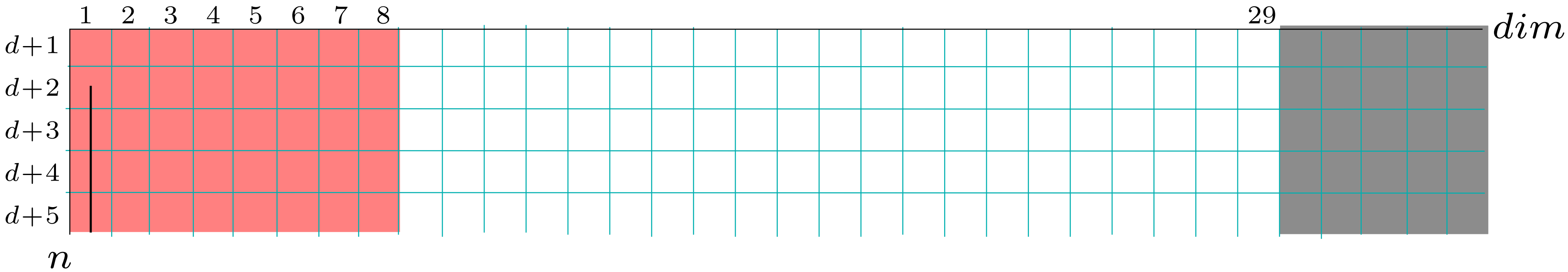

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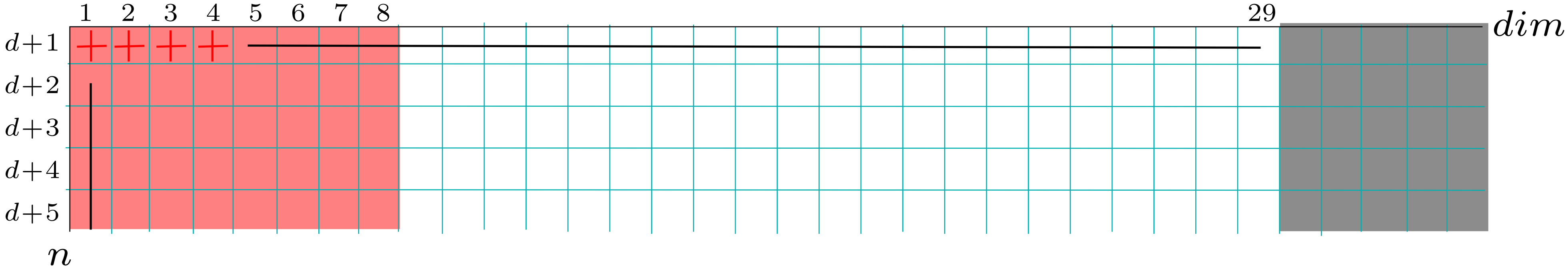
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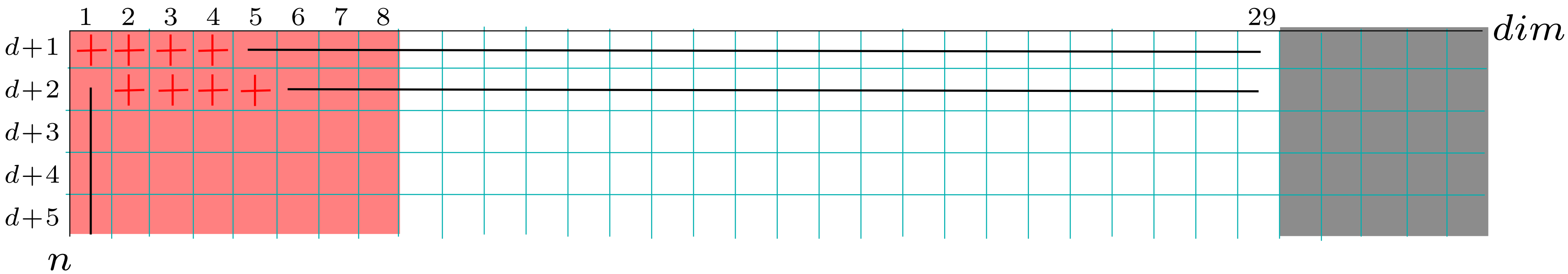

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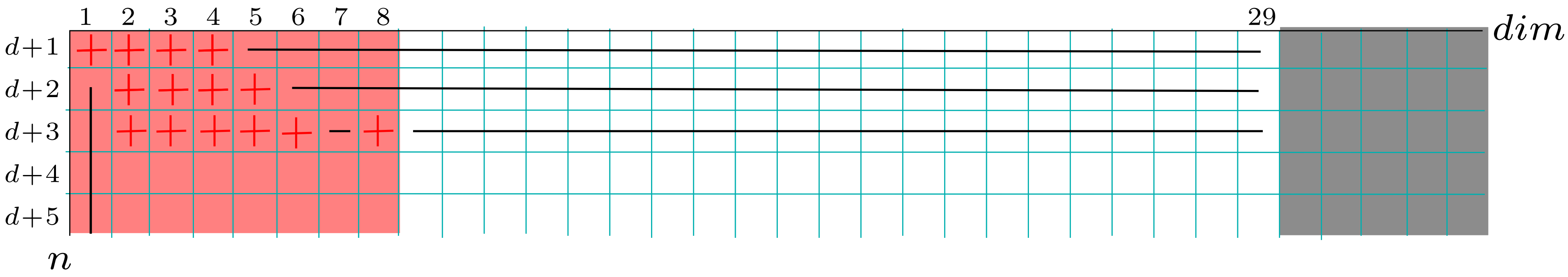

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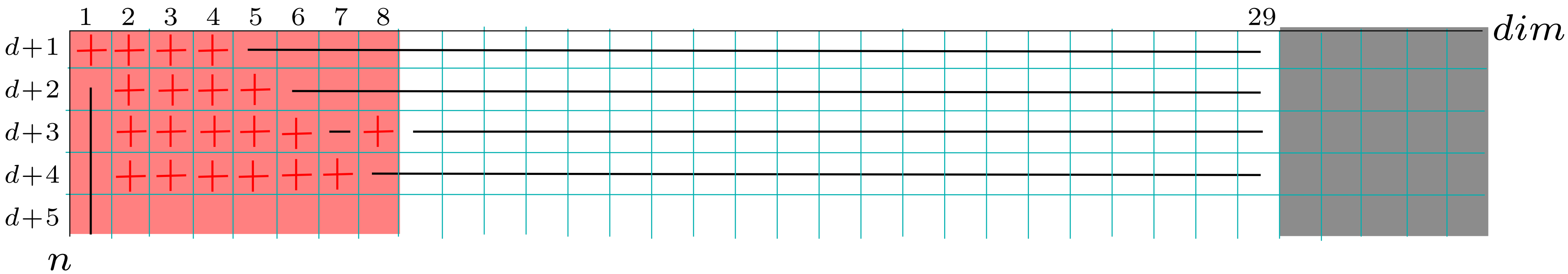

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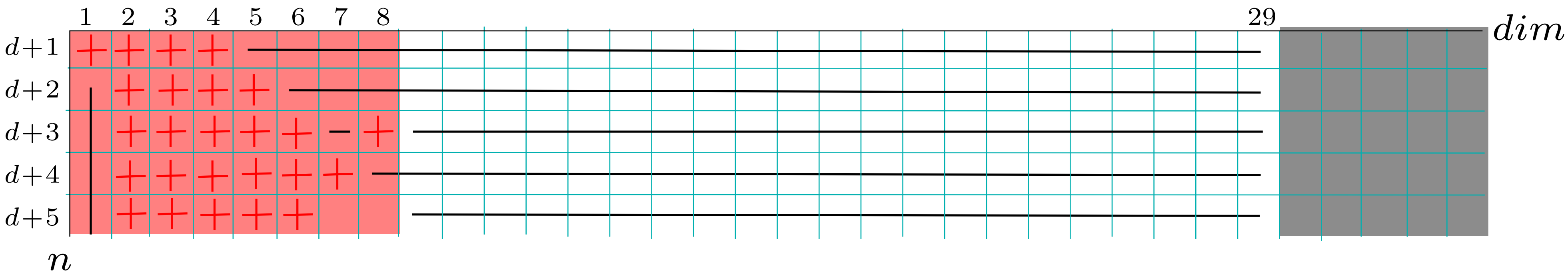
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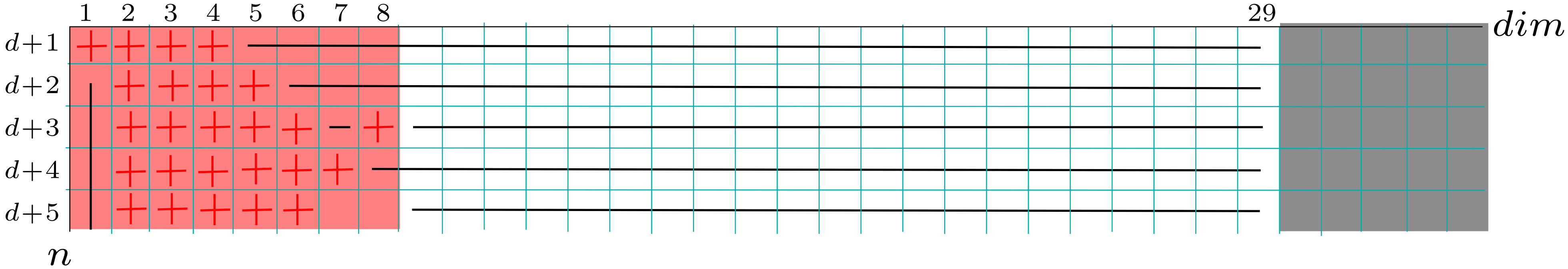

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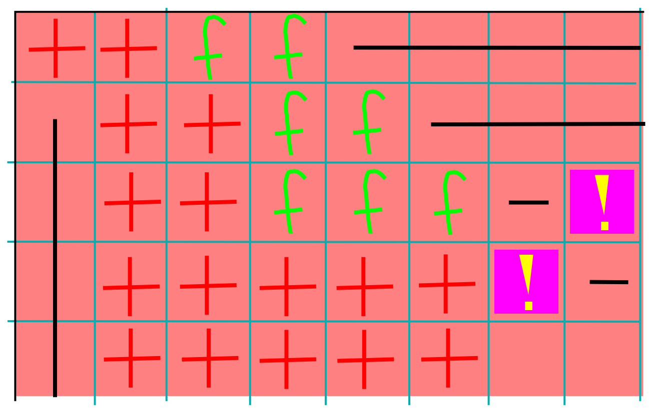

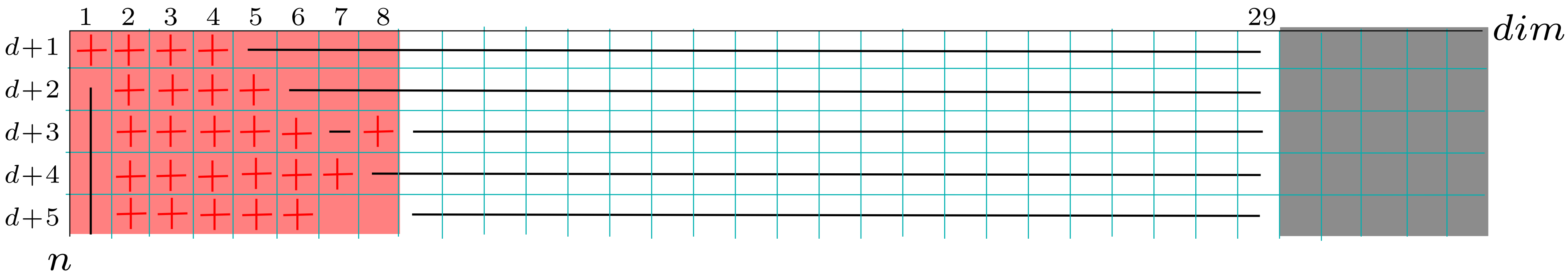

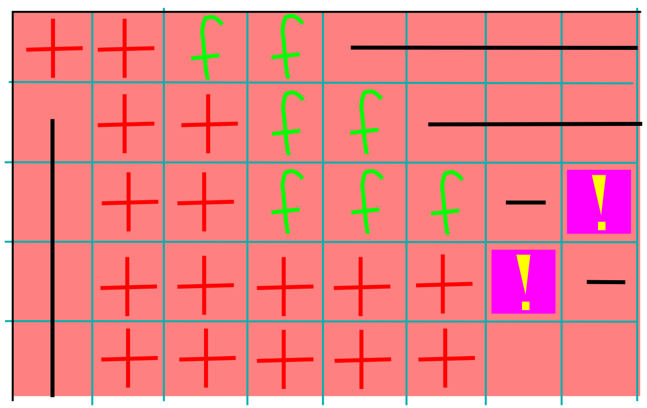

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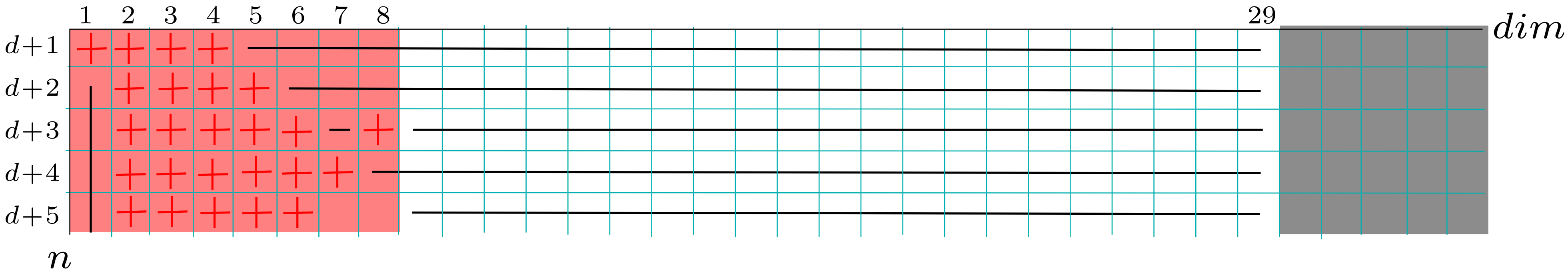
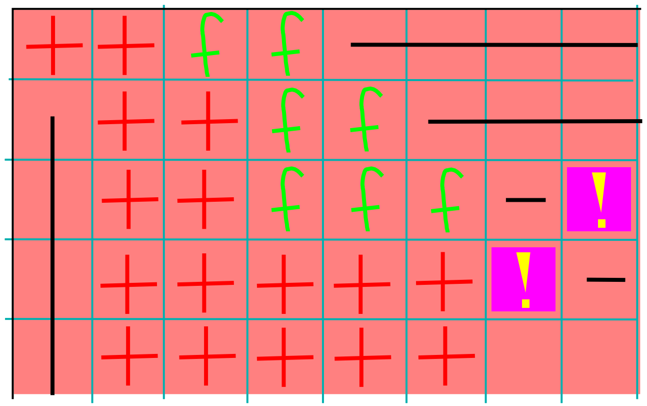
2) use previous cases





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Inductive algorithm?

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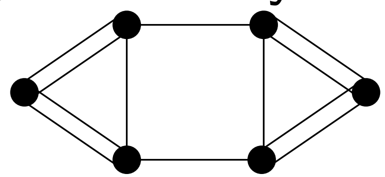
- (T,F '06): If all Lannér subdiagrams are of order 2, then $d \leq 13$.
  (for compact or simple finite volume polytopes).

Finite volume polytopes

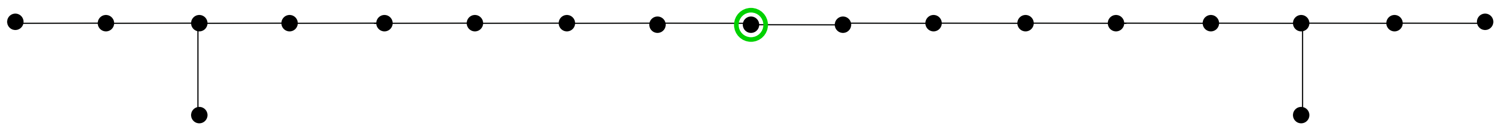
- combinatorics: not "simple" but "simple in edges"
  (a k-face is contained in $d-k$ facets unless $k=0$).

- missing face $\leftrightarrow$ Lannér or quasi-Lannér subdiagram
  (i.e. diagram of a simplex with some vertices at $\partial\mathbb{H}^d$).

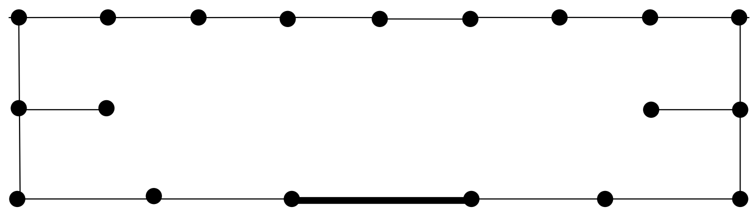
- $n = d + 1$, simplices. $d \leq 9$, fin. many for $d \geq 3$.
- $n = d + 2$, ◦ $\Delta^i \times \Delta^j \Rightarrow$ · prisms, $d \leq 5$, fin. many for $d > 3$.

  · $\Delta^2 \times \Delta^2 \Rightarrow$ 

  ◦ pyramid over $\Delta^i \times \Delta^j$, $d = 3, \ldots, 13$ and 17(fin. many).



- $n = d + 3$, $d \leq 16$. A unique example in $d = 16$:



Reflection subgroups of reflection groups

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Question. What about finite index reflection subgroups?

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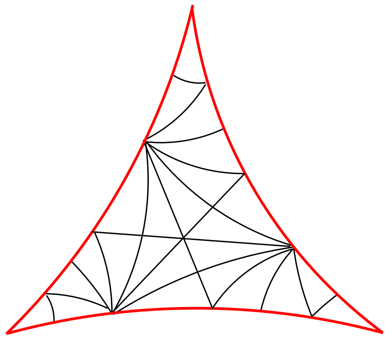
Question. What about finite index reflection subgroups?

"yes" $\leftrightarrow$ tiling of a Coxeter polytope by Coxeter polytopes.

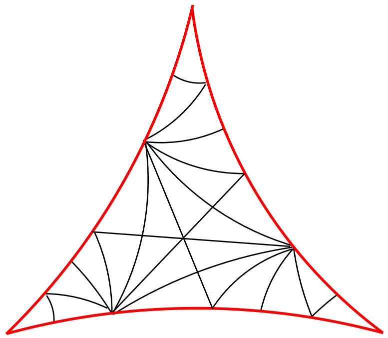
Reflection subgroups of reflection groups

Example:



$(\frac{\pi}{7}, \frac{\pi}{7}, \frac{\pi}{7})$ tiled by 24 copies of $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{7})$

Reflection subgroups of reflection groups

Example:  $(\frac{\pi}{7}, \frac{\pi}{7}, \frac{\pi}{7})$ tiled by 24 copies of $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{7})$

Example: A reflection group generated by $(\frac{\pi}{5}, \frac{\pi}{5}, \frac{\pi}{5})$ has no finite index reflection subgroups.

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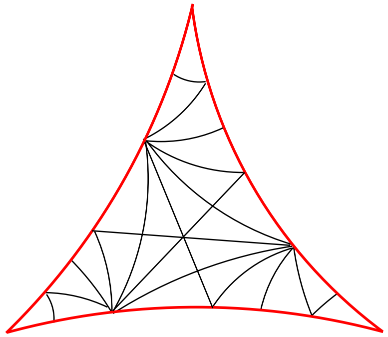
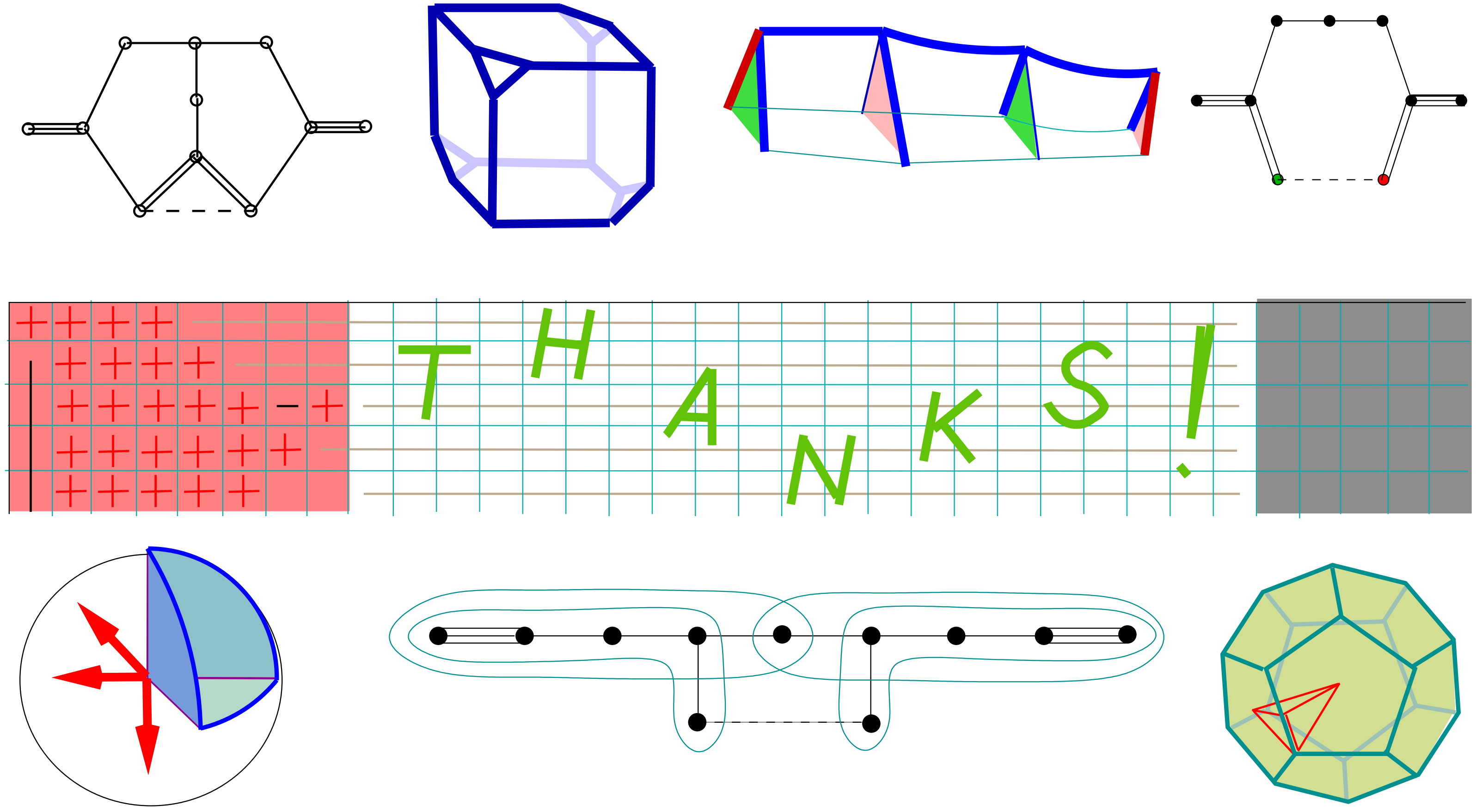
Example:  $(\frac{\pi}{7},\frac{\pi}{7},\frac{\pi}{7})$ tiled by 24 copies of $(\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{7})$

Example: A reflection group generated by $(\frac{\pi}{5},\frac{\pi}{5},\frac{\pi}{5})$ has no finite index reflection subgroups.

Thm. (T,F '03) G infinite indecomposable group, $H \subset G$ a finite index reflection subgroup. Then $rk\ H \geq rk\ G$.

($rk\ G$ is a number of reflections generating G).


THANKS!