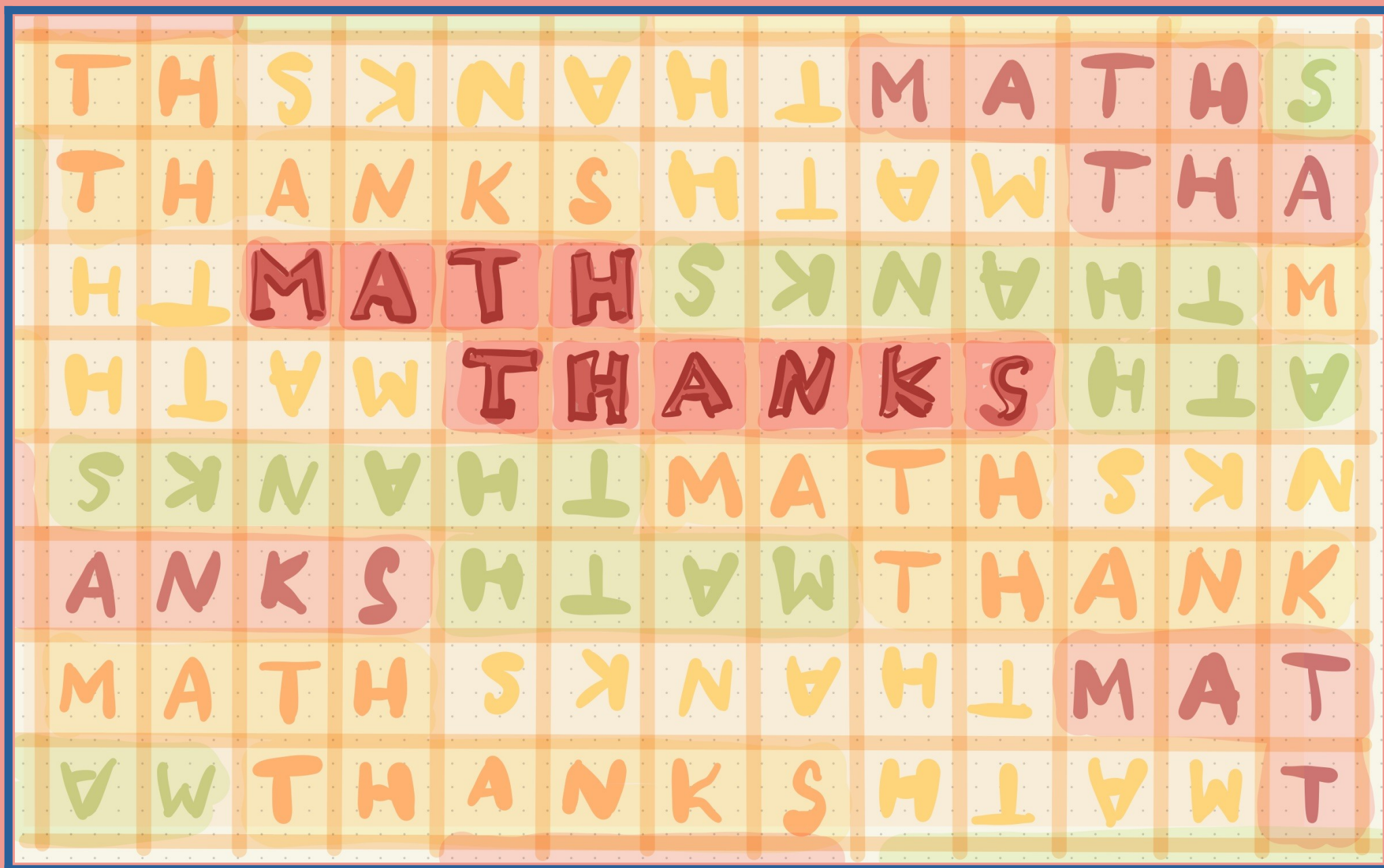




1. “Thanks slides” in Teaching Geometry online (2020-21)



1.1. Tilings on the Euclidean plane.



1.2. 3-D Geometry.

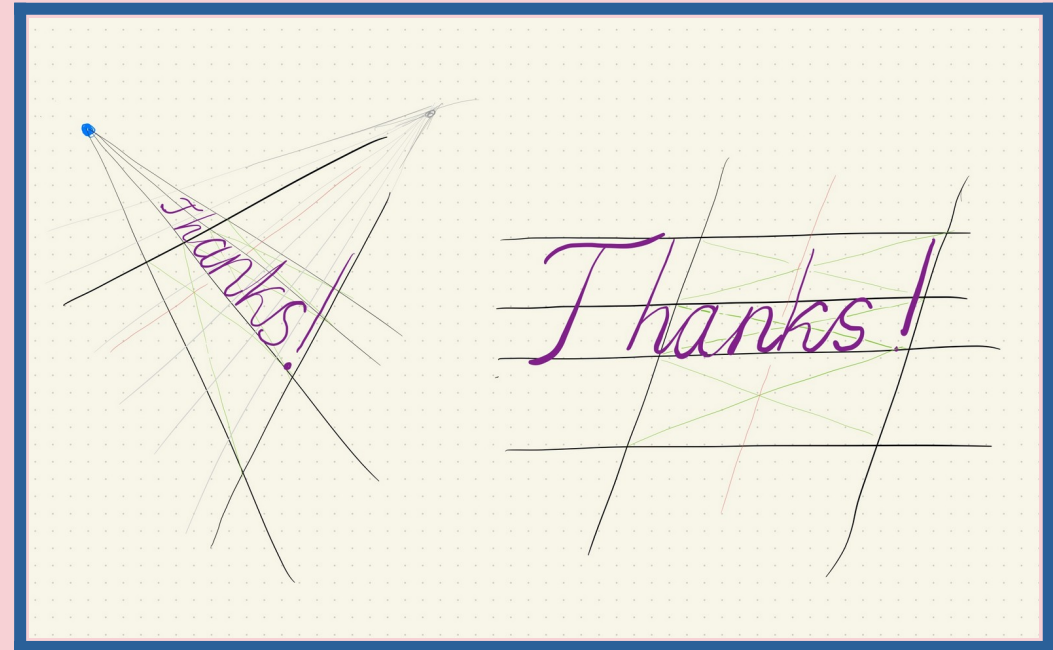
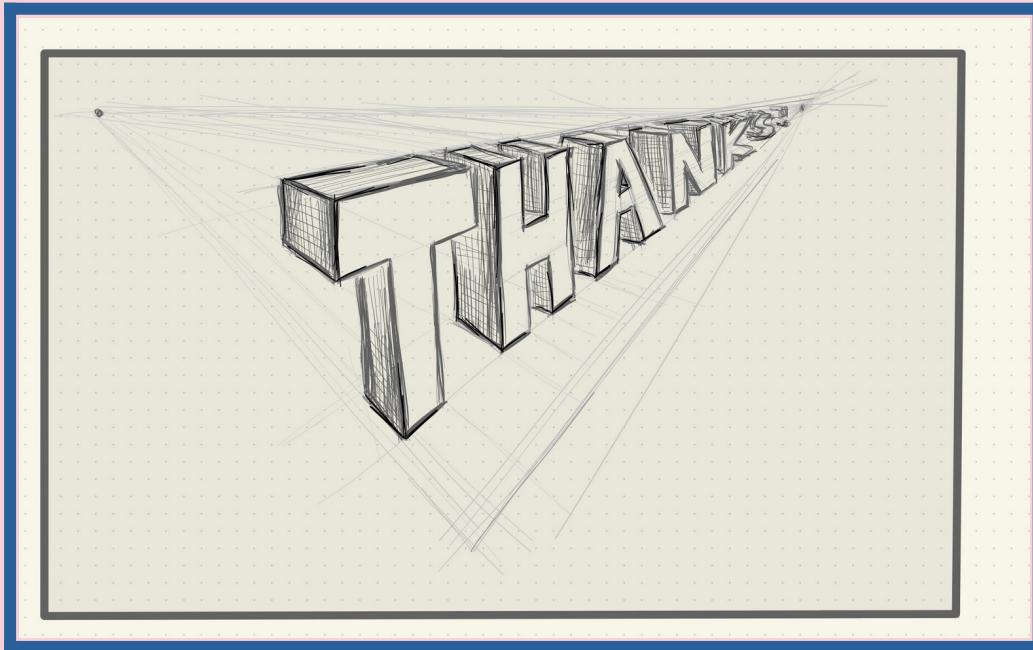
"Three flies are flying in a room. What is the probability they all are in one plane at 10:00?"



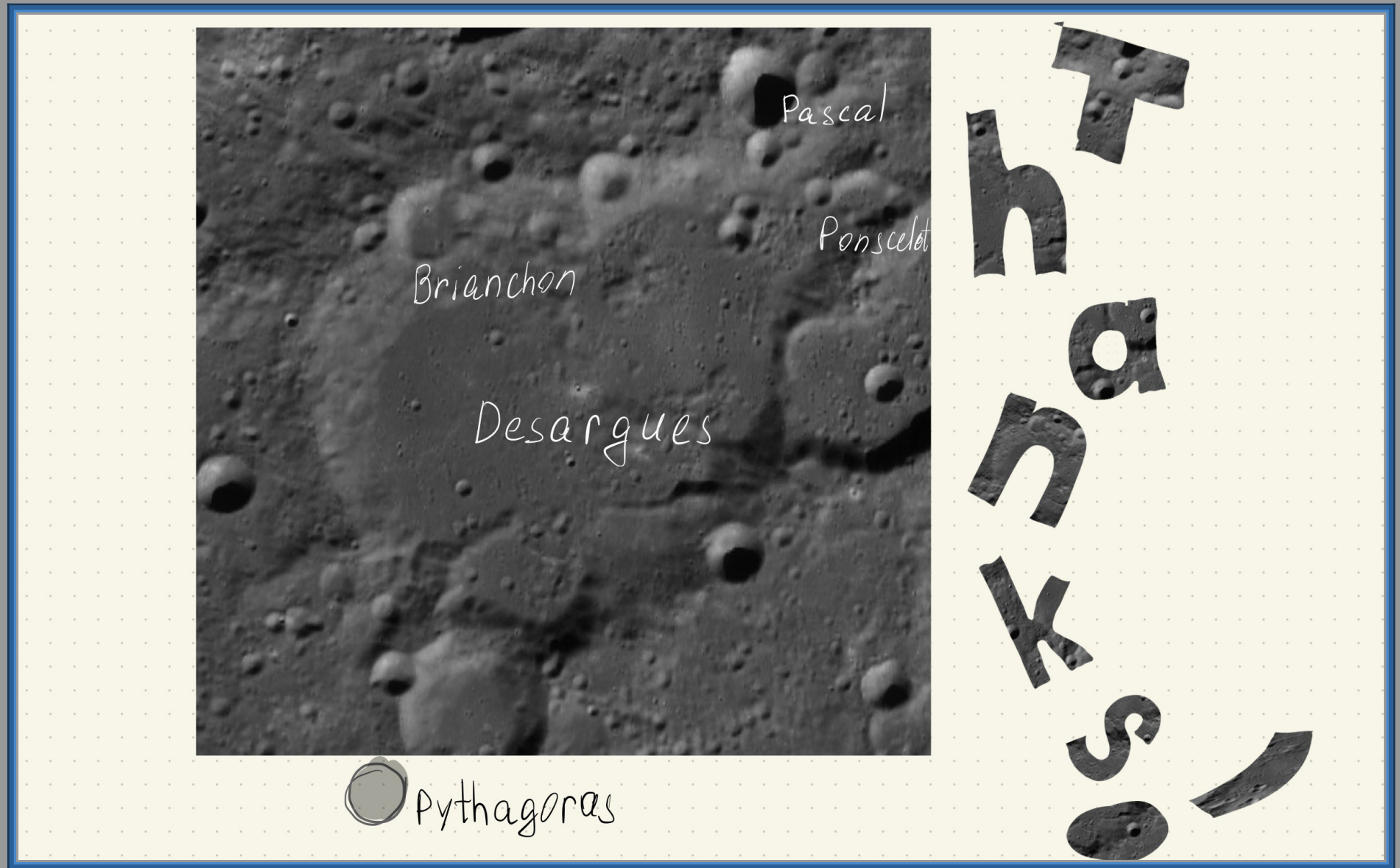
1.3. Discrete group actions and Fundamental domains.
Where to look for examples?



1.4. Affine Geometry.



1.5-1.6. Two lectures on projective geometry.
One can send two points to the line at infinity! Just to simplify the picture...



1.7. Classical theorems of projective geometry:

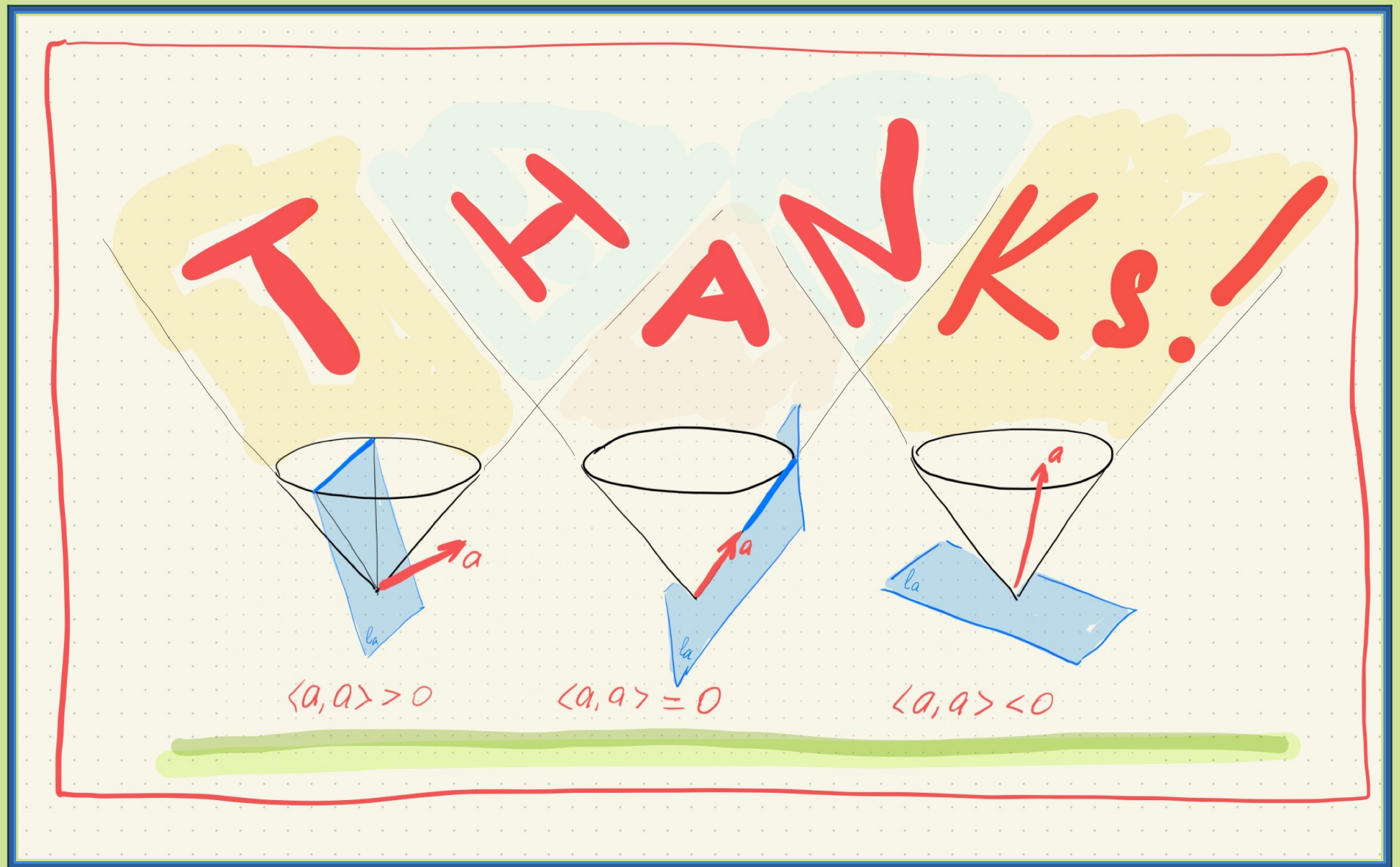
Pascal, Brianchon, Desargues, Poncelet – these all are the names of Lunar craters!



1.8. Hierarchy of Geometries.



1.9. Möbius transformations:
attractors and repellers.

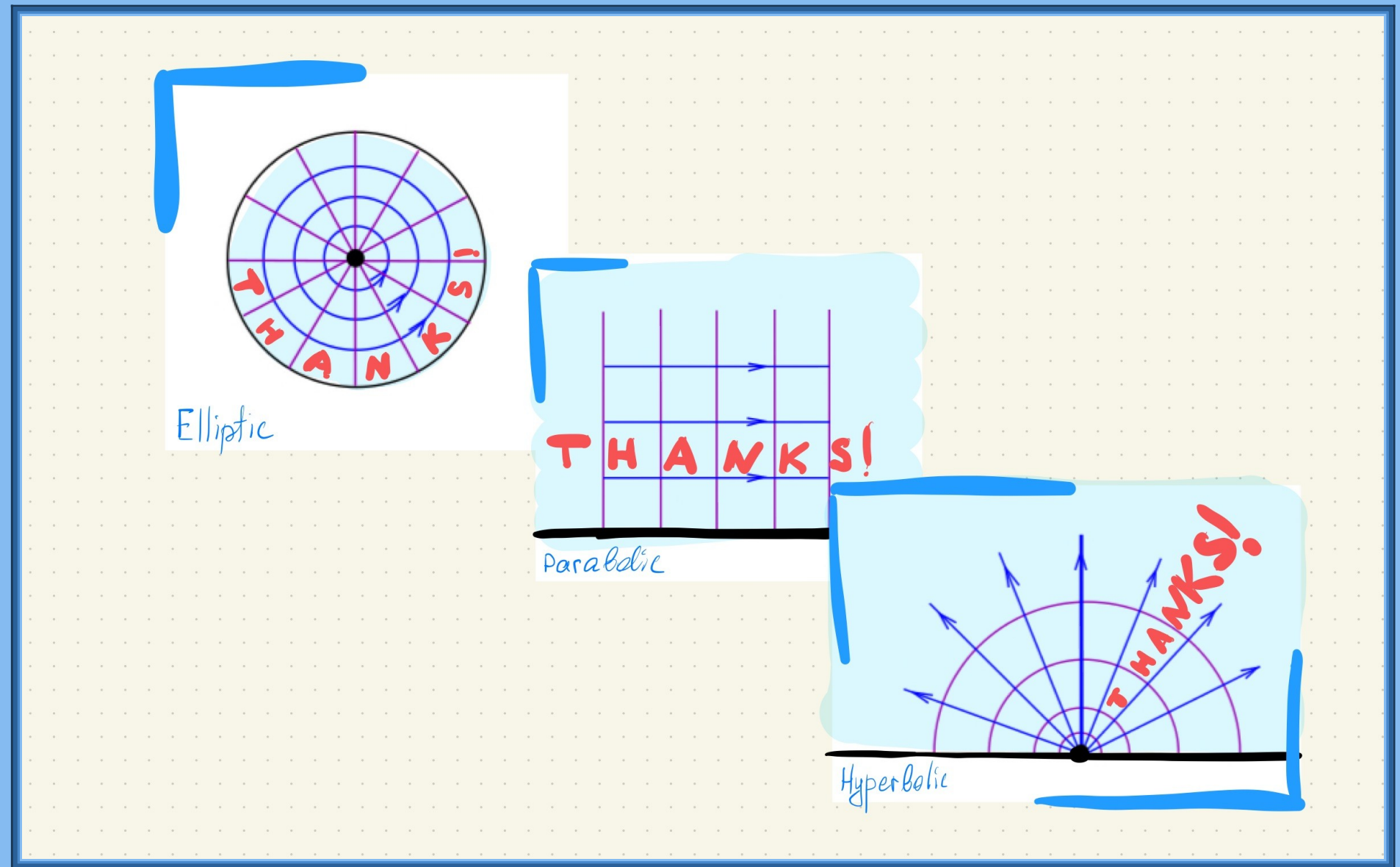


1.10. Vector model of hyperbolic plane:
vectors and pseudo-orthogonal planes.

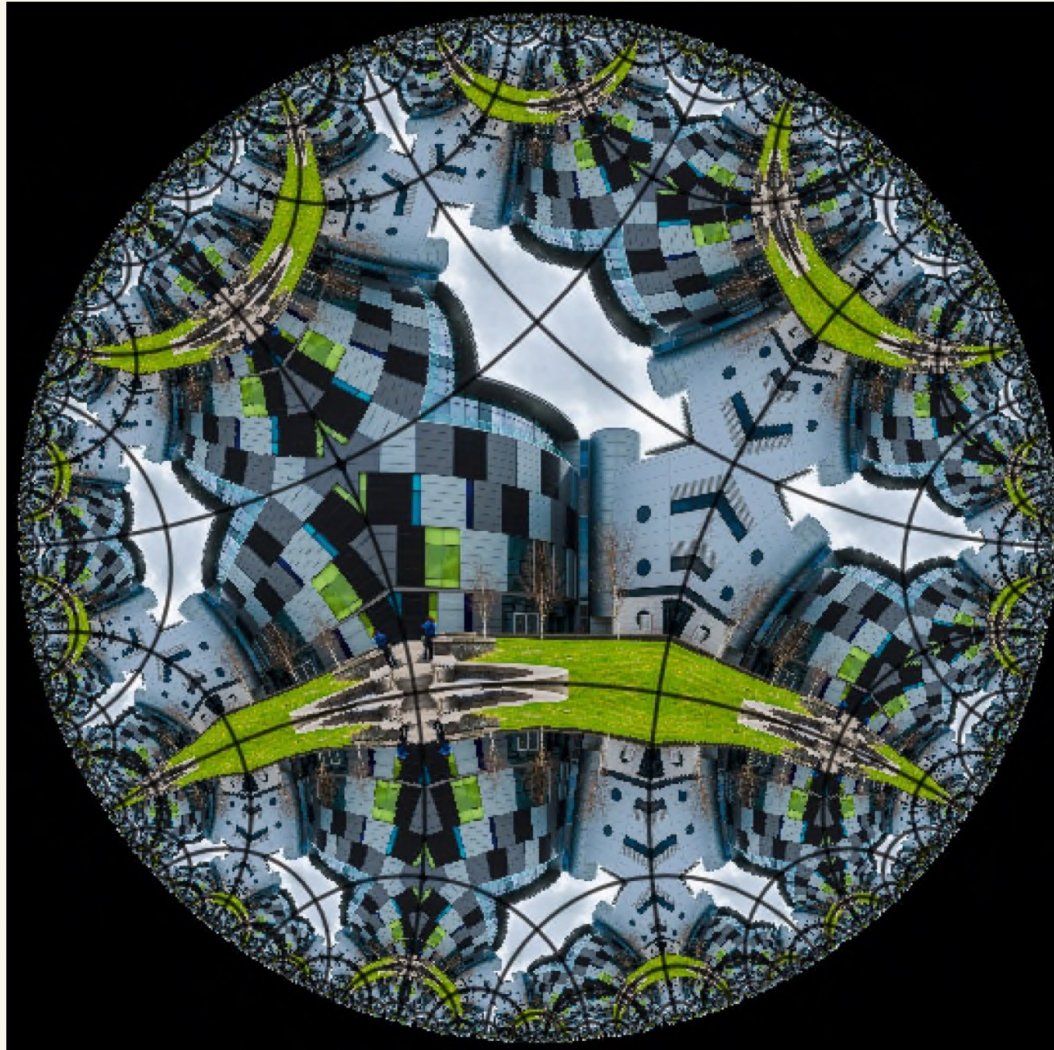


1.12 Hyperbolic geometry in the Poincaré disc:

Is it true that the perpendicular bisectors in a hyperbolic triangle have a common point?

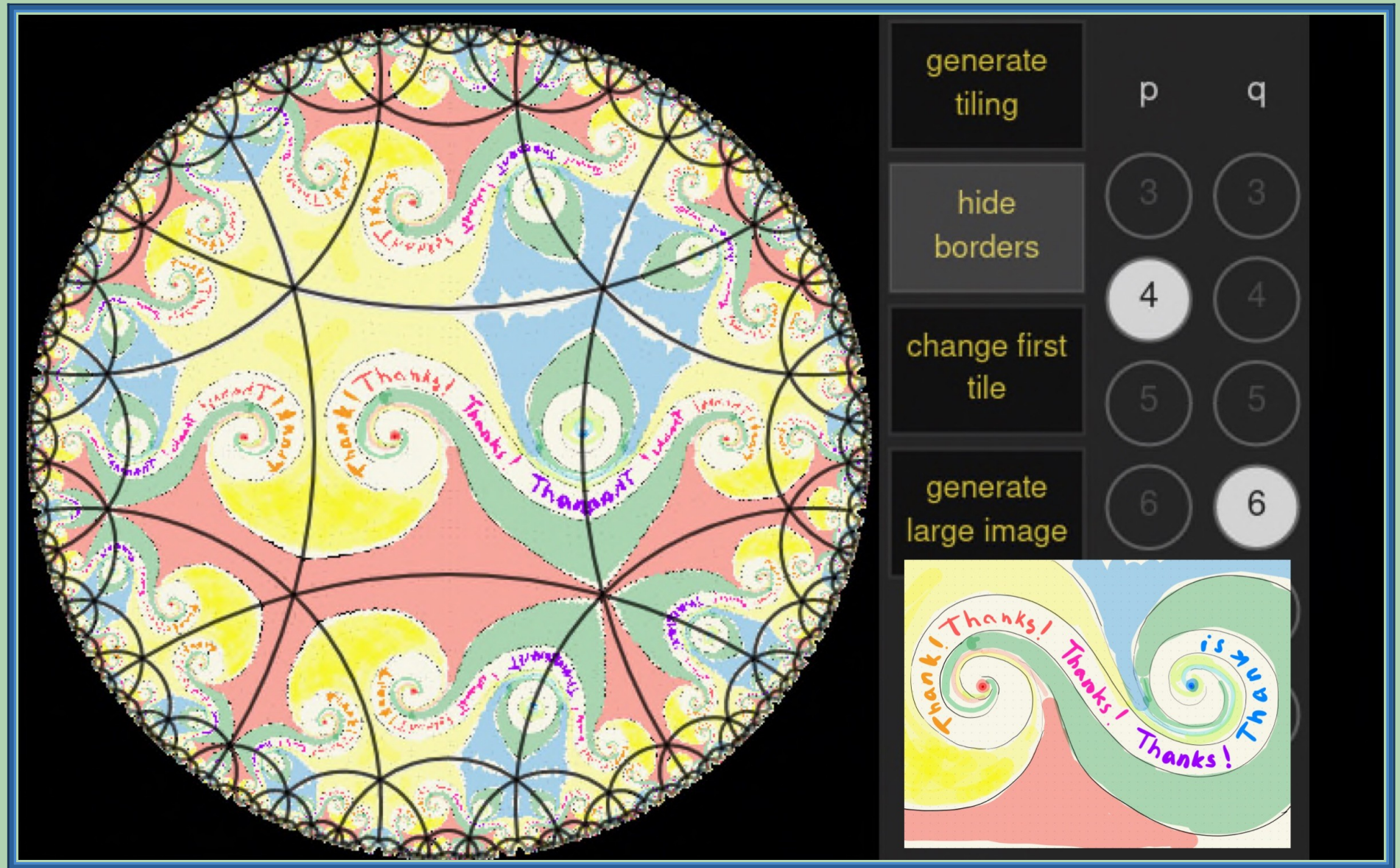


1.12. Types of orientation-preserving isometries on the hyperbolic plane.



1.13. Equidistant curves look like banana!
(which of the two pictures contains more banana?)

Image produced with Malin Christersson's applet
<https://www.malinc.se/m/ImageTiling.php>

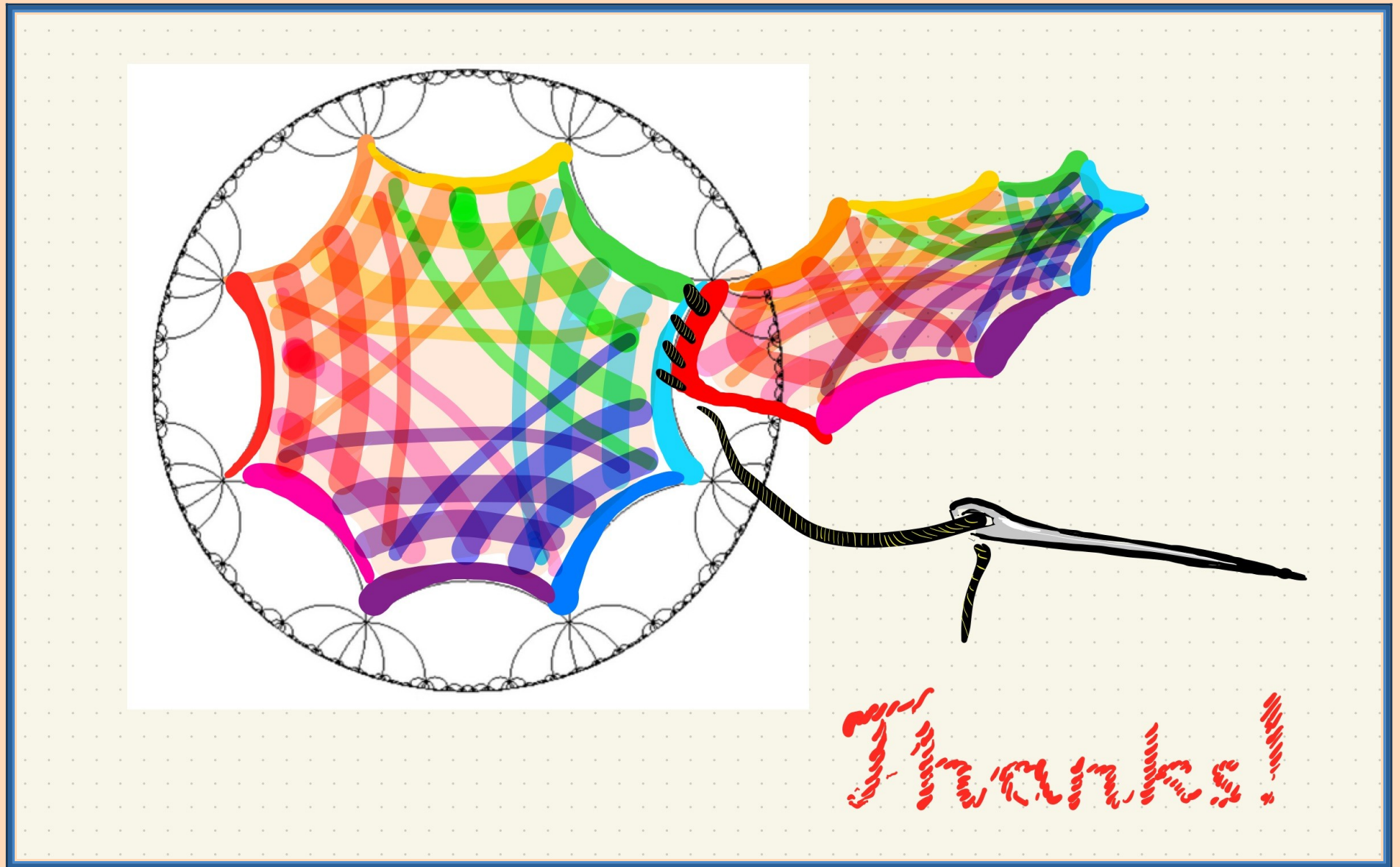


1.14. Tilings on the hyperbolic plane.

Image produced with Malin Christer's applet
<https://www.malinc.se/m/ImageTiling.php>

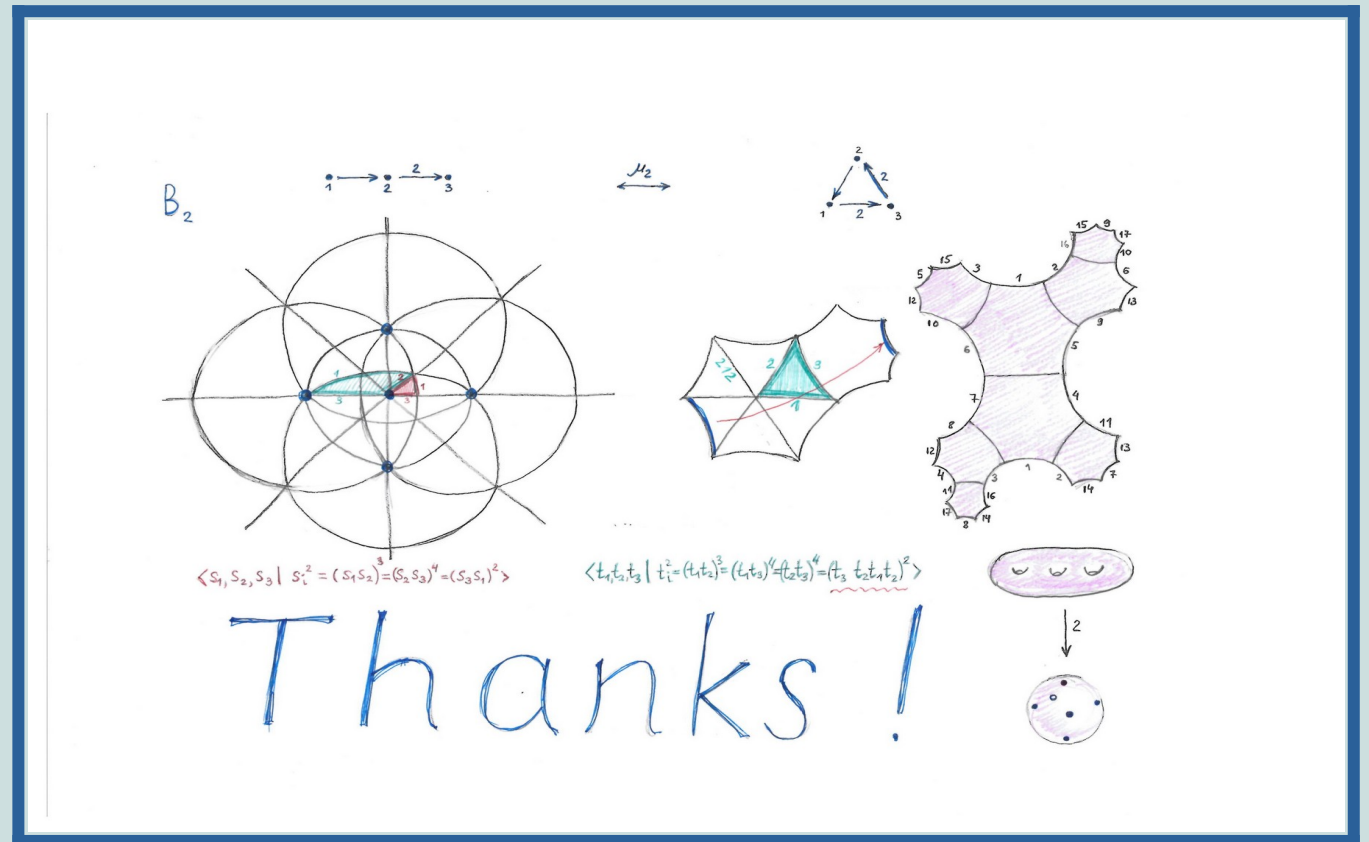


1.15. Comparing geometries...

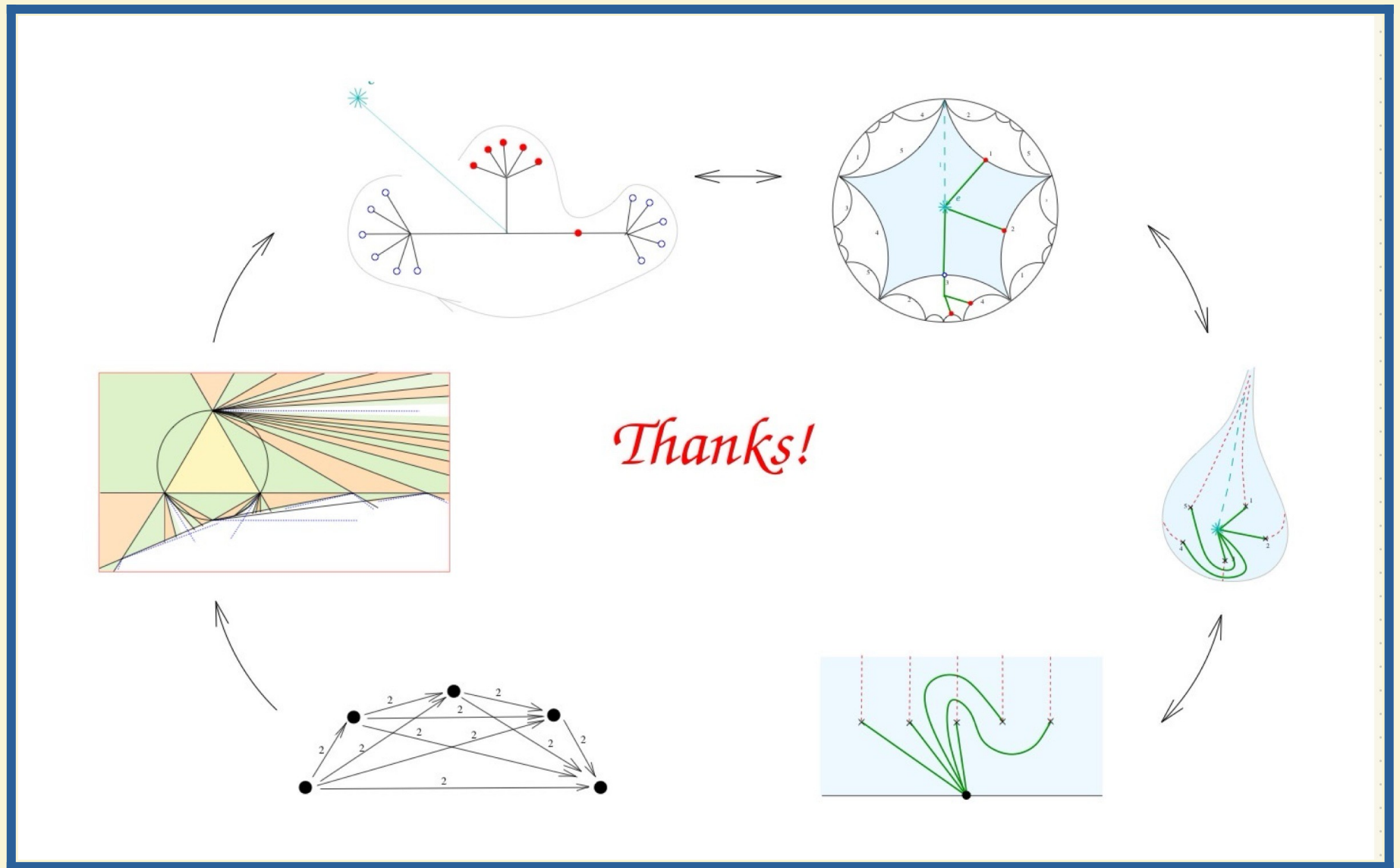


- 1.16. Poincaré theorem:
Discrete group action defined by the side pairing in the octagon.

2. “Thanks pages” from research talks

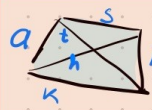
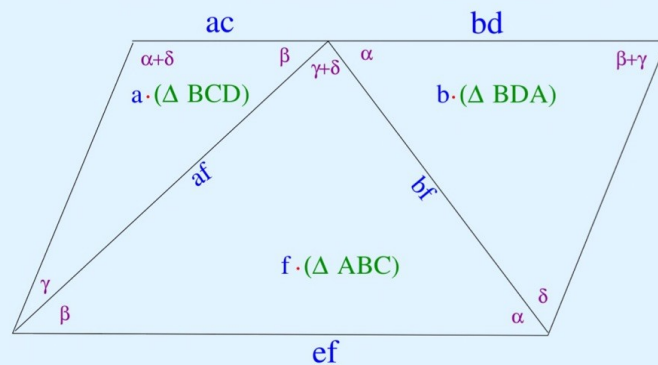
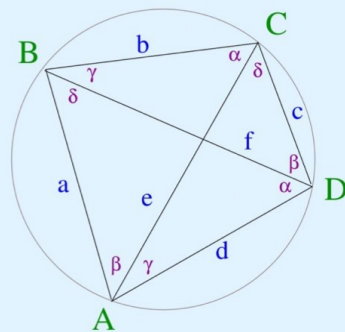


2.1. Coxeter groups, quiver mutations and hyperbolic manifolds.

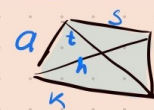


2.2. Quiver mutations, reflection groups and curves on punctured disc.

⑥ Back to Ptolemy: Proof without words



$$th = an + ks!$$



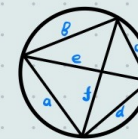
Bonus:

For non-cyclic ABCD
 $ef \leq ac + bd$

Ptolemy spaces =
 Metric spaces
 with this property

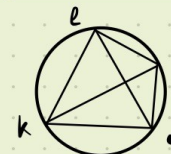
Metric space is
 CAT(0) $\Leftrightarrow$
 Ptolemy + Busemann
 convex

Ptolemy



$$ef = ac + bd$$

Plücker



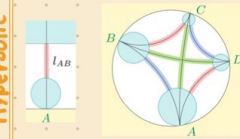
$$p_{ik}p_{jl} = p_{ij}p_{kl} + p_{jk}p_{li}$$

Frieze Patterns



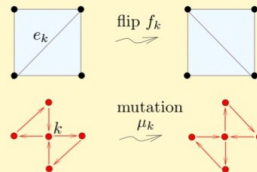
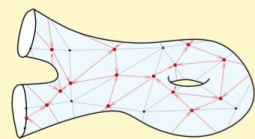
$$ad - bc = 1$$

Hyperbolic



$$\lambda_{AB} = e^{l_{AB}/2}$$

Cluster algebra
 from
 triangulated
 surface



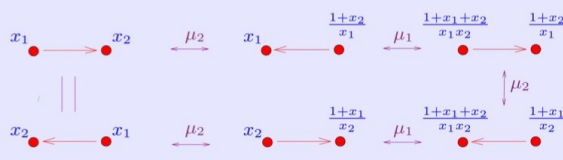
Grassmannian
 cluster algebras

Cluster category

Example 3.

Cluster algebra of type A_2 :
 5 seeds, 5 cluster variables

(the seed after 5 mutations
 coincides with the initial seed
 up to a permutation of the vertices)

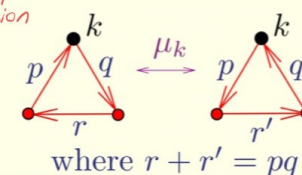


Cluster algebra

Exchange relation

$$u'_k = \frac{\prod_{i \rightarrow k} u_i^{b_{ik}} + \prod_{j \leftarrow k} u_j^{b_{kj}}}{u_k}$$

Quiver mutation



2.3. Ptolemy relation and friends.

Ptolemy theorem (on blue), its appearance in Grassmannians, in Conway-Coxeter's friezes and in hyperbolic geometry - all unified in cluster algebras.

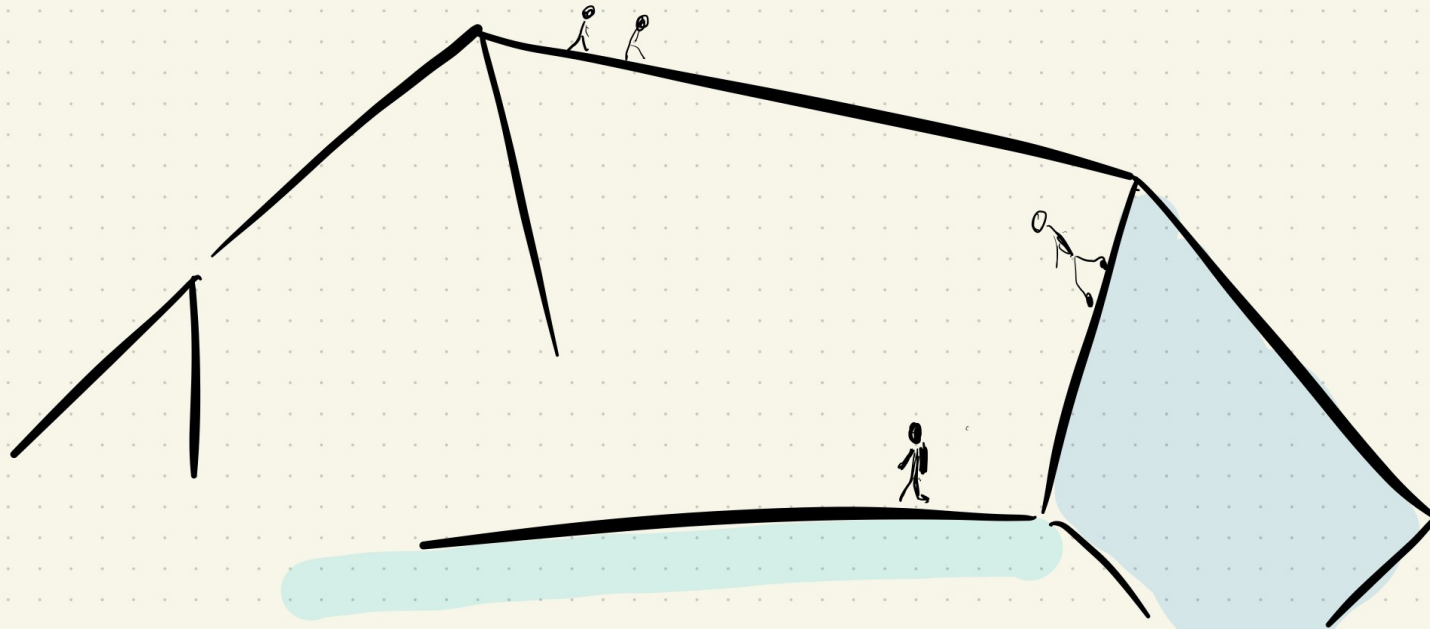
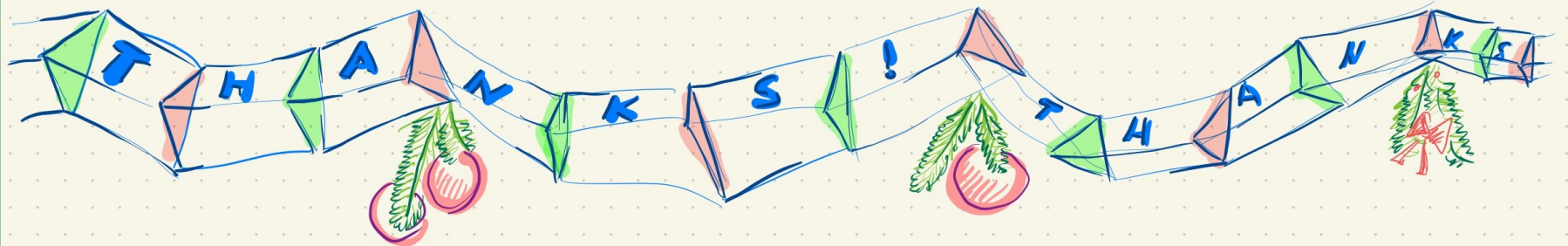


- 2.4. Bases for cluster algebras from orbifolds.
(Bracelet using two orbifold points is drawn – this is an element of the constructed basis).

$d = \dim P$, $n = \# \text{facets}$, $p = \# \text{dotted edges}$

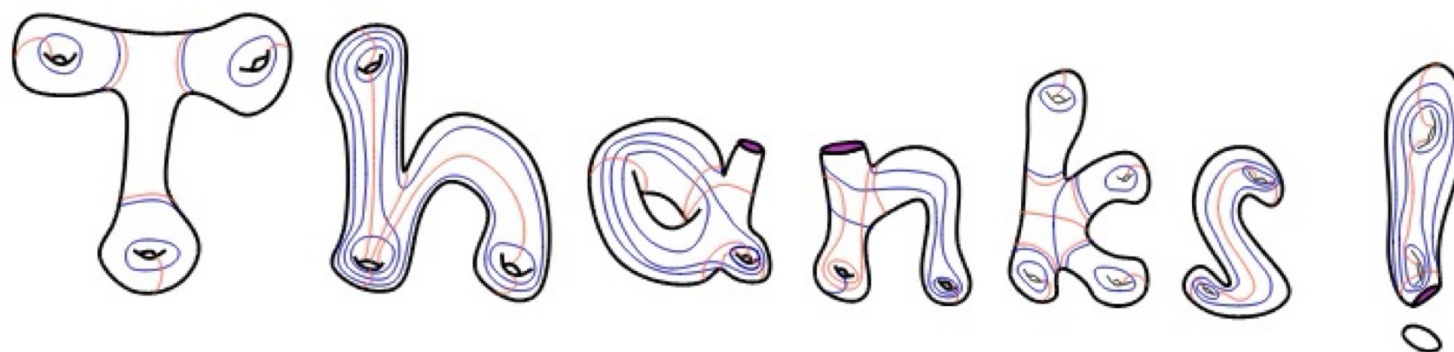
Thm: $|\mathcal{P}| < \infty$, algorithm

$\mathcal{P} = \{P \mid d \geq 4, p \leq n - d - 2\}$

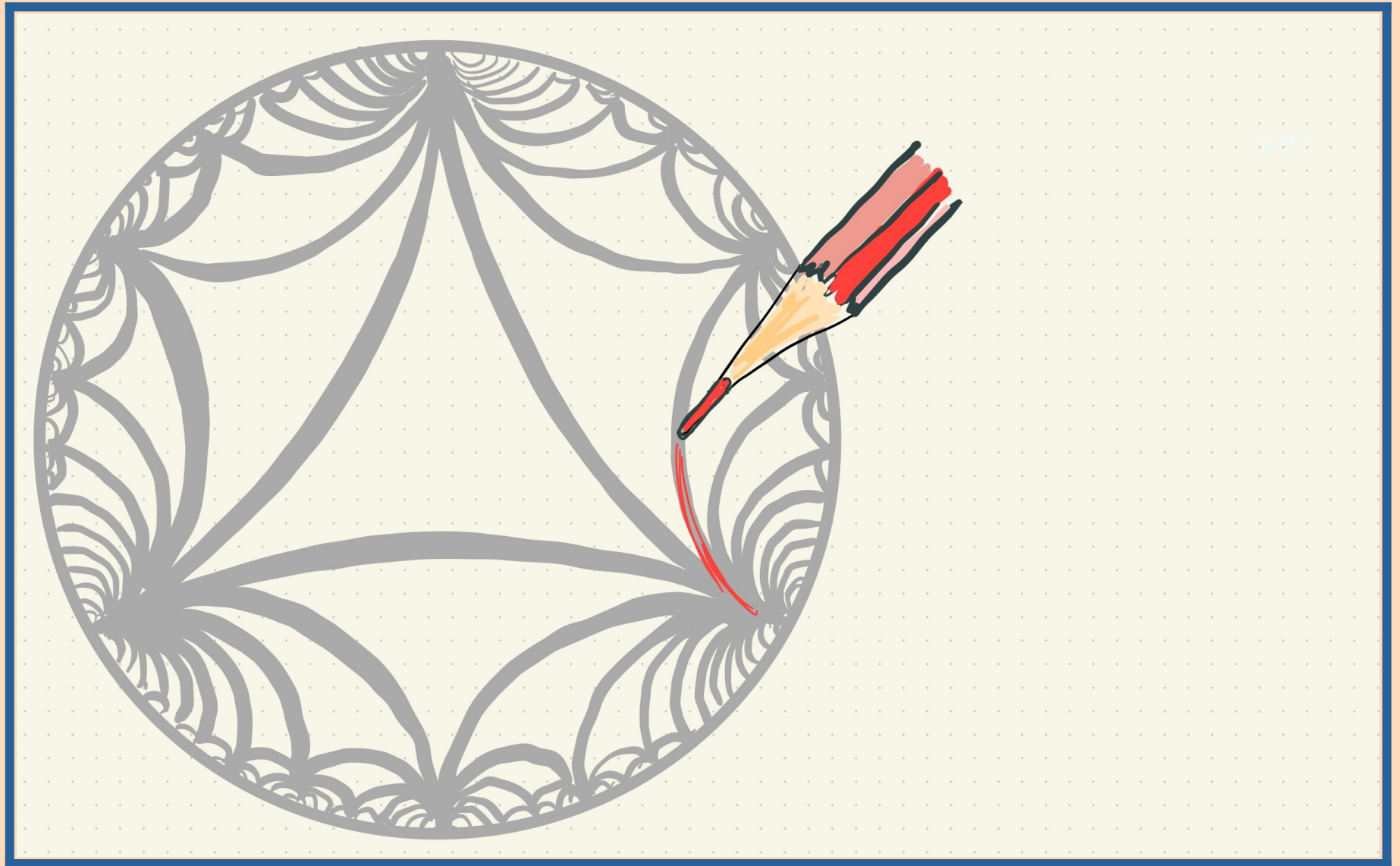


2.5. Hyperbolic Coxeter polytopes.

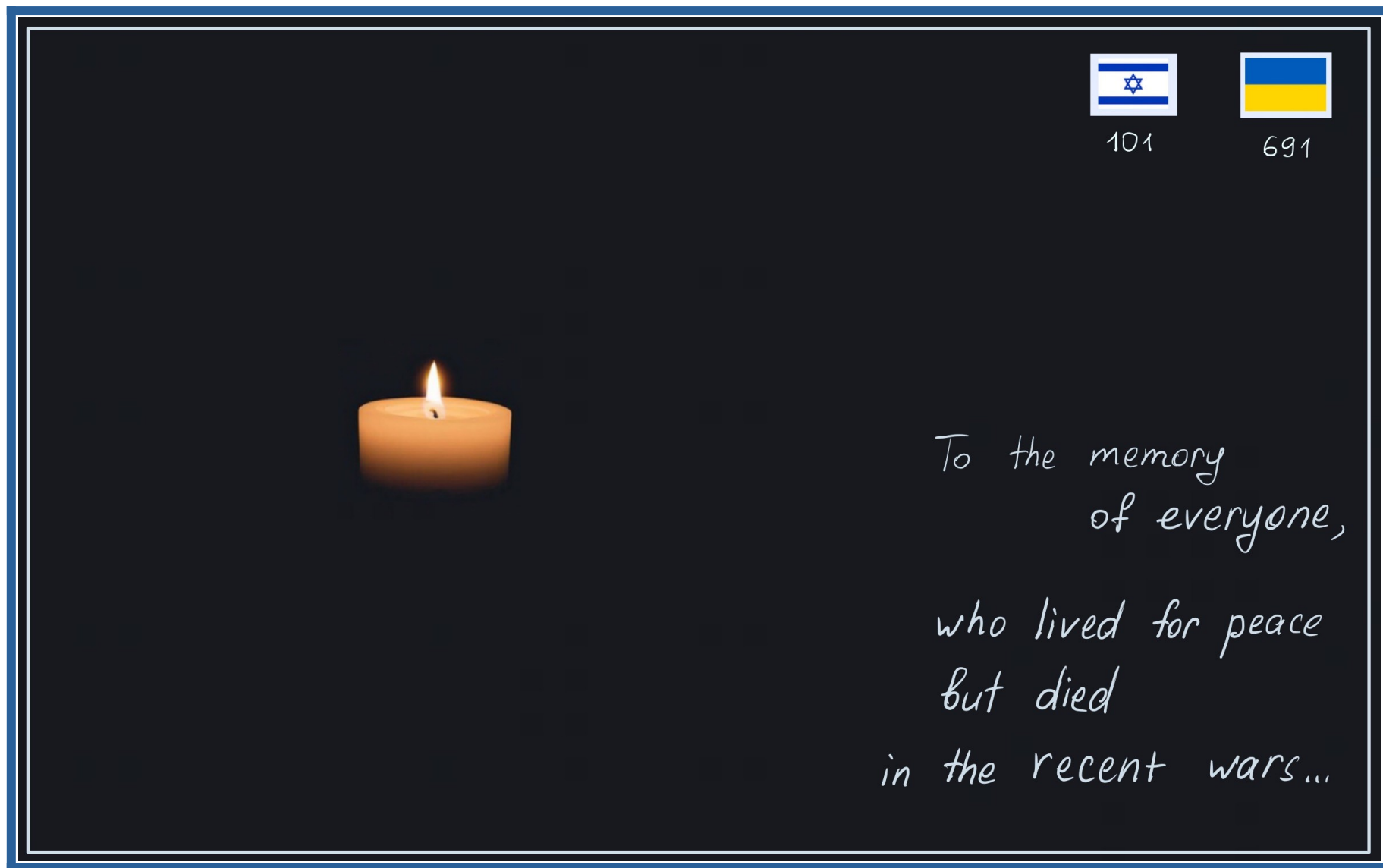
(Makarov's garland and an algorithm listing polytopes with relatively small number of facets).



2.6. Double pants decomposition.
(Surfaces decomposed into 3-holed spheres in two ways).



*** When I was preparing a talk on a 3D generalization of the Farey graph...



*** When I was preparing a talk on a 3D generalization of the Farey graph...
it was 100 days since 07.10.2023. Many killed on 7th October... Hostages still kept in Gaza...
I express my solidarity and support to people of Israel, to people of Ukraine,
and to everyone who is living for peace...

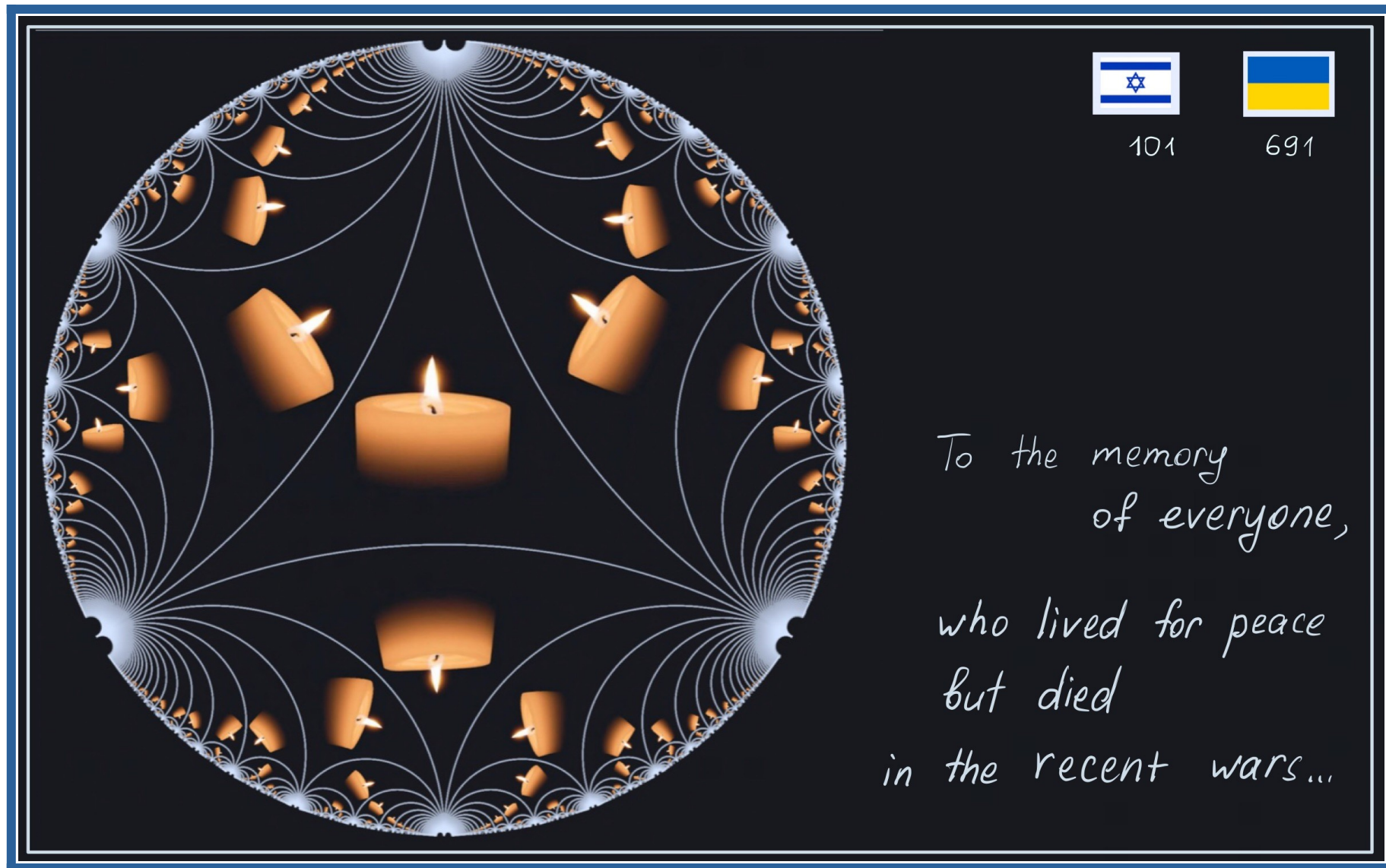


Image produced with Malin Christersson's [applet](https://www.malinc.se/m/ImageTiling.php)

<https://www.malinc.se/m/ImageTiling.php>

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Image produced with Malin Christersson's [applet](https://www.malinc.se/m/ImageTiling.php)

<https://www.malinc.se/m/ImageTiling.php>

*** When I was preparing a talk on a 3D generalization of the Farey graph...
it was 100 days since 07.10.2023. Many killed on 7th October... Hostages are still kept in Gaza...

And just to relieve the stress in the audience before the Tea break



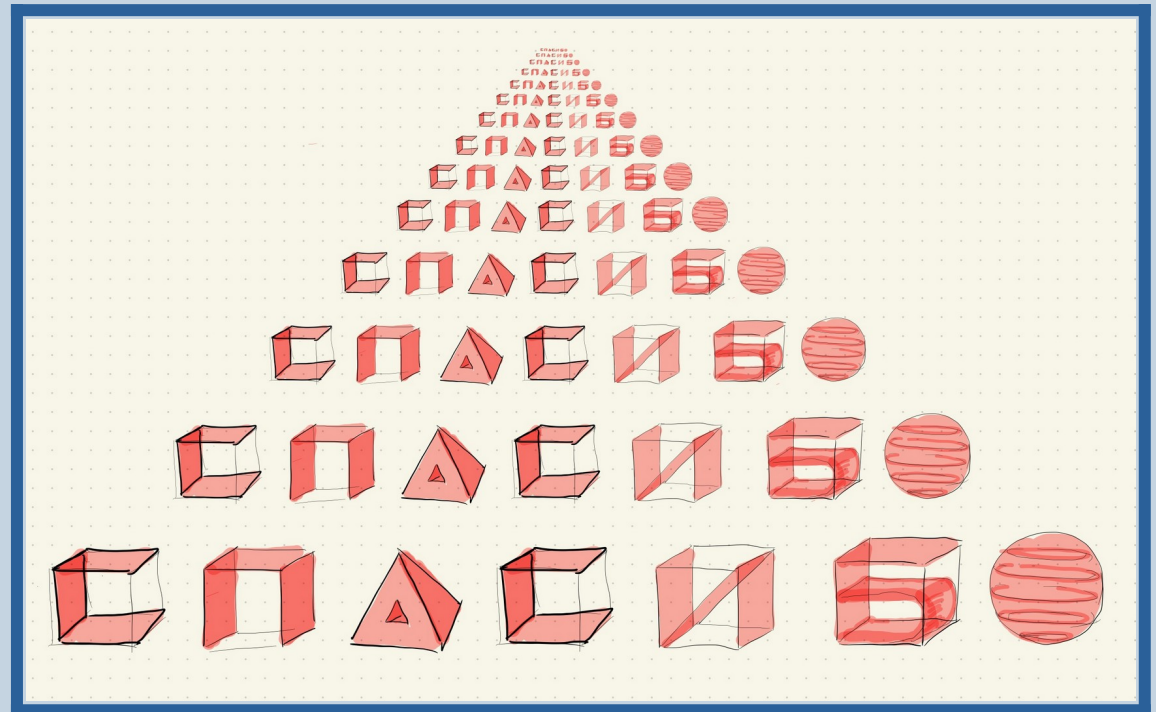
Image produced with Malin Christersson's [applet](https://www.malinc.se/m/ImageTiling.php)
<https://www.malinc.se/m/ImageTiling.php>

2.7. Music of the spheres.
 (Euler's tonnetz, its variations on the Farey graphs, and music on some hyperbolic surfaces...)

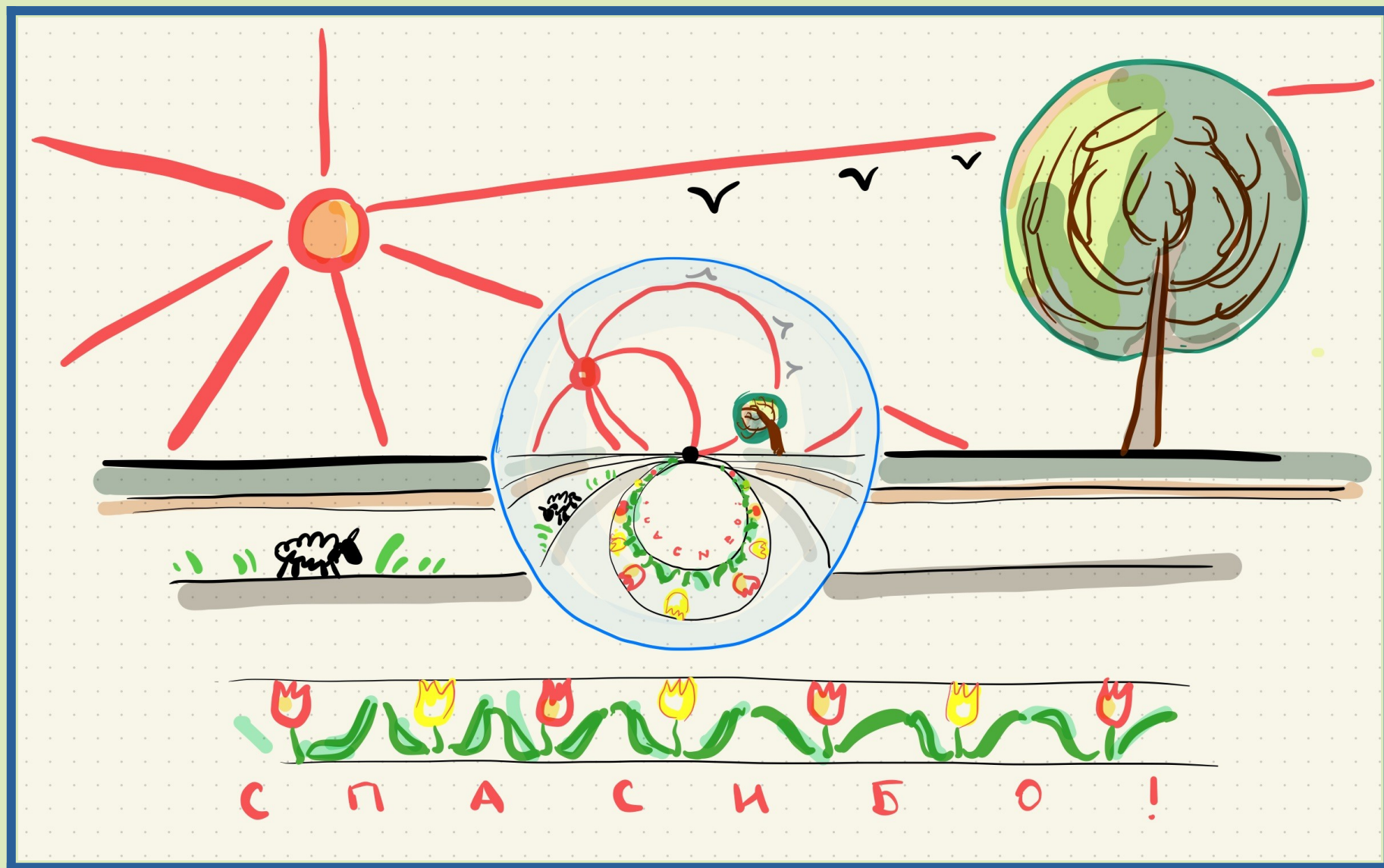
3. “Спасибо” pages from online workshops for Ukrainian kids (2022-...)

“Спасибо” is “Thanks” in Russian.

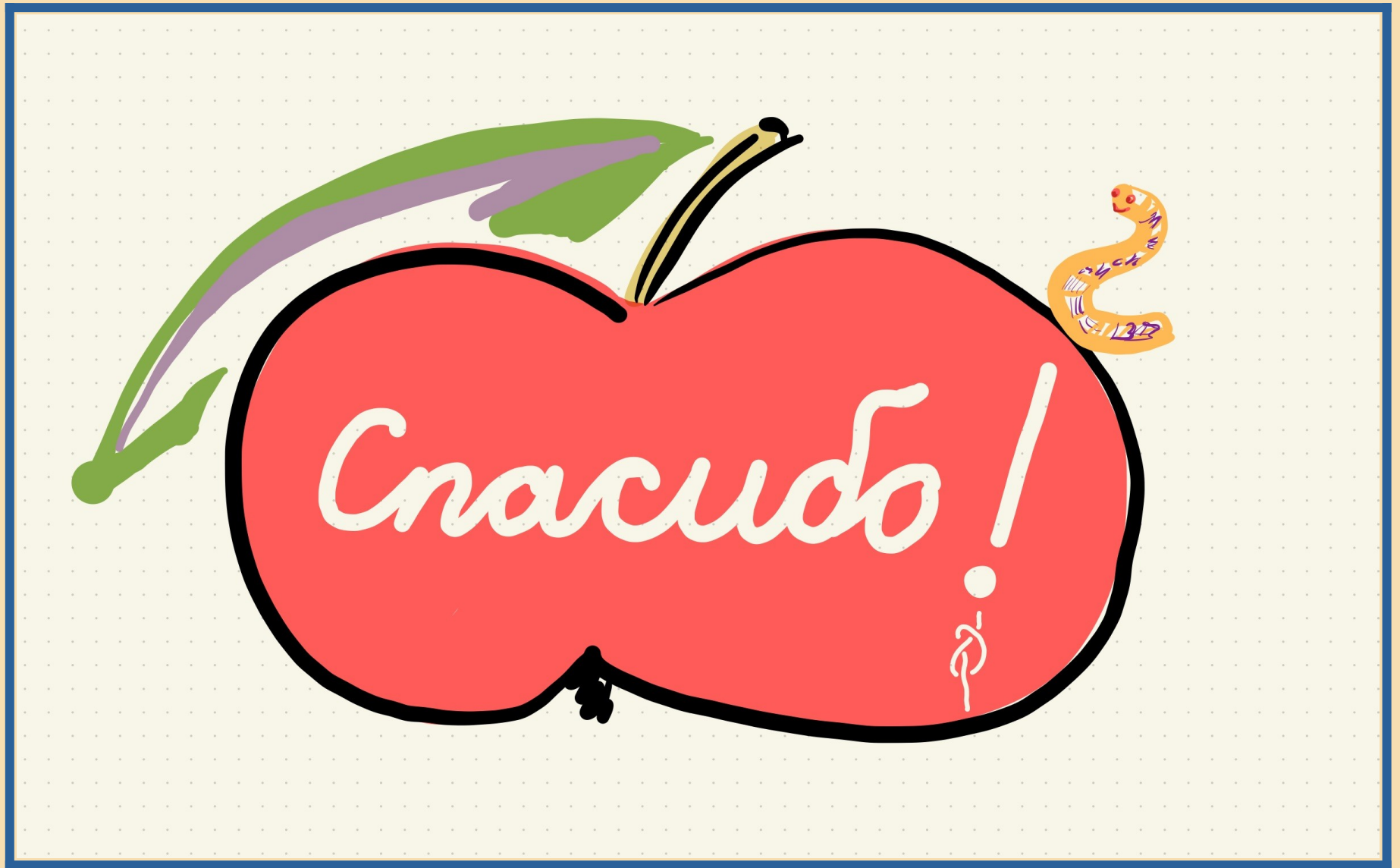
In these workshops with 11-14 years olds we try to enjoy topics of elementary, modern and classical mathematics through problem solving.



3.1. Questions on proportionality and volumes.

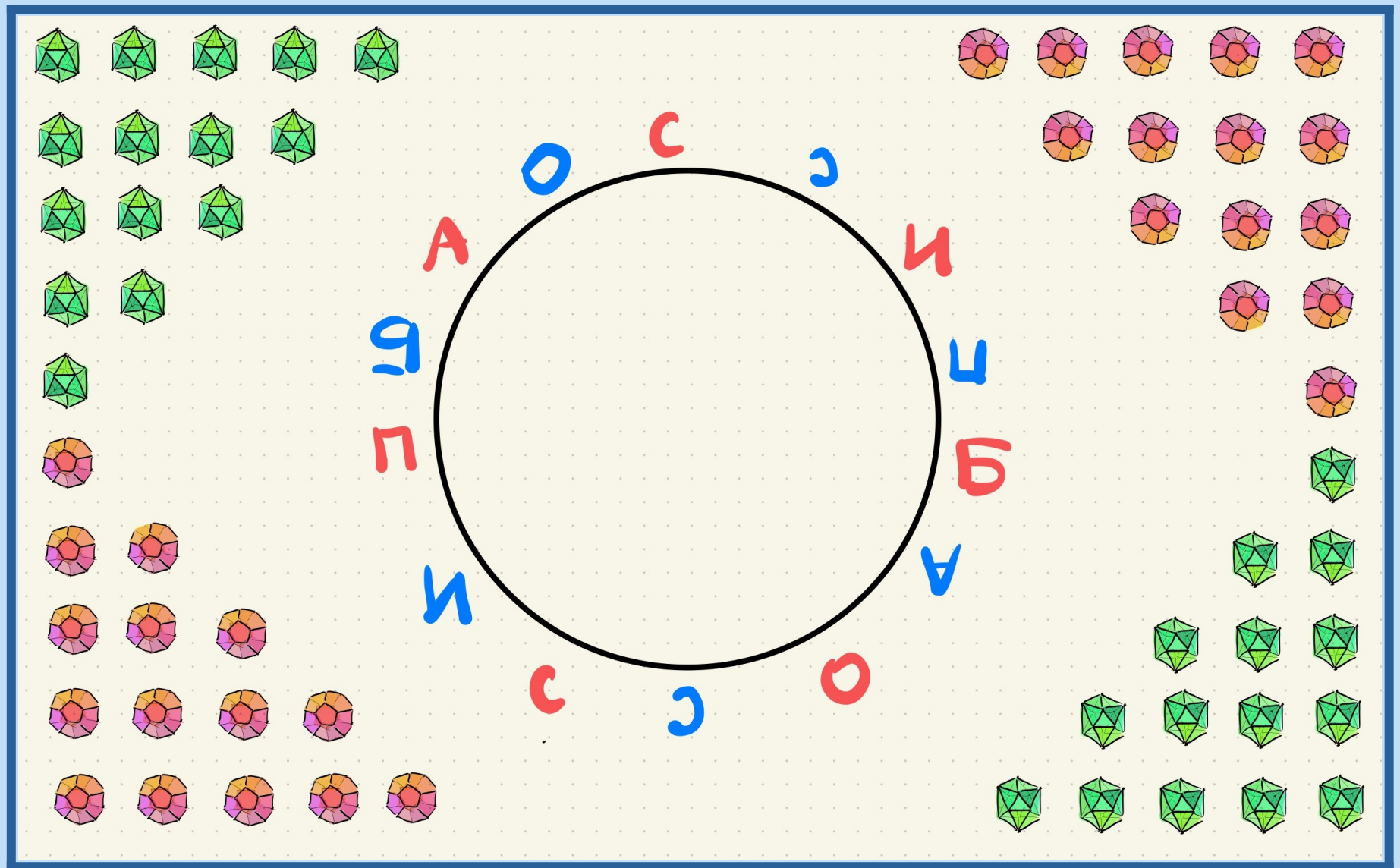


3.2. Inversion.

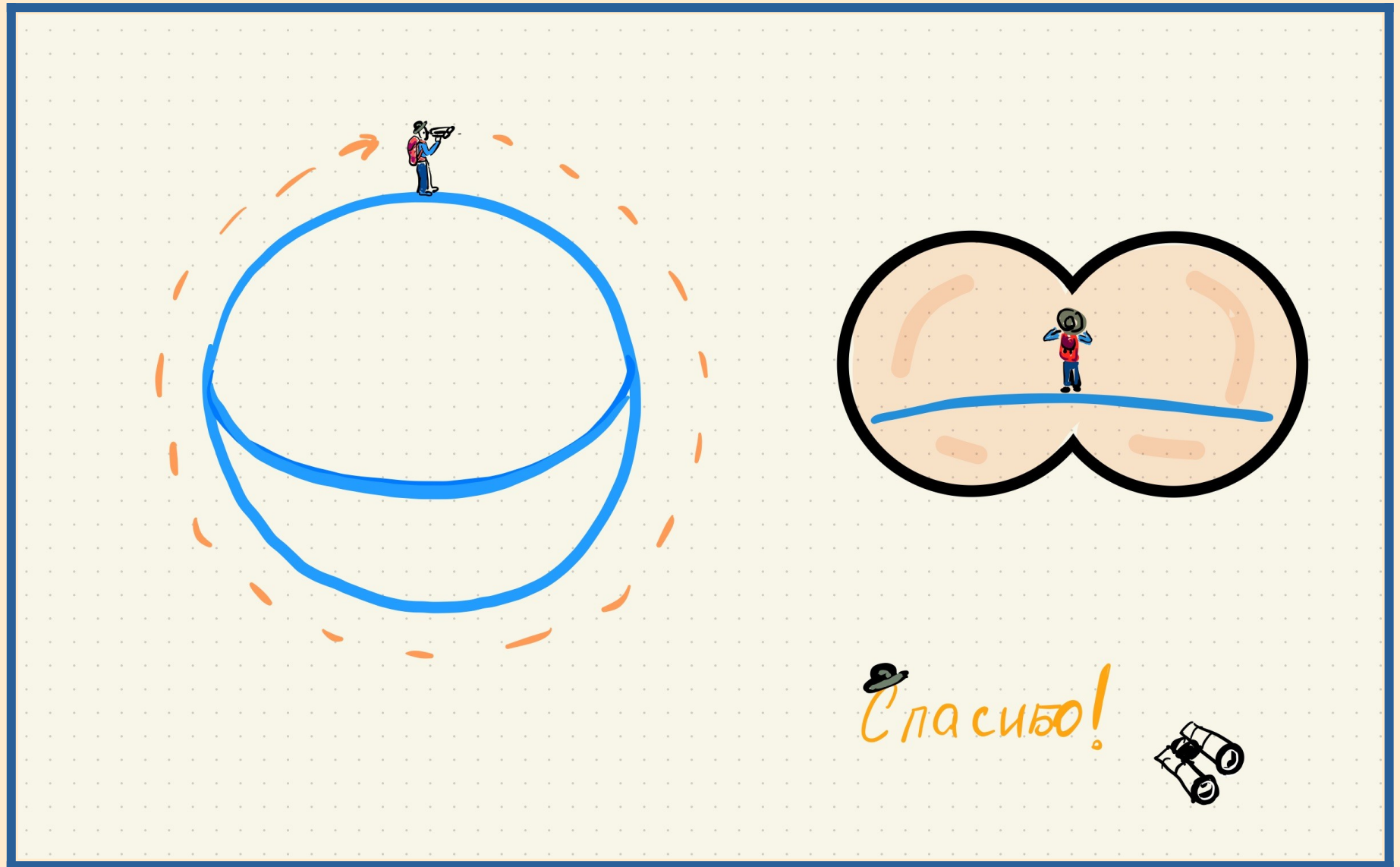


3.3. Topology.

A worm made a knotted tunnel in an apple. What kind of a topological surface we got?



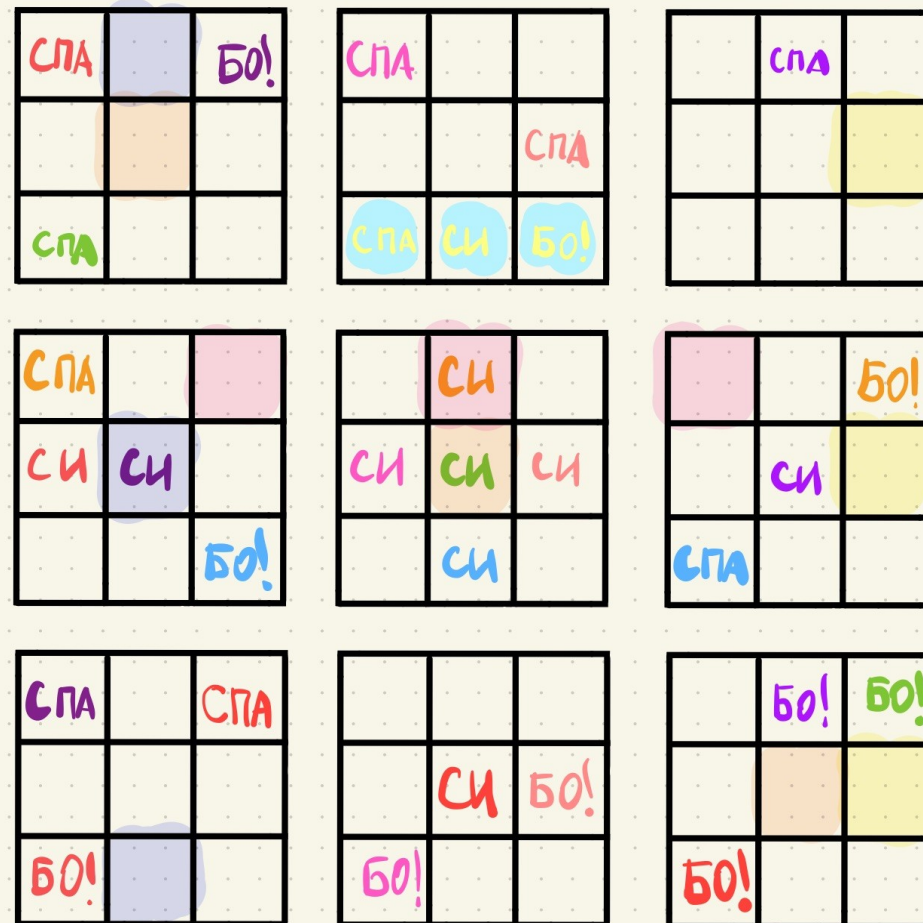
3.4. Mathematical games.
Symmetries.



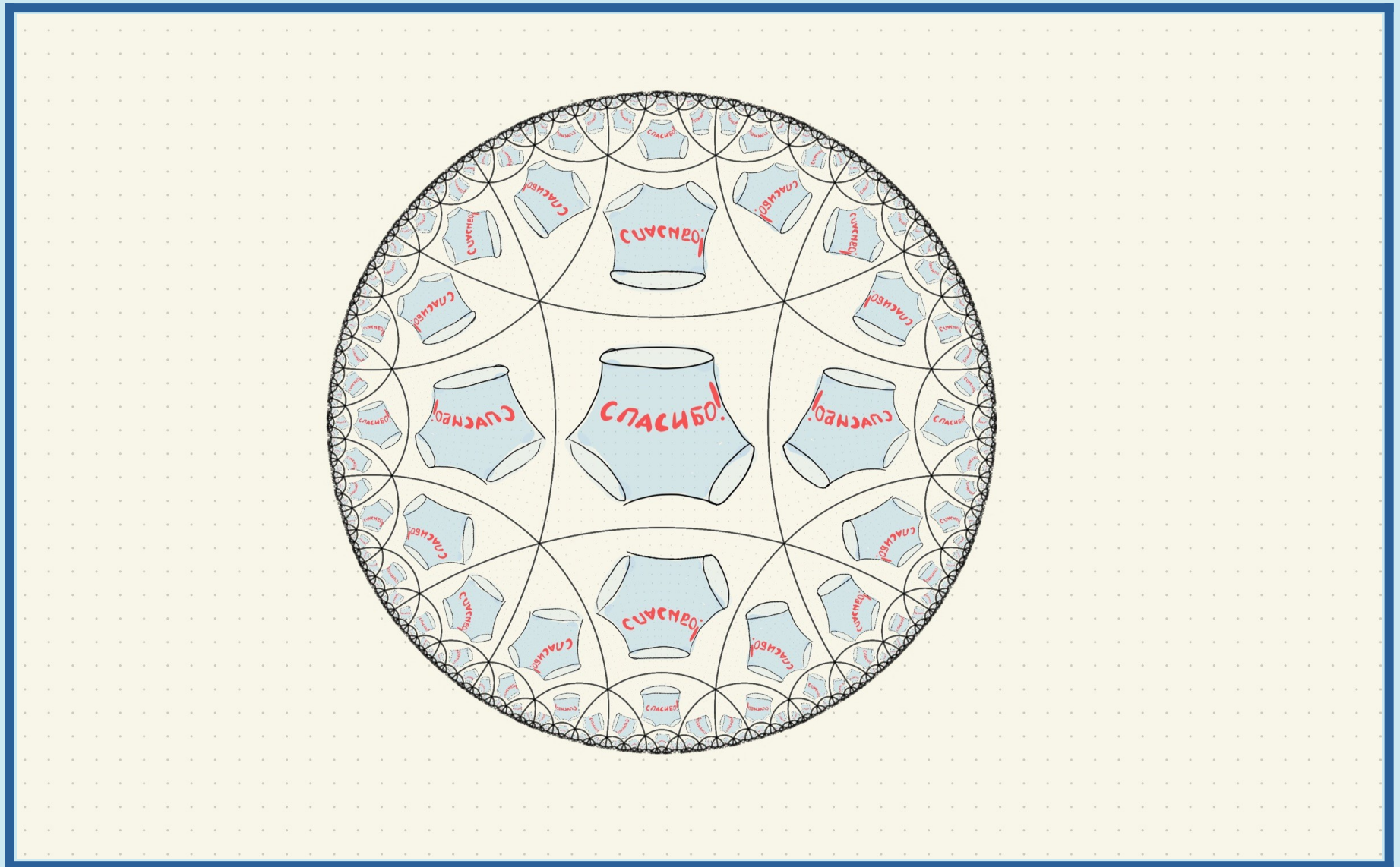
3.5. Spherical geometry.

What would a tourist see in binoculars?

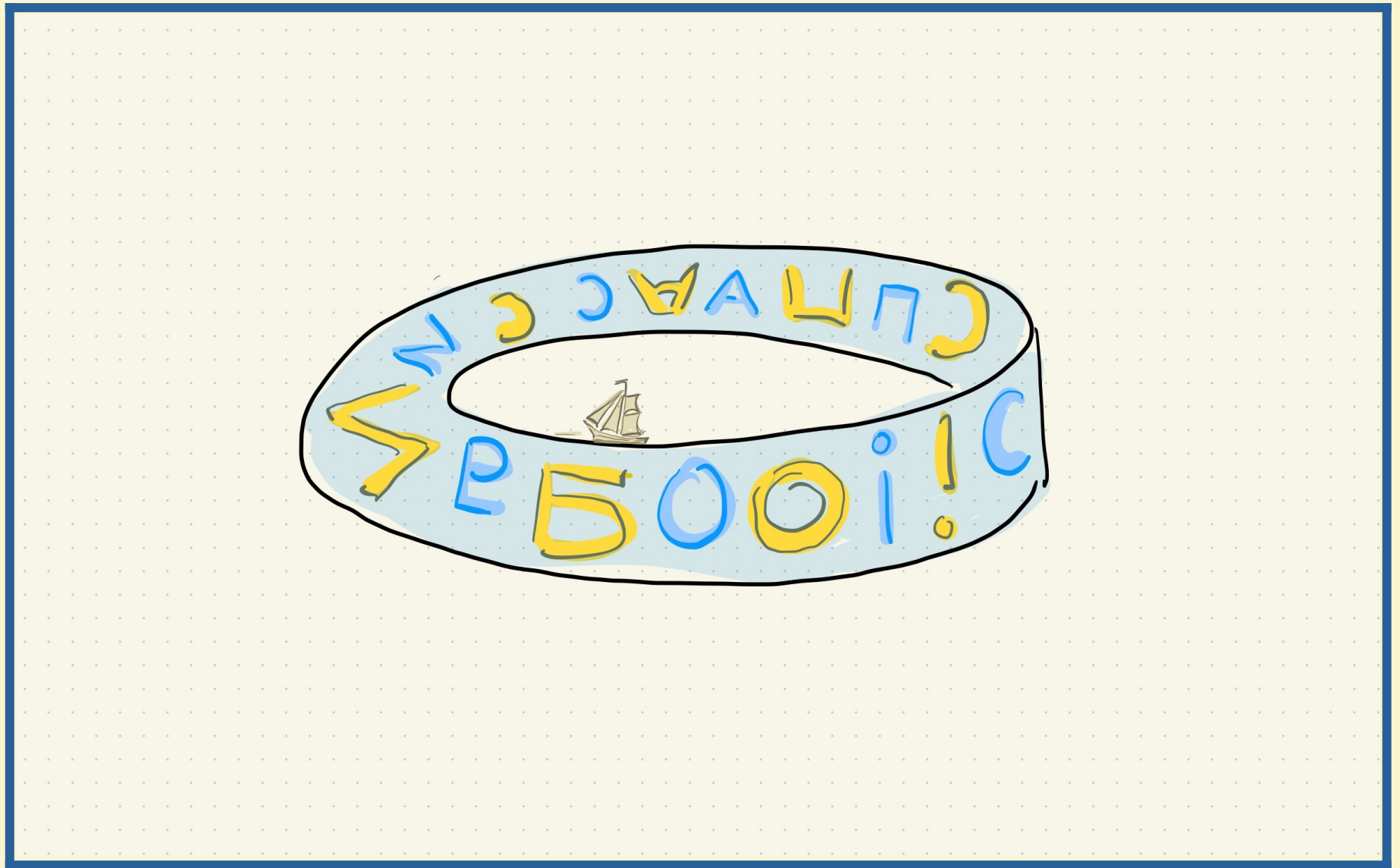
(That day I needed to repeat the same workshop twice back to back for different attendees)



3.6. Higher dimensions.
Tic Tac Toe in 3D.



3.7. Reflection groups in hyperbolic geometry. Hyperbolic surfaces.



3.8. Möbius band.

(Questions about finding pirate's treasures on a plane, torus and Möbius band)

The Kiss Precise

by Fredrick Soddy (Nature **137**, p.1021, 1936)

For pairs of lips to kiss maybe
Involves no trigonometry.
'Tis not so when four circles kiss
Each one the other three.
To bring this off the four must be
As three in one or one in three.
If one in three, beyond a doubt
Each gets three kisses from without.
If three in one, then is that one
Thrice kissed internally.

Four circles to the kissing come.
The smaller are the benter.
The bend is just the inverse of
The distance from the center.
Though their intrigue left Euclid dumb
There's now no need for rule of thumb.
Since zero bend's a dead straight line
And concave bends have minus sign,
*The sum of the squares of all four bends
Is half the square of their sum.*

To spy out spherical affairs
An oscular surveyor
Might find the task laborious,
The sphere is much the gayer,
And now besides the pair of pairs
A fifth sphere in the kissing shares.
Yet, signs and zero as before,
For each to kiss the other four
*The square of the sum of all five bends
Is thrice the sum of their squares.*

The Kiss Precise (generalized)

by Thorold Gosset (Nature **139**, p.62, 1937)

And let us not confine our cares
To simple circles, planes and spheres,
But rise to hyper flats and bends
Where kissing multiple appears,
In n -ic space the kissing pairs
Are hyperspheres, and Truth declares -
As $n + 2$ such osculate
Each with an $n + 1$ fold mate
*The square of the sum of all the bends
Is n times the sum of their squares.*

Уточненный поцелуй

Перевод [Анны Феликсон](#)

Тригонометрию забудь
при поцелуе пары губ.
Но при касании кругов,
учти, закон совсем другой.
Четыре круга, всяк из них
коснулся трех из остальных.
Коснулись все извне – один
вписался трех других среди,
или снаружи от троих
четвертый круг опишет штрих.

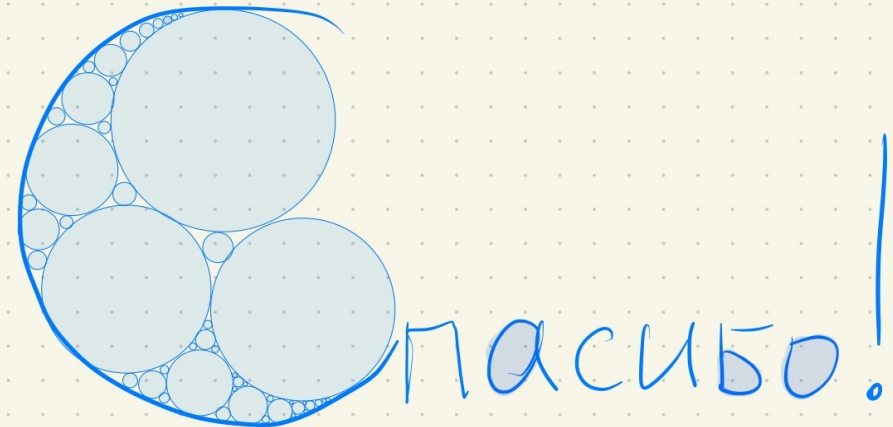
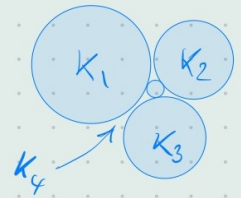
Круг меньший – он кривей иных в том четверном лобзании.
Обратна круга кривизна до центра расстоянию.
Поставим минус если вдруг
вокруг других описан круг;
а кривизна простой прямой
дает в ответе мертвый ноль.
*Сложи квадраты четырех
кривизн, удвоив их, получишь
квадрат от суммы четырех
кривизн.* Евклид не смог бы лучше.

Взаимное касанье сфер
прекрасней всех земных афер,
ведь к поцелую пары пар
присовокуплен пятый шар.
Нули и знак кривизн учти
в касании шаров пяти.
*Квадраты всех пяти кривизн
сложи, утрой и сверь:
квадрат от суммы всех пяти
кривизн этих сфер.*

Теперь заради перемен
представь себе размерность n ,
в ней гипersферы n плюс две,
и неприметно для людей
коснулась каждая из них
 n плюс одной - всех остальных.
*Квадраты n плюс двух кривизн
сложив, на n помножим -
получим сумму всех кривизн
в квадрате непреложно.*

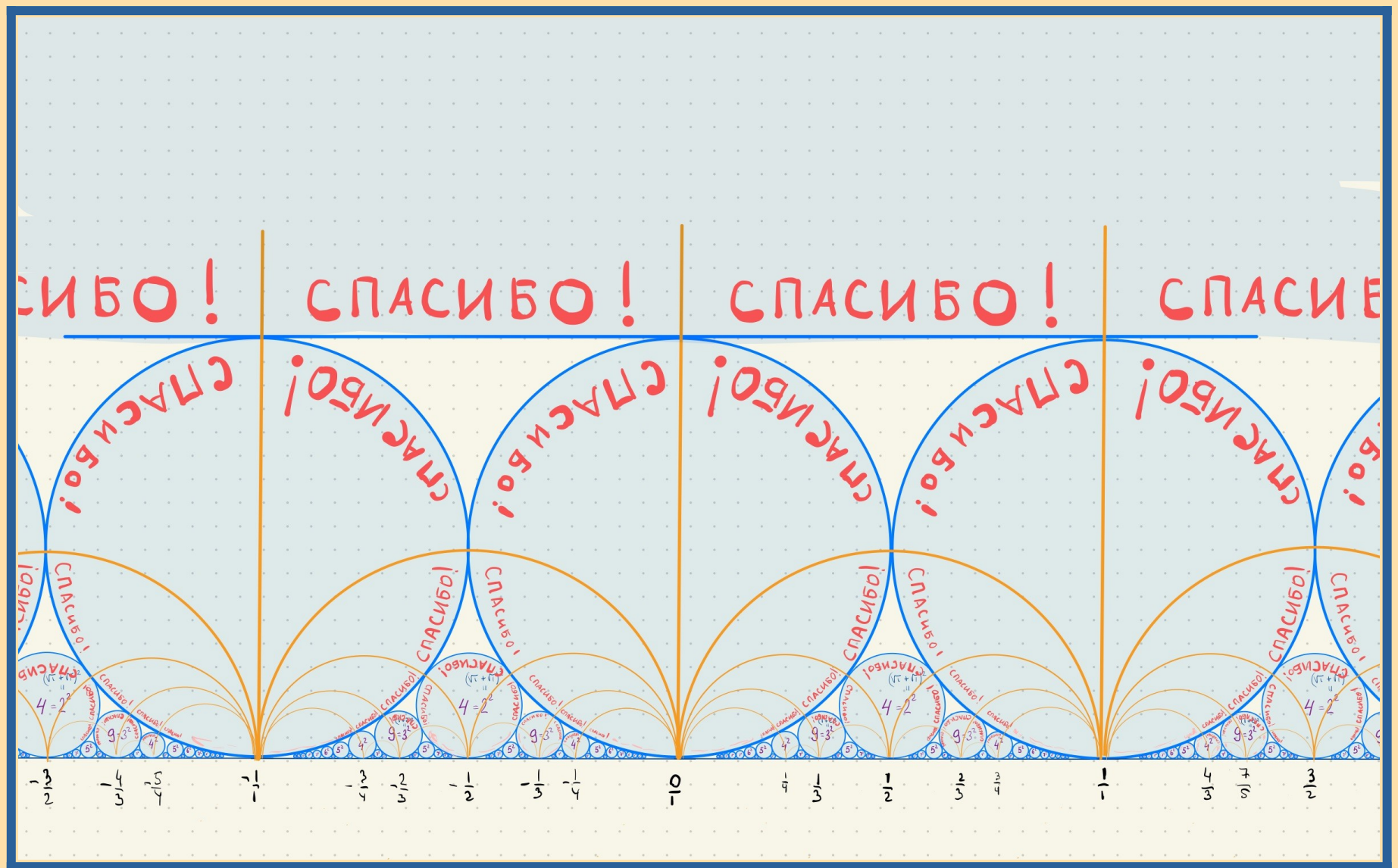
25.01.2024

$$2(k_1^2 + k_2^2 + k_3^2 + k_4^2) = (k_1 + k_2 + k_3 + k_4)^2$$

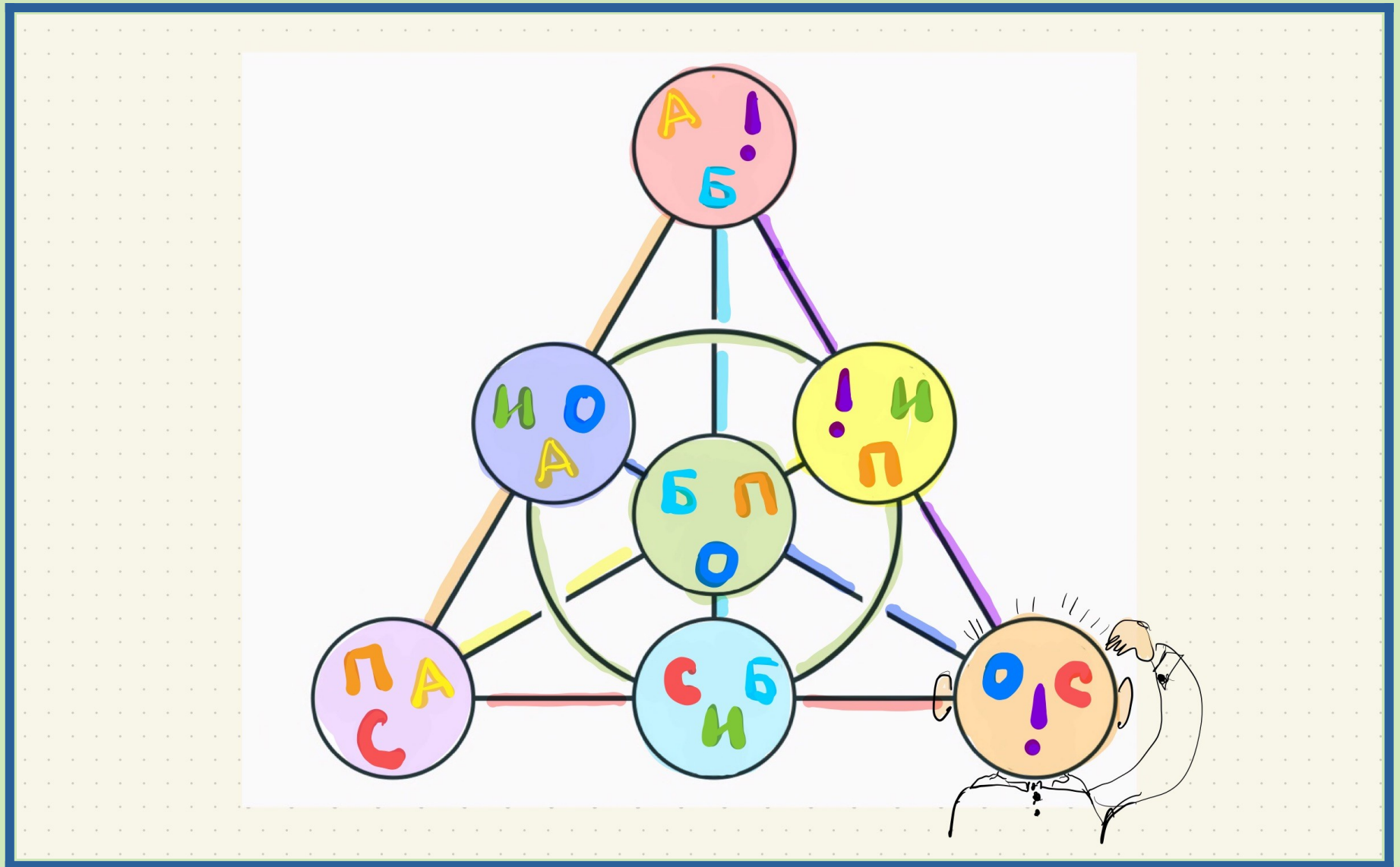


3.9. Descartes' Circle Theorem.

“The Kiss Precise” by Soddy and Gosset – and my translation to Russian (made for the workshop).



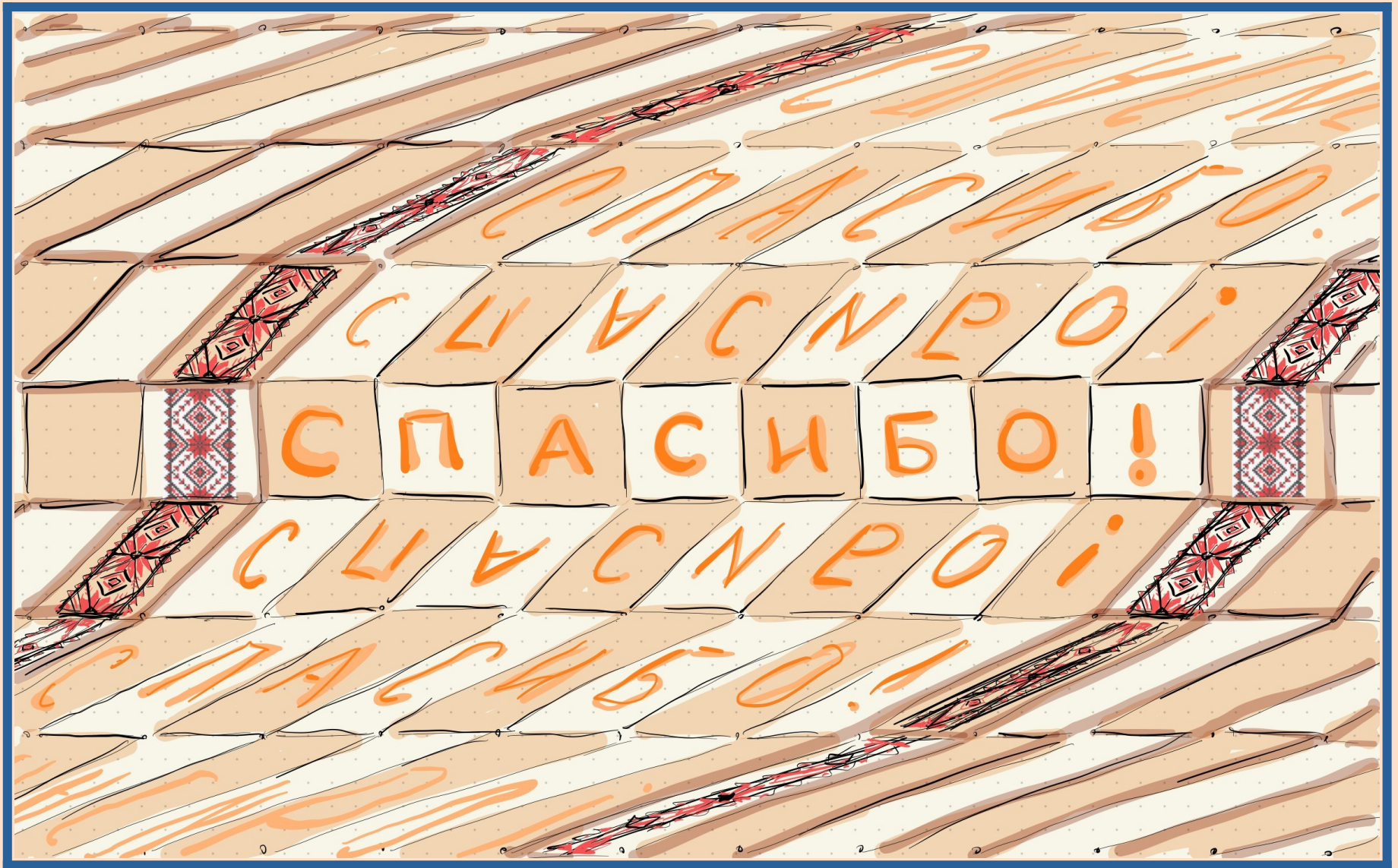
3.10. Farey graph, Ford's circles – and their curvatures.



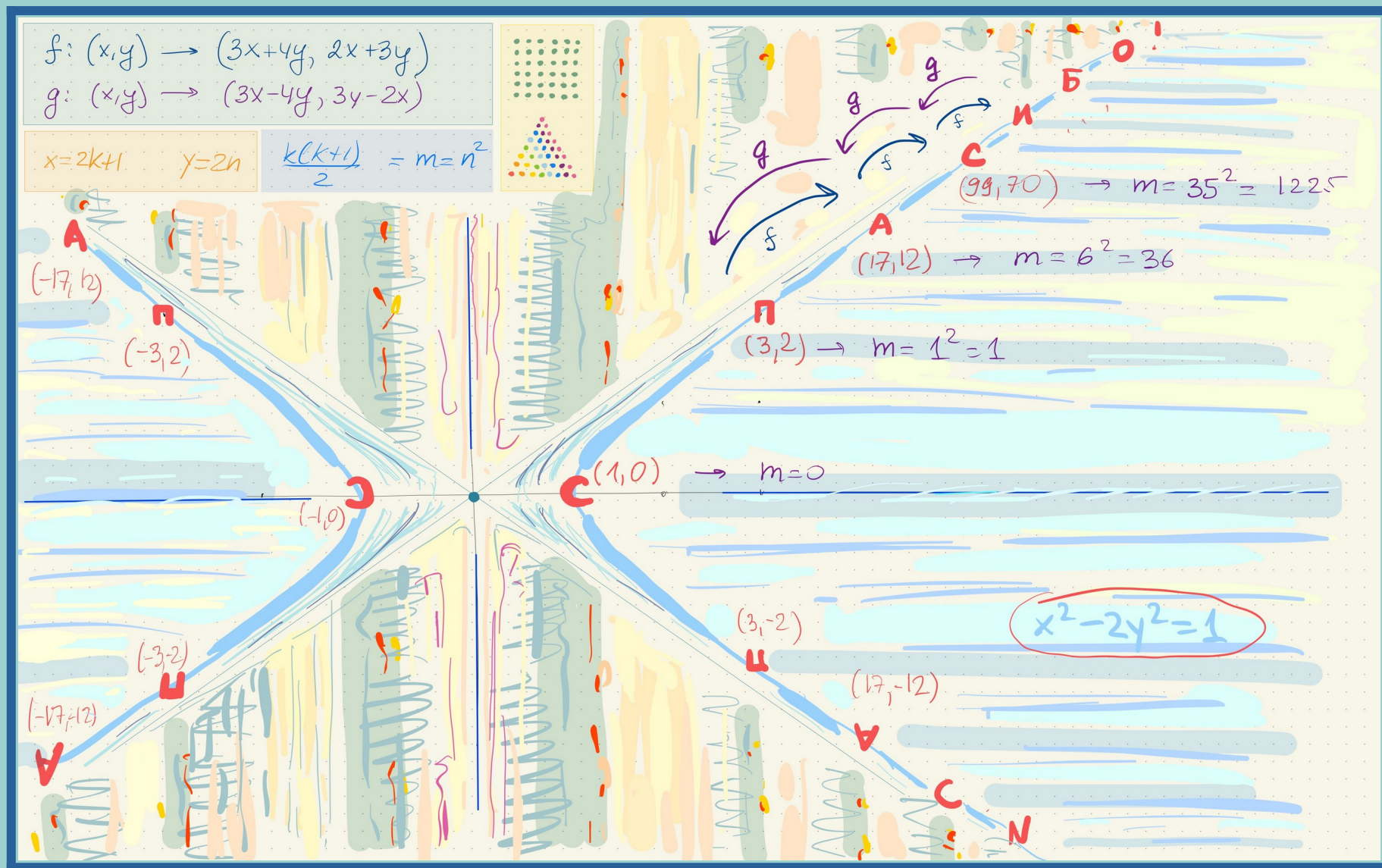
3.11. Playing Dobble.

(A perfect 3-Dobble deck on the Fano plane).

Notice: each symbol appears on exactly 3 cards – every time in a new position!



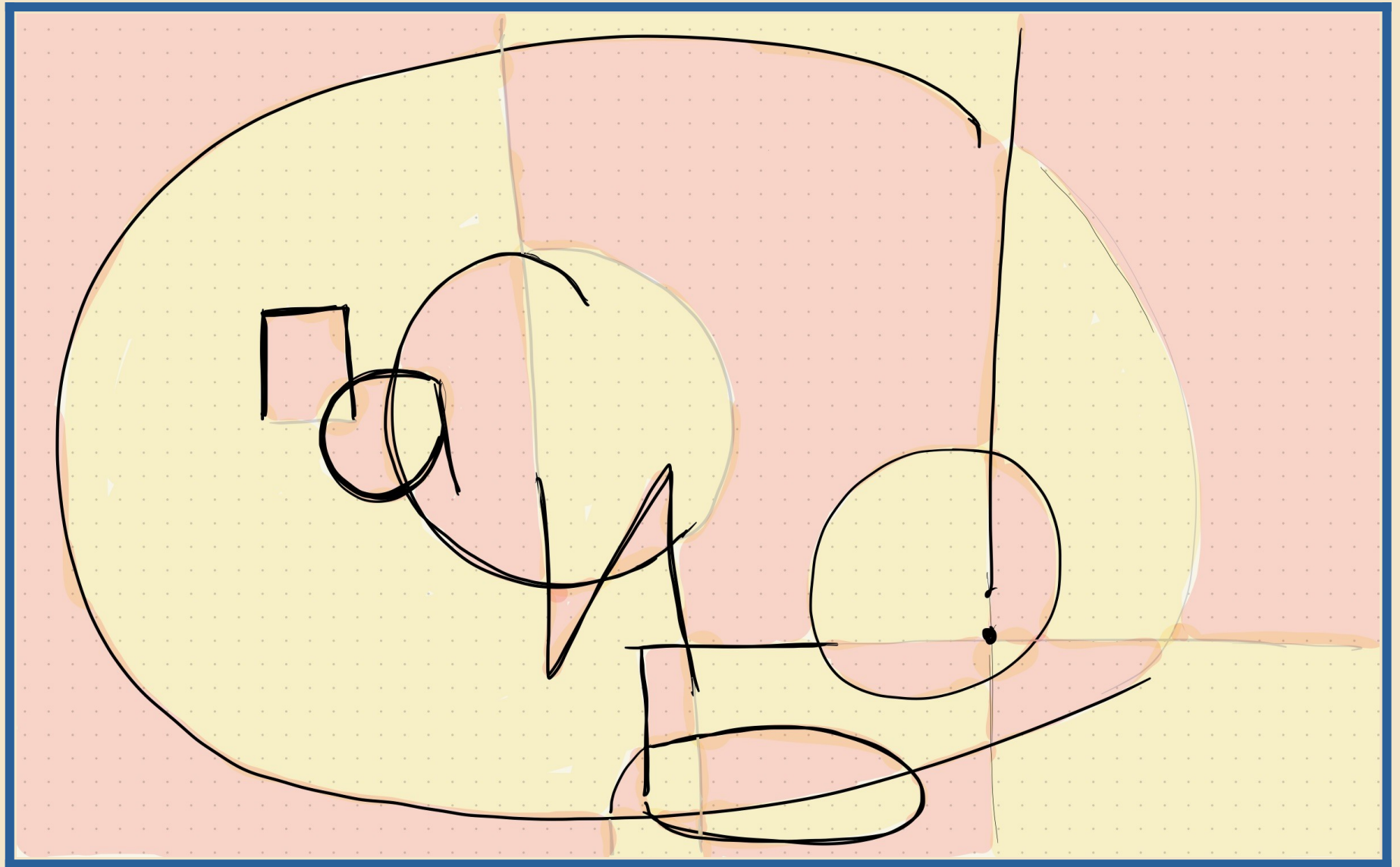
3.12. Semi-invariants in problem solving.



3.13. Pell's equation.



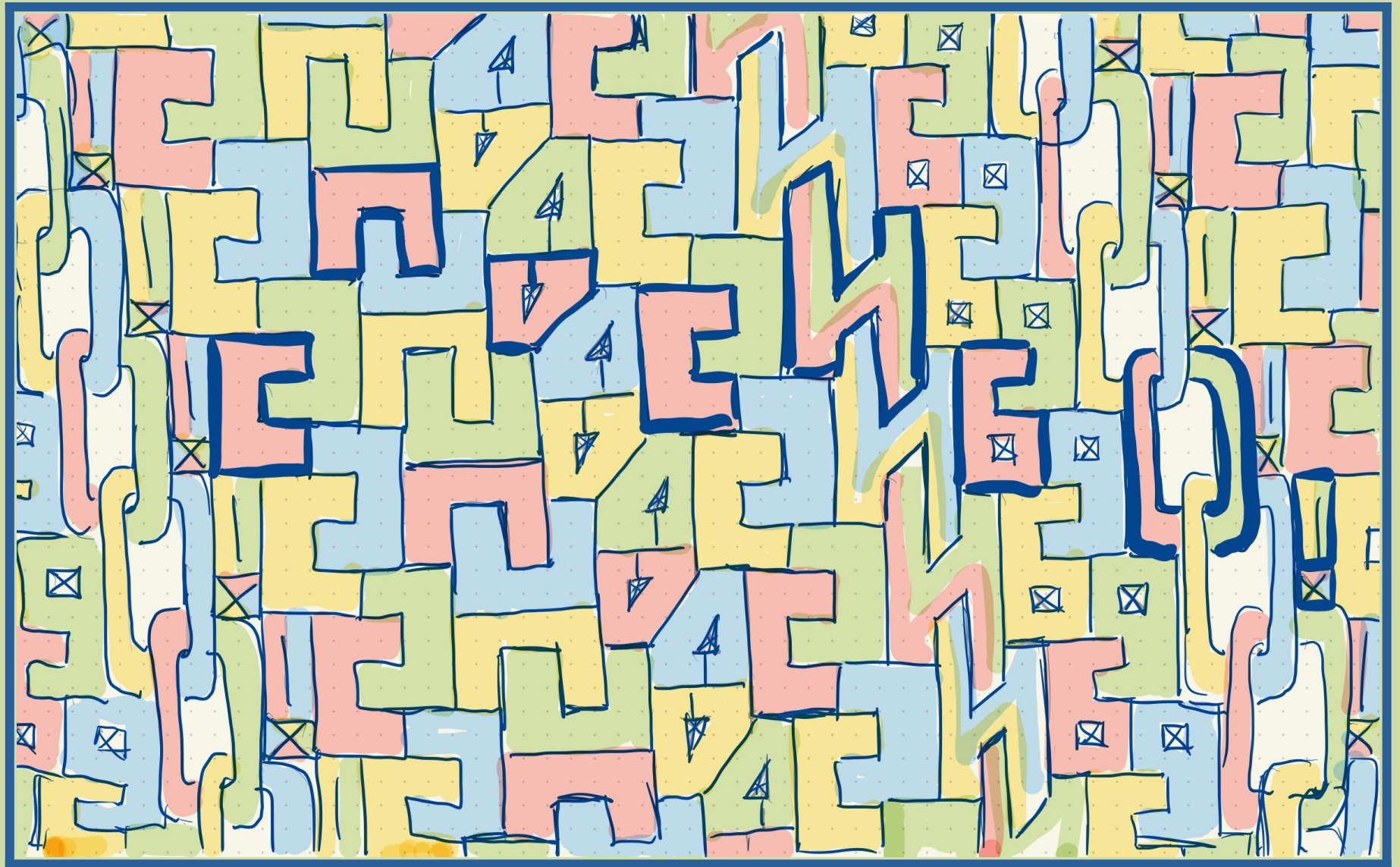
3.14. Hyperbolic geometry – Upper half-plane.



3.15. Induction.

The plane is crossed by finitely many lines and circles.

Show that it can be coloured black and white so that the colours will alternate.



3.16. Tilings on the plane.



3.17. Geometry in the cup of tea.

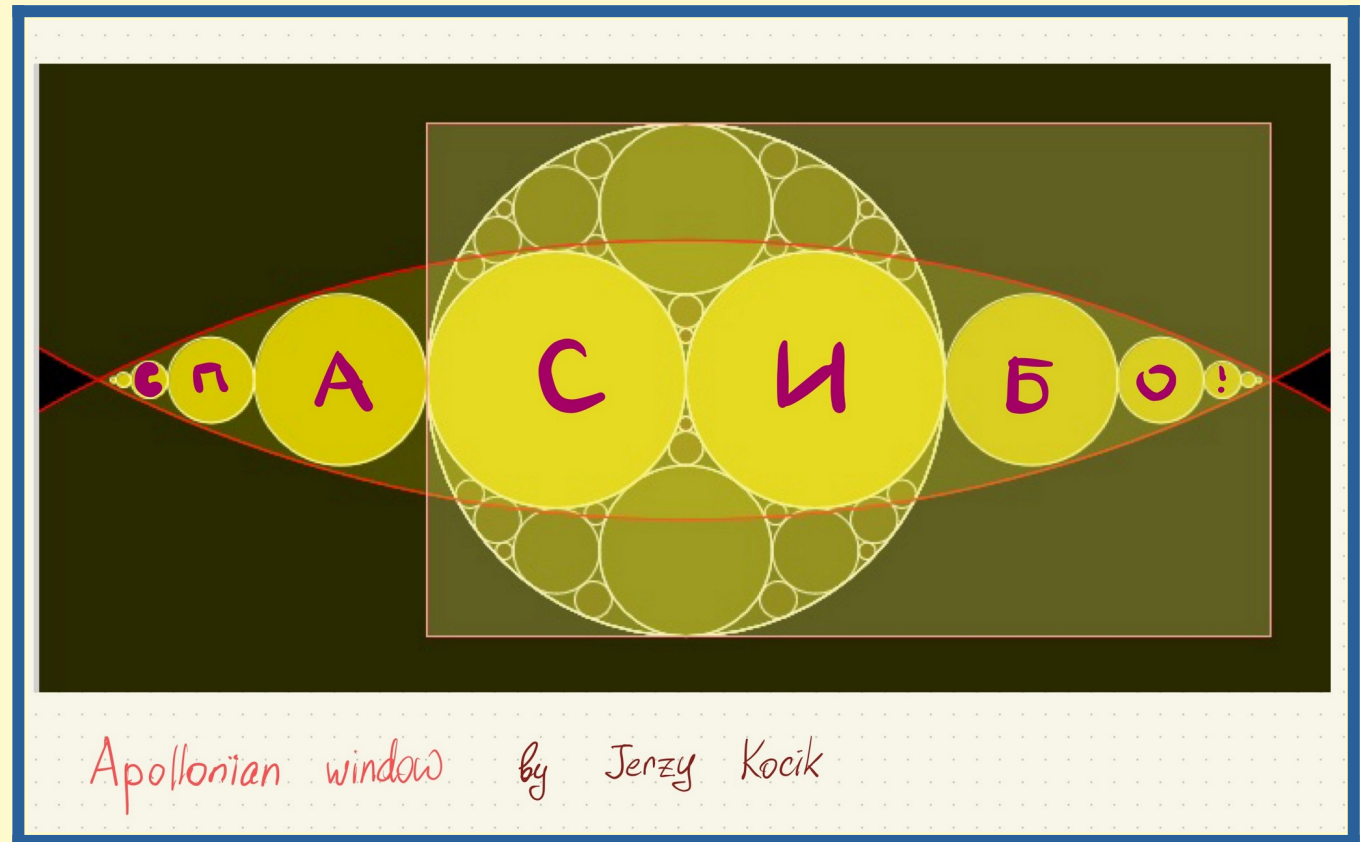
Given a cup of tea and a cup of milk, transfer a spoon of tea into the milk, stir and then transfer a spoon of the mixture back into the tea.

Is there now more milk in the tea or tea in the milk?

4. Appendix: **Thanks based on works by others...**

The easiest way to get beautiful thanks is to include somebody else's art.

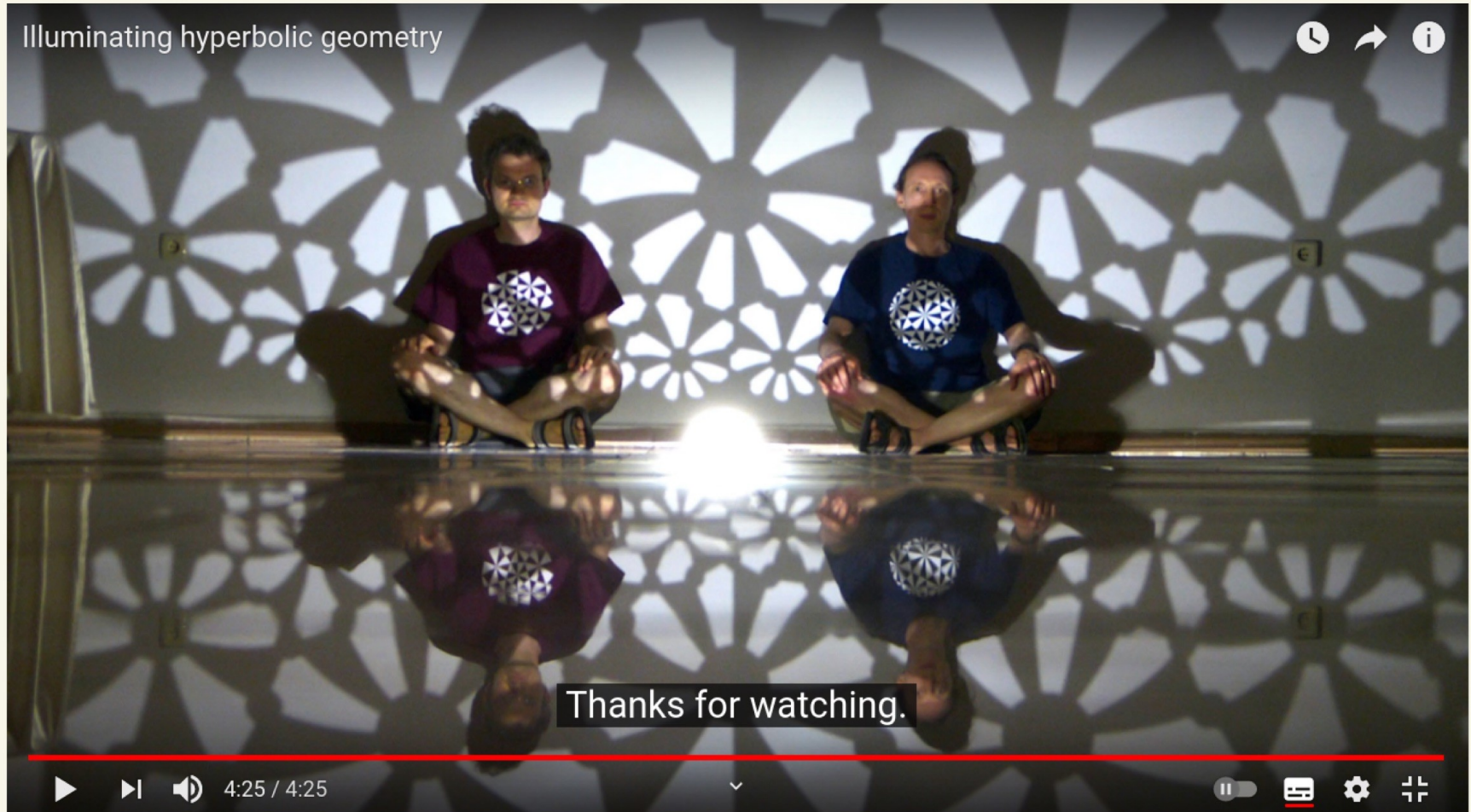
(of course, referring to the author – but still, what is about the copyright?)



2.1. Apollonian packing and many other things (including golden ratio and Fibonacci).
(Image by Jerzy Kocik).

Henry Segerman

Saul Schleimer



- 4.2. Hyperbolic geometry.
(used a screenshot from "Illuminating hyperbolic geometry" video
by Henry Segerman and Saul Schleimer)



used image
by Jos Leys

Thanks!

- 4.3. Hyperbolic geometry – end of spring term (right before Easter holidays).
(used an image by Jos Leys)



Education is what remains
after one has forgotten
everything he learned in
school.

Albert Einstein

THANKS!

M.C. Escher "Reptiles"
Lithograph, 1943

- 4.4. Last lecture in a year long Geometry course.
(used a lithograph by M.C. Escher "Reptiles", 1943)

Thanks:

- to all my teachers, colleagues, friends and relatives from whom I learned to enjoy math, art and the life;
- to every who enjoyed some of the above together with me;
- and to everyone who reacted when I tried to share...`

Thank you all!