2.5). CHEBYSHEV EQUIOSCILLATION THEOREM

Thm 2.3 – Let $f \in C[a,b]$. Then $p_n \in \mathcal{P}_n$ is a minimax polynomial of degree n for f if and only if f, p_n have an alternating set of length $n+2$.

not a non-uniform alternating set.

actually was fully proved by Kirchberger (ca. 1902) in his PhD thesis under Hilbert.

Proof:-

($\Leftarrow$) Suppose that f, p_n have an alternating set of length $n+2$, with $|f(x_i) - p_n(x_i)| = |E| = \|f - p_n\|_\infty$.

Then De la Vallée Poussin (Thm 2.2) $\Rightarrow$ $\|f - p_n^*\|_\infty \geq |E|$.

But $\|f - p_n^*\|_\infty \leq |E|$ by definition of p_n^*, so $\|f - p_n\|_\infty = \|f - p_n^*\|_\infty$, i.e. p_n is a minimax polynomial.

($\Rightarrow$) We will assume that f, p_n have no alternating set of length $n+2$, and show that p_n cannot be the best approximation.

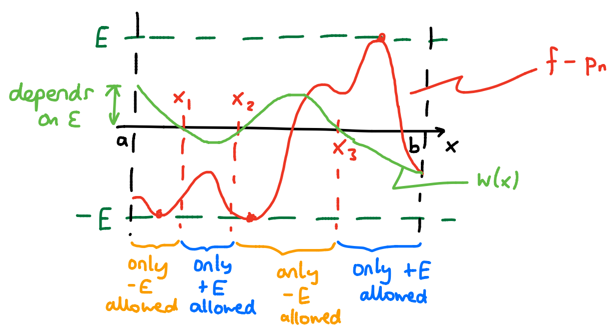
Let $E = \|f - p_n\|_\infty$, and assume (w.l.o.g.) that the leftmost extremum has $f - p_n = -E$.

Then there are numbers $x_1, \ldots, x_k$ such that $a < x_1 < \ldots < x_k < b$ with $k \leq n$, such that

1). $f(x) - p_n(x) < E$ for $x \in [a, x_1] \cup [x_2, x_3] \cup [x_4, x_5] \cup \ldots$

2). $f(x) - p_n(x) > -E$ for $x \in [x_1, x_2] \cup [x_3, x_4] \cup \ldots$

e.g. here I can find suitable points x_1, x_2, x_3 ($k=3$):



Now let $w(x) = (x_1 - x)(x_2 - x) \ldots (x_k - x)$.

Then
$q_n(x) = p_n(x) + \varepsilon w(x)$ will be a better approximation than p_n for small enough $\varepsilon > 0$

you always add something of the correct sign.

(because all of the extrema will be reduced). Hence p_n cannot be minimax. ∎

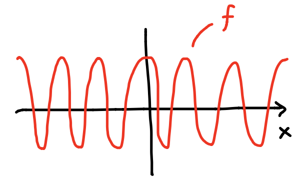
[Note: if there were an alternating set of length $n+2$, we could find $x_1, \ldots, x_{n+1}$ so $w(x)$ would have degree $n+1$. Then q_n would not be in $\mathcal{P}_n$ so the argument would fail. $\longrightarrow$ i.e. we would need a higher degree polynomial to do better.]

Note: The minimax polynomial may have degree $< n$, and may have an alternating set of length $> n+2$.

Example: $f(x) = \cos(x)$ on $[-n\pi, n\pi]$, find minimax polynomial of degree n.

There is an obvious alternating set of length $2n+1$ when $p_n(x) = 0$.

By Thm 2.3, we must have $p_n^*(x) = 0$.



Corollary 2.4 – The minimax polynomial $p_n^* \in \mathcal{P}_n$ for $f \in C[a,b]$ is unique.
(uniqueness)

Proof: Let both p_n^*, q_n^* be minimax polynomials. Then

because both are minimax.

$$\left\| f - \frac{p_n^* + q_n^*}{2} \right\|_\infty = \left\| \frac{f - p_n^*}{2} + \frac{f - q_n^*}{2} \right\|_\infty \leqslant \tfrac{1}{2}\|f - p_n^*\|_\infty + \tfrac{1}{2}\|f - q_n^*\|_\infty = \|f - p_n^*\|_\infty.$$

i.e. $\frac{1}{2}(p_n^* + q_n^*)$ is a minimax polynomial.

By Thm 2.3, there are $n+2$ alternating points at which $\frac{1}{2}(f - p_n^*) + \frac{1}{2}(f - q_n^*) = \|f - p_n^*\|_\infty$.

At each of these points, $f - p_n^*$ and $f - q_n^*$ are both $\|f - p_n^*\|_\infty$ or both $-\|f - p_n^*\|_\infty$.

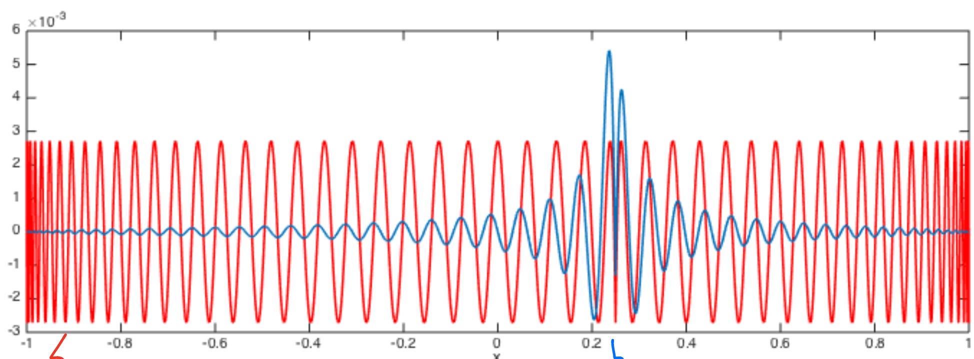
So $f - p_n^*$ and $f - q_n^*$ agree at $n+2$ points, so

$$(f - p_n^*) - (f - q_n^*) = q_n^* - p_n^* = 0 \text{ at these } n+2 \text{ points.}$$

Since $q_n^* - p_n^* \in \mathcal{P}_n$, we must have $q_n^* - p_n^* = 0$. $\blacksquare$

Remark: Minimax approximations are not used much in practice, despite being nice in theory.
Note that e.g. Chebyshev interpolants may do better at most locations.

e.g. $f(x) = |x - \tfrac{1}{4}|$ on $[-1, 1]$.



$f - p_{100}^*$

$f - q_{100}$ (Chebyshev interpolant)