1). PIECEWISE POLYNOMIAL INTERPOLATION

Key idea:- Avoid oscillations by using lots of low-degree polynomial interpolants.
- useful if you aren't free to select the points, but sacrifices smoothness.

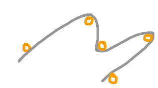
Let $s(x)$ be a function which interpolates $f \in C[a,b]$ at the knots $x_0 < x_1 < \ldots < x_n$, consisting of n polynomials $s_i(x)$ in each interval $[x_i, x_{i+1}]$. Then $s(x)$ is called a spline of degree N if it is $N-1$ times continuously differentiable.

This means that the pieces satisfy the following matching conditions at the interior knots:

automatic from interpolation requirement →

$$\left.\begin{aligned} s_{i-1}(x_i) &= s_i(x_i) \\ s'_{i-1}(x_i) &= s'_i(x_i) \\ &\vdots \\ s^{(N-1)}_{i-1}(x_i) &= s^{(N-1)}_i(x_i). \end{aligned}\right\} \text{ for } i = 1, \ldots, n-1.$$

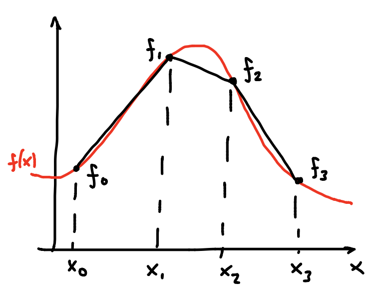
In other words, splines have a certain degree of smoothness.



Remark:-
- Splines are useful in computer aided design (e.g. car or aeroplane bodies).
- Originally a draftsman's "spline" was a strip of flexible wood/metal held by pegs and traced by hand to produce a smooth curve. → will see cubic splines later

1.1). LINEAR SPLINES (N = 1).

This is simply piecewise-linear interpolation, with no derivative conditions.



The requirements are just

$$\begin{aligned} s_0(x_0) &= f_0, & s_0(x_1) &= f_1 \\ s_1(x_1) &= f_1, & s_1(x_2) &= f_2 \\ &\vdots & &\vdots \\ s_{n-1}(x_{n-1}) &= f_{n-1}, & s_{n-1}(x_n) &= f_n. \end{aligned}$$

This uniquely defines each $s_i(x)$ by

$$s_i(x) = f_i + \left(\frac{f_{i+1} - f_i}{x_{i+1} - x_i}\right)(x - x_i)$$

and hence

$$s(x) = \begin{cases} s_0(x), & x_0 \le x \le x_1, \\ \vdots & \\ s_{n-1}(x), & x_{n-1} \le x \le x_n. \end{cases}$$

By taking more and more points, functions in $C^2[a,b]$ can be approximated arbitrarily well by linear splines...

Thm 1.1 :- Let $f \in C^2[a,b]$ and $s(x)$ be a linear spline that interpolates f at the knots $x_0 < \ldots < x_n$. If $h = \max_{1 \le i \le n}(x_i - x_{i-1})$ then

$$\|f - s\|_\infty \le \tfrac{1}{8} h^2 \|f''\|_\infty.$$

Proof:- In each interval $[x_i, x_{i+1}]$ s is a linear interpolant, so by Thm 0.1,

$$f(x) - s_i(x) = \frac{w_2(x) f''(\xi)}{2} \quad \text{for some } \xi \in [x_i, x_{i+1}].$$

$$= (x-x_i)(x-x_{i+1}) \frac{f''(\xi)}{2}.$$

So

$$|f(x) - s_i(x)| \leq |(x-x_i)(x-x_{i+1})| \frac{|f''(\xi)|}{2} = \underbrace{(x-x_i)(x_{i+1}-x)}_{} \frac{|f''(\xi)|}{2}$$

by symmetry, maximum is at mid-point $x = \frac{x_i + x_{i+1}}{2}$.

$$\leq \left(\frac{x_i + x_{i+1}}{2} - x_i\right)\left(x_{i+1} - \frac{x_i + x_{i+1}}{2}\right) \frac{|f''(\xi)|}{2}$$

$$= \frac{1}{4}(x_{i+1} - x_i)^2 \frac{|f''(\xi)|}{2}.$$

Hence

$$\|f - s\|_\infty \leq \frac{1}{8} h^2 \|f''\|_\infty. \quad ▨$$

Clearly to minimize error we should add more knots where the function has largest second derivative.

see this in the sketch above.

Linear splines are the "flattest" among all functions interpolating f at a given set of knots, in the sense that the average slope $\|s'\|_2$ is the smallest.

" $\left(\int_a^b |s'(x)|^2 dx\right)^{1/2}$.

Thm 1.2 :- Let s be a linear spline interpolating $f \in C[a,b]$ at the knots $x_0 < \dots < x_n$. Then

$$\|s'\|_2 \leq \|f'\|_2.$$

← dates back to J.C. Holladay in 1950s.

Proof:- Note that

$$\|f' - s'\|_2^2 = \int_a^b |f' - s'|^2 dx$$

$$= \int_a^b \{(f')^2 - 2f's' + (s')^2\} dx$$

$$= \|f'\|_2^2 + \|s'\|_2^2 - 2\int_a^b f's' \, dx$$

$$= \|f'\|_2^2 - \|s'\|_2^2 - 2\int_a^b (f' - s') s' \, dx.$$

so

$$\|f'\|_2^2 = \|f' - s'\|_2^2 + \|s'\|_2^2 + 2\underbrace{\int_a^b (f' - s') s' \, dx}_{\circledast}.$$

If $\circledast$ vanishes then we are done.

Integrate by parts:

$$\circledast = \int_a^b \{(f-s)s'\}' dx - \int_a^b (f-s) s'' dx.$$ split into separate intervals

$$= \sum_{i=1}^n \int_{x_i}^{x_{i+1}} [(f - s_i) s_i']' dx - \sum_{i=1}^n \int_{x_i}^{x_{i+1}} (f - s_i) \underbrace{s_i''}_{=0} dx$$

$$= \sum_{i=1}^n \left\{ (f - s_i) s_i' \Big|_{x = x_{i+1}} - (f - s_i) s_i' \Big|_{x = x_i} \right\}$$

$$= 0 \quad \text{because } s_i(x_i) = f(x_i) \text{ and } s_i(x_{i+1}) = f(x_{i+1}). \quad ▨$$