Quadratic splines are less used in practice (no nice variational property), so we move to

1.2) CUBIC SPLINES

A cubic spline ($N=3$) is made up of cubic pieces

$$s_i(x) = a_i + b_i x + c_i x^2 + d_i x^3, \qquad i = 0, \dots, n-1,$$

that satisfy the following conditions:

1). Interpolation at the knots

$$\left.\begin{array}{l} s_i(x_i) = f_i \\ s_i(x_{i+1}) = f_{i+1} \end{array}\right\} \text{ for } i = 0, \dots, n-1. \longrightarrow 2n \text{ conditions}$$

2). Matching first derivatives at interior knots:

$$s_{i-1}'(x_i) = s_i'(x_i) \quad \text{for } i = 1, \dots, n-1. \longrightarrow n-1 \text{ conditions}$$

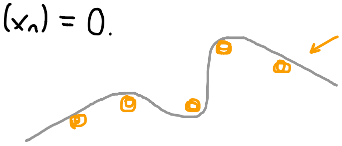
3). Matching second derivatives at interior knots:

$$s_{i-1}''(x_i) = s_i''(x_i) \quad \text{for } i = 1, \dots, n-1. \longrightarrow n-1 \text{ conditions}$$

We have $4n$ unknowns (a_i, b_i, c_i, d_i) but only $4n-2$ conditions, so we need 2 extra conditions, whatever the value of n. $\longrightarrow$ so we can't impose continuity of s''' at every knot.

There are several ways of choosing the extra two conditions – some popular ones are:

- Natural cubic spline: $s_0''(x_0) = s_{n-1}''(x_n) = 0$.



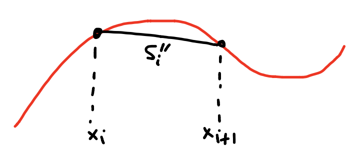
like the physical spline: ends are straight.

- Complete/clamped cubic spline: $\begin{cases} s_0'(x_0) = f'(x_0) \\ s_{n-1}'(x_n) = f'(x_n) \end{cases}$ $\longleftarrow$ needs extra info about f.

- Not-a-knot cubic spline: $\begin{cases} s_0'''(x_1) = s_1'''(x_1) \\ s_{n-2}'''(x_{n-1}) = s_{n-1}'''(x_{n-1}) \end{cases}$

Unlike the linear spline, the different s_i are now coupled and cannot be computed independently. We will show how to formulate a linear system (will also let us decide existence and uniqueness).
makes cubic splines convenient.

Let $M_i = s_i''(x_i)$ — called the moments of the cubic spline.



Since s_i is cubic, s_i'' is linear, so in $[x_i, x_{i+1}]$ we can write

$$s_i''(x) = \left(\frac{x_{i+1}-x}{h_i}\right) M_i + \left(\frac{x-x_i}{h_i}\right) M_{i+1}, \qquad \text{where } h_i = x_{i+1} - x_i.$$

Note that we have imposed the condition that the second derivative is continuous.

Now integrate twice:

$$s_i(x) = \frac{(x_{i+1}-x)^3}{6h_i} M_i + \frac{(x-x_i)^3}{6h_i} M_{i+1} + \alpha_i (x - x_i) + \beta_i (x_{i+1} - x).$$

To determine α_i and β_i, we can apply the interpolation conditions:

$$f_i = s_i(x_i) = \tfrac{1}{6}h_i^2 M_i + h_i\beta_i \qquad\Rightarrow\qquad \beta_i = \frac{f_i - \frac{1}{6}h_i^2 M_i}{h_i}$$

$$f_{i+1} = s_i(x_{i+1}) = \tfrac{1}{6}h_i^2 M_{i+1} + h_i\alpha_i \qquad\qquad \alpha_i = \frac{f_{i+1} - \frac{1}{6}h_i^2 M_{i+1}}{h_i}.$$

We still have to apply the first derivative condition. For each piece, we have

$$\begin{aligned} s_i'(x) &= \frac{-(x_{i+1}-x)^2}{2h_i}M_i + \frac{(x-x_i)^2}{2h_i}M_{i+1} + \alpha_i - \beta_i \\ &= \frac{-(x_{i+1}-x)^2}{2h_i}M_i + \frac{(x-x_i)^2}{2h_i}M_{i+1} + \frac{1}{h_i}(f_{i+1}-f_i) + \tfrac{1}{6}h_i(M_i - M_{i+1}). \end{aligned}$$

Changing the index gives

$$s_{i-1}'(x) = -\frac{(x_i - x)^2}{2h_{i-1}}M_{i-1} + \frac{(x-x_{i-1})^2}{2h_{i-1}}M_i + \frac{1}{h_{i-1}}(f_i - f_{i-1}) + \tfrac{1}{6}h_{i-1}(M_{i-1} - M_i).$$

So

$$s_{i-1}'(x_i) = s_i'(x_i)$$

$$\Leftrightarrow \quad \frac{h_{i-1}^2}{2h_{i-1}}M_i + \frac{1}{h_{i-1}}(f_i - f_{i-1}) + \tfrac{1}{6}h_{i-1}(M_{i-1} - M_i) = -\frac{h_i^2}{2h_i}M_i + \frac{1}{h_i}(f_{i+1}-f_i) + \tfrac{1}{6}h_i(M_i - M_{i+1})$$

$$\Leftrightarrow \quad h_{i-1}M_{i-1} + 2(h_{i-1}+h_i)M_i + h_iM_{i+1} = \underbrace{6\left(\frac{f_{i+1}-f_i}{h_i} - \frac{f_i - f_{i-1}}{h_{i-1}}\right)}_{b_i} \quad \text{for } i = 1, \dots, n-1.$$

Together with the natural spline boundary conditions $M_0 = M_n = 0$, this gives a tridiagonal system of linear equations,

$$\begin{bmatrix} 1 & 0 & 0 & & \cdots & 0 \\ h_0 & 2(h_0+h_1) & h_1 & 0 & & \vdots \\ 0 & h_1 & 2(h_1+h_2) & h_2 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \\ \vdots & & & & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ 0 & \cdots & & & 0 & 1 \end{bmatrix} \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ \vdots \\ M_{n-1} \\ M_n \end{bmatrix} = \begin{bmatrix} 0 \\ b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ 0 \end{bmatrix}$$

The system is diagonally dominant since $2(h_{i-1}+h_i) > h_{i-1}+h_i$ so has a unique solution.

[A matrix A is strictly diagonally dominant if $|a_{ii}| > \sum_{j\neq i}|a_{ij}|$ for all rows i.

Proof that every s.d.d. matrix is nonsingular:

Suppose not. Then there exists a vector $\vec{u} \neq 0$ such that $A\vec{u} = \vec{0}$, and $\vec{u}$ has some element u_i of largest magnitude.

We have

$$\begin{aligned} \sum_j a_{ij}u_j = 0 \quad &\Leftrightarrow \quad a_{ii}u_i = -\sum_{j\neq i} a_{ij}u_j \\ &\Leftrightarrow \quad a_{ii} = -\sum_{j\neq i}\left(\frac{u_j}{u_i}\right)a_{ij} \\ &\Leftrightarrow \quad |a_{ii}| \leq \sum_{j\neq i} \underbrace{\left|\frac{u_j}{u_i}\right|}_{\leq 1} . |a_{ij}| \leq \sum_{j\neq i}|a_{ij}| \quad \text{— contradiction.} \quad \blacksquare \end{aligned}$$

Example: Fit a natural cubic spline through the data $(0,0)$, $(1,1)$, $(2,8)$.

Here $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $h_0 = h_1 = 1$.

In $[0,1]$:

$$s_0(x) = \frac{(1-x)^3}{6} M_0 + \frac{x^3}{6} M_1 + \alpha_0 x + \beta_0 (1-x) \quad M_0 = 0 \ \text{(natural c. spl.)}$$

$$= \frac{x^3}{6} M_1 + \alpha_0 x + \beta_0 (1-x)$$

and

$$f_0 = s_0(x_0) \Rightarrow 0 = \beta_0$$

$$f_1 = s_0(x_1) \Rightarrow 1 = \tfrac{1}{6} M_1 + \alpha_0 \qquad \Rightarrow \quad s_0(x) = \frac{x^3}{6} M_1 + \left(1 - \tfrac{1}{6} M_1\right) x.$$

In $[1,2]$:

$$s_1(x) = \frac{(2-x)^3}{6} M_1 + \frac{(x-1)^3}{6} M_2 + \alpha_1 (x-1) + \beta_1 (2-x) \quad M_2 = 0$$

$$= \frac{(2-x)^3}{6} M_1 + \alpha_1 (x-1) + \beta_1 (2-x)$$

and

$$f_1 = s_1(x_1) \Rightarrow 1 = \tfrac{1}{6} M_1 + \beta_1$$

$$f_2 = s_1(x_2) \Rightarrow 8 = \alpha_1 \qquad \Rightarrow \quad s_1(x) = \frac{(2-x)^3}{6} M_1 + 8(x-1) + \left(1 - \tfrac{1}{6} M_1\right)(2-x).$$

Finally the first derivative condition gives

$$s_0'(x_1) = s_1'(x_1)$$

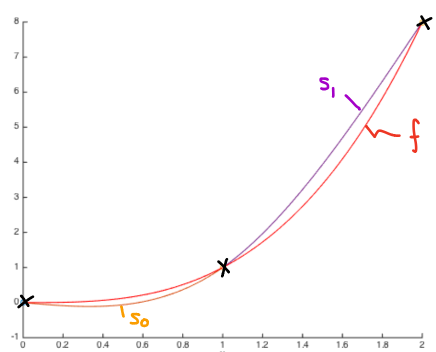
$$\Leftrightarrow \frac{3x_1^2}{6} M_1 + \left(1 - \tfrac{1}{6} M_1\right) x_1 = -\frac{(2-x_1)^2}{2} M_1 + 8 - 1 + \tfrac{1}{6} M_1$$

$$\Leftrightarrow \tfrac{1}{2} M_1 + 1 - \tfrac{1}{6} M_1 = -\tfrac{1}{2} M_1 + 7 + \tfrac{1}{6} M_1$$

$$\Leftrightarrow \tfrac{2}{3} M_1 = 6 \quad \Leftrightarrow \quad M_1 = 9.$$

So our natural cubic spline is

$$s(x) = \begin{cases} \tfrac{3}{2} x^3 - \tfrac{1}{2} x & \text{for } 0 \le x \le 1, \\ \tfrac{3}{2} (2-x)^3 + 8(x-1) - \tfrac{1}{2}(2-x) & \text{for } 1 \le x \le 2. \end{cases}$$



Observe that our spline is not $s(x) = x^3$, even though the data come from $f(x) = x^3$, i.e. natural cubic splines do not reproduce cubic functions.

unlike linear splines.

this leads to a lower order of convergence than e.g. complete c. spl.

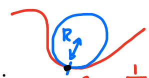
This is because $f(x) = x^3$ does not satisfy the end conditions $f''(x_0) = f''(x_n) = 0$.

Thm 1.3. (Holladay) Let s be the natural cubic spline interpolating $f \in C^2[a,b]$ at the knots $x_0 < \ldots < x_n$. Then
$$\|s''\|_2 \leqslant \|f''\|_2 .$$

- i.e. s gives the smallest value of the integral $\left(\int_a^b |s''(x)|^2 dx\right)^{1/2}$ over the class of admissible functions: it is the "smoothest" interpolant. ← cf. Thm 1.2 for linear splines.

- Physical interpretation:-

The local curvature of a graph $y = f(x)$ is $\kappa = \dfrac{y''}{(1+|y'|^2)^{3/2}}$.



$\kappa = \frac{1}{R}$ where R = radius of curvature

The total internal strain energy stored in an elastic beam is proportional to $\int \kappa^2 dx$, so if $|y'|$ is small, the shape of minimum energy will minimise $\int (y'')^2 dx$, i.e. will be a cubic spline. → so this is close to the shape taken by a drafting spline.

Proof of Thm 1.3:- As in Thm 1.2, start with
$$\begin{aligned}
\|f''-s''\|_2^2 &= \int_a^b |f''-s''|^2 dx \\
&= \int_a^b \{(f'')^2 - 2f''s'' + (s'')^2\} dx \\
&= \|f''\|_2^2 + \|s''\|_2^2 - 2\int_a^b f''s'' dx \\
&= \|f''\|_2^2 - \|s''\|_2^2 - 2\int_a^b (f''-s'')s'' dx.
\end{aligned}$$

Our task is to show that the last term vanishes. ← Same idea as Thm 1.2 !
$$\begin{aligned}
\int_a^b (f''-s'')s'' dx &= \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} (f''-s_i'')s_i'' dx \\
&= \underbrace{\sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} [(f'-s_i')s_i'']' dx}_{①} - \underbrace{\sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} (f'-s_i')\overset{A_i \text{ (constant for each } s_i)}{s_i'''}\, dx}_{②}
\end{aligned}$$

Now
$$\begin{aligned}
① = \sum_{i=0}^{n-1} \underbrace{(f'(x)-s_i'(x))s_i''(x)}_{=0 \text{ at interior nodes}}\Big|_{x_i}^{x_{i+1}} &= (f'(x_n)-s_{n-1}'(x_n))s_{n-1}''(x_n) - (f'(x_0)-s_0'(x_0))s_0''(x_0) \\
&= 0 \quad \text{for natural cubic spline } (s_{n-1}''(x_n) = s_0''(x_0) = 0)
\end{aligned}$$

leaving
$$\begin{aligned}
② &= \sum_{i=0}^{n-1} A_i \int_{x_i}^{x_{i+1}} (f-s_i)' dx \\
&= \sum_{i=0}^{n-1} A_i \{(f(x_{i+1}) - s_i(x_{i+1})) - (f(x_i) - s_i(x_i))\} \overset{\text{by interpolation conditions}}{=} 0. \qquad \blacksquare
\end{aligned}$$

In fact, the natural cubic spline is the only function that achieves $\|f''-s''\|_2^2 = 0$:-

Since the integrand in $\int_a^b (f''-s'')^2 dx$ is non-negative and continuous, we must have
$$\begin{aligned}
(f''-s'')^2 = 0 \quad &\Leftrightarrow \quad f'' = s'' \\
&\Leftrightarrow \quad f(x) = s(x) + cx + d.
\end{aligned}$$
The end conditions $s(x_0) = f(x_0)$ and $s(x_n) = f(x_n)$ show that $c = d = 0$, so only the natural cubic spline achieves $\|f''\|_2 = \|s''\|_2$.