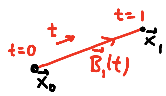
2.2). Bezier Curves

Bernstein polynomials also appear in the parametric formulae for Bézier curves.

Recall: $B_n(f,x) = \sum_{k=0}^{n} f(\tfrac{k}{n})\binom{n}{k} x^k (1-x)^{n-k}$.

If we replace $f(\tfrac{k}{n})$ by positions $\vec{x}_k \in \mathbb{R}^2$ and x by t we get parametric expressions for Bézier curves:

e.g. $\vec{B}_1(t) = (1-t)\vec{x}_0 + t\vec{x}_1$.

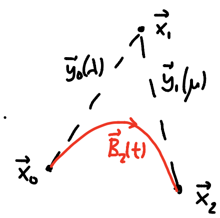


This is just linear interpolation, $\vec{B}_1(0) = \vec{x}_0$, $\vec{B}_1(1) = \vec{x}_1$.

e.g. $\vec{B}_2(t) = (1-t)^2\vec{x}_0 + 2(1-t)t\,\vec{x}_1 + t^2\vec{x}_2$.

When $t=0$, $\vec{B}_2(0) = \vec{x}_0$

$t=1$, $\vec{B}_2(1) = \vec{x}_2$



Claim: The tangents at $\vec{x}_0$ and $\vec{x}_2$ intersect at $\vec{x}_1$.

To see this, differentiate w.r.t. t:

$$\vec{B}_2' = -2(1-t)\vec{x}_0 + 2(1-2t)\vec{x}_1 + 2t\,\vec{x}_2$$

so the tangent direction at $t=0$ is

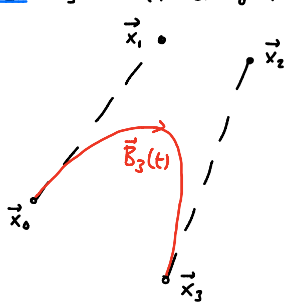
$$\vec{B}_2'(0) = -2\vec{x}_0 + 2\vec{x}_1 \quad\Rightarrow\quad \text{tangent line } \vec{y}_0(\lambda) = \vec{x}_0 + 2\lambda(\vec{x}_1 - \vec{x}_0)$$

and at $t=1$ is

$$\vec{B}_2'(1) = -2\vec{x}_1 + 2\vec{x}_2 \quad\Rightarrow\quad \text{tangent line } \vec{y}_1(\mu) = \vec{x}_2 + 2\mu(\vec{x}_2 - \vec{x}_1).$$

Notice that when $\lambda = \frac{1}{2}$ and $\mu = -\frac{1}{2}$ we get $\vec{y}_0 = \vec{y}_1 = \vec{x}_1$, so this is the intersection point.

e.g. $\vec{B}_3(t) = (1-t)^3\vec{x}_0 + 3t(1-t)^2\vec{x}_1 + 3t^2(1-t)\vec{x}_2 + t^3\vec{x}_3$.



Again, $\vec{B}_3(0) = \vec{x}_0$, $\vec{B}_3(1) = \vec{x}_3$.

This time, the tangent to $\vec{B}_3$ at $\vec{x}_0$ passes through $\vec{x}_1$, and the tangent at $\vec{x}_3$ through $\vec{x}_2$. → Problem sheet!

Properties

1). A Bézier curve always lies within the convex hull of the control points x_i, because $\vec{B}_n(t)$ is a weighted average of the $\vec{x}_i$.

the smallest convex set containing all of the points.

2). If the $\vec{x}_i$ lie on a curve γ, then $\vec{B}_n(t)$ will converge to γ as $n \to \infty$ (cf. last lecture).

3). Bézier curves have a variation diminishing property: If a straight line is drawn through the curve, the number of intersections with the curve will be less than or equal to the number of intersections with the control polygon. (Proof omitted)

Example:- Find the parametric equations of the Bézier curve with control points $(0,0)$, $(0,1)$, $(1,\frac{1}{2})$, $(-1,\frac{1}{2})$.

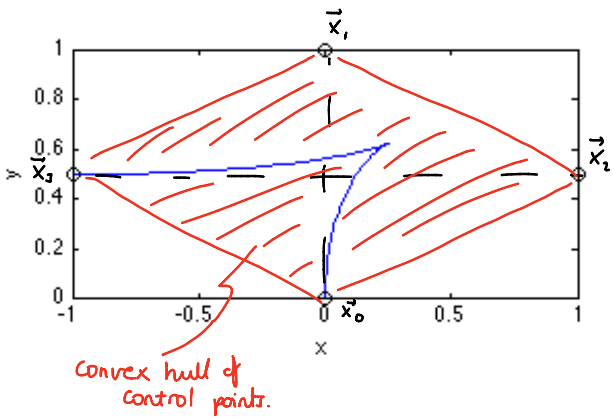
Note: the order matters!

We have

$$x(t) = (1-t)^3(0) + 3t(1-t)^2(0) + 3t^2(1-t)(1) + t^3(-1) = t^2(3-4t)$$

$$y(t) = (1-t)^3(0) + 3t(1-t)^2(1) + 3t^2(1-t)(\tfrac{1}{2}) + t^3(\tfrac{1}{2}) = t\left(3 - 6t + 3t^2 + \tfrac{3}{2}t - \tfrac{3}{2}t^2 + \tfrac{1}{2}t^2\right)$$

$$= t\left(2t^2 - \tfrac{9}{2}t + 3\right)$$



Note that the curve has a cusp where $x'(t) = y'(t) = 0$

$\Leftrightarrow \; 3(2t-1)(t-1) = 6t(1-2t) = 0 \;\Leftrightarrow\; t = \frac{1}{2}$

which corresponds to

$x(\frac{1}{2}) = \frac{1}{4}$

$y(\frac{1}{2}) = \frac{5}{8}$.

Note: Bézier curves can also have self-intersections, e.g.

You can see the variation diminishing property.

Applications:-

- Bézier curves (and splines) were developed independently by Pierre Bézier (1910-99) of Renault cars and Paul de Casteljau of Citröen.
- They are fundamental to computer aided design and manufacturing.

Example:- Bézier curves are used in scalable fonts, e.g. Type 1 or TrueType fonts, and in Adobe Postscript.

used because of de Casteljau's efficient algorithm for computing them.

You can easily draw cubic Bézier curves in Postscript, e.g. here is a postscript file for the previous example:

```
%!PS
newpath
400 200 moveto
400 400 600 300 200 300 curveto
stroke
```

$\vec{x}_0$ = 400 200, $\vec{x}_1$ = 400 400, $\vec{x}_2$ = 600 300, $\vec{x}_3$ = 200 300