Introduction to MHD (magnetohydrodynamics)

Anthony Yeates

STFC Summer School, 2-Sep-2025

What is MHD?

Magnetohydrodynamics (**MHD**) is the standard continuum model for an electrically conducting fluid in the presence of a magnetic field.

– applies to liquid metals and plasmas.

– main variables:

$\boldsymbol{v}(\boldsymbol{x}, t)$ — **velocity field** [m/s] or [cm/s]

$\rho(\boldsymbol{x}, t)$ — (mass) **density** [kg/m^3] or [g/cm^3]

$p(\boldsymbol{x}, t)$ — (fluid) **pressure** [Pa] or [Ba]

$\boldsymbol{B}(\boldsymbol{x}, t)$ — **magnetic field*** [T] or [G]

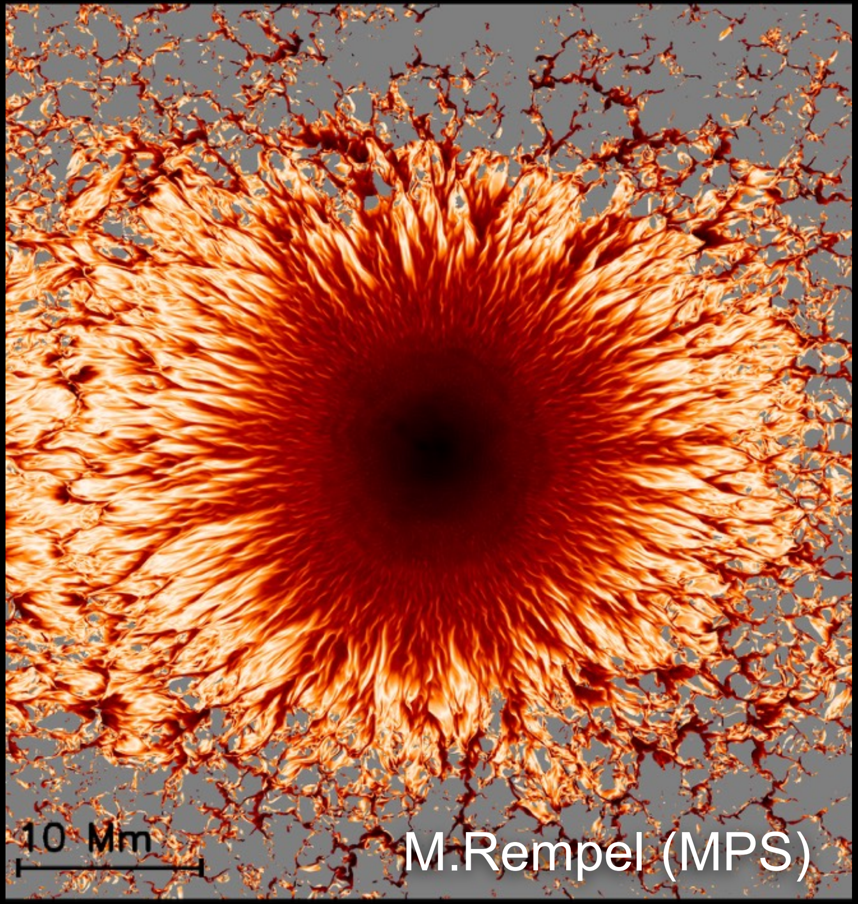
*technically "magnetic flux density"

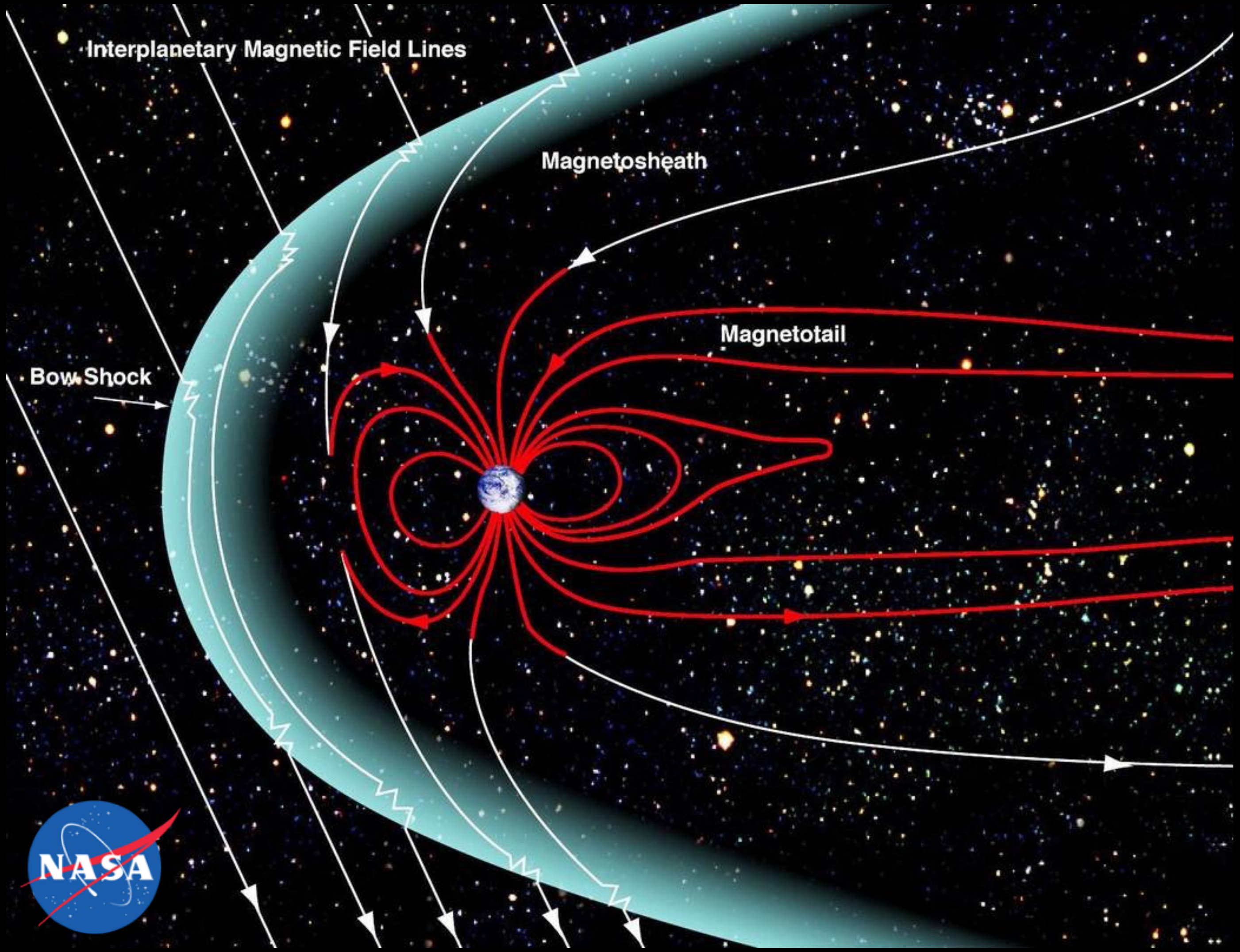
– key assumptions:

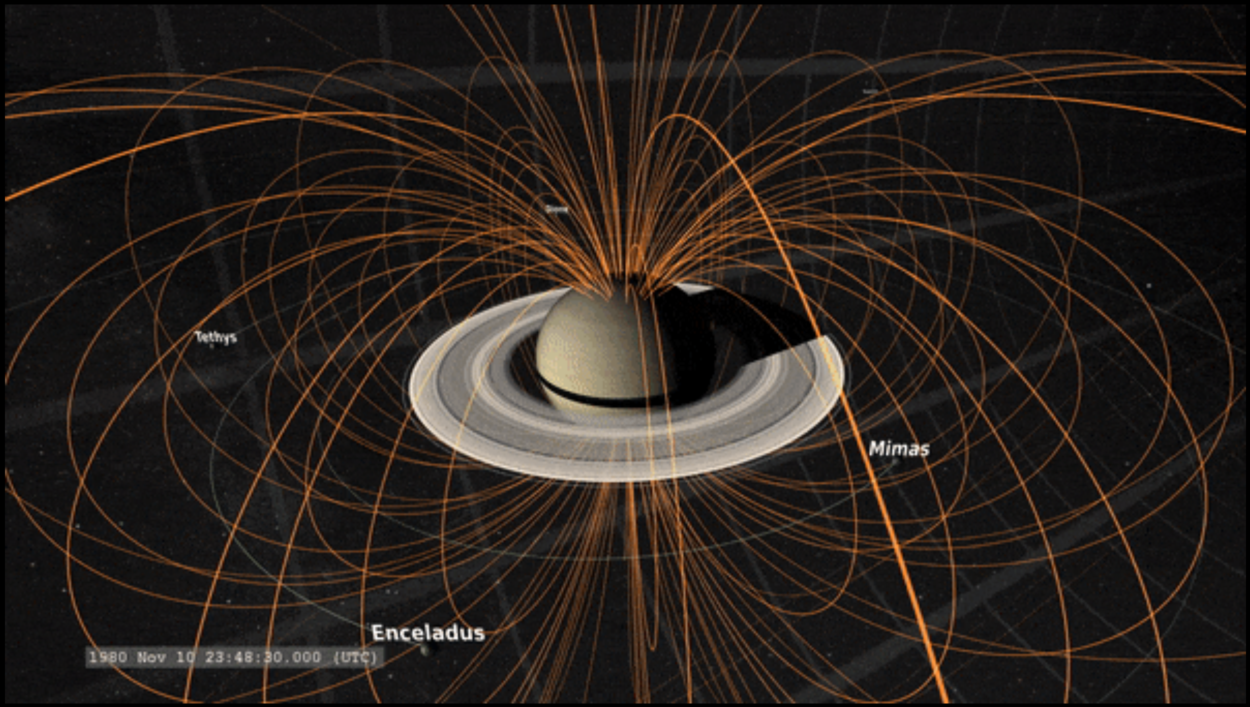
– sufficiently collisional [for fluid model to make sense]
– single fluid [neglect electron mass]
– speeds are sub-relativistic [neglect displacement current…]

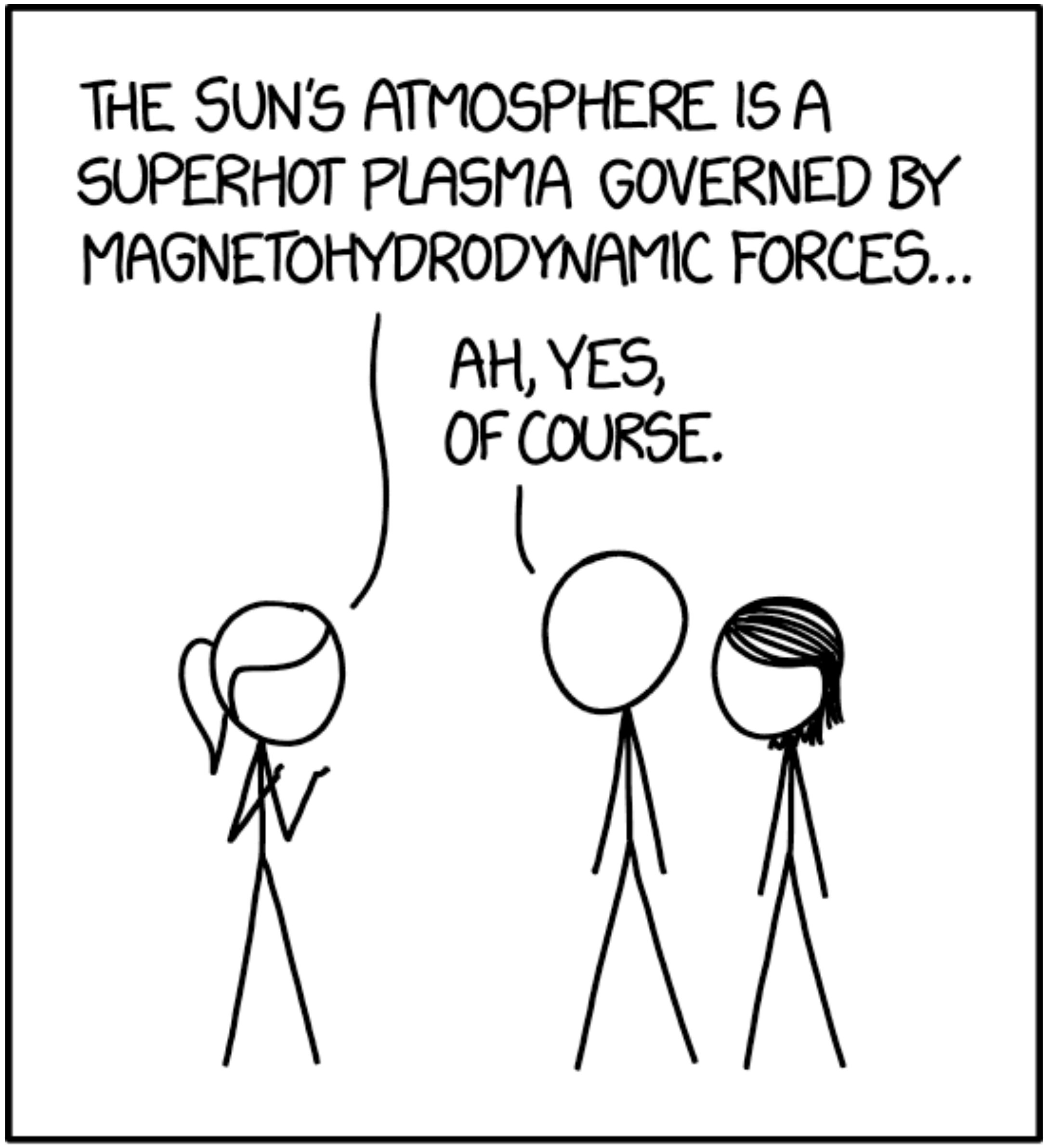
two-way interaction between magnetic field and fluid motions

Applications









WHENEVER I HEAR THE WORD "MAGNETOHYDRODYNAMIC" MY BRAIN JUST REPLACES IT WITH "MAGIC."

https://xkcd.com/1851/

"Magnetohydrodynamics combines the intuitive nature of Maxwell's equations with the easy solvability of the Navier-Stokes equations."

ELECTROMAGNETIC EQUATIONS

FLUID EQUATIONS

EQUILIBRIA

Electromagnetic equations

1. Maxwell's equations:

$$\nabla \cdot \boldsymbol{E} = \frac{\rho_c}{\varepsilon_0} \qquad \nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad \text{(Faraday's Law)}$$

$$\nabla \cdot \boldsymbol{B} = 0 \qquad \nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j} + \epsilon_0 \mu_0 \frac{\partial \boldsymbol{E}}{\partial t} \qquad \text{(Ampère's Law)}$$

2. Ohm's law:

neglect the displacement current [valid for **sub-relativistic** motions]

$$\boldsymbol{j} = \sigma \left(\underbrace{\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}}_{\text{electric field in the frame of the moving fluid}} \right)$$

electric field in the frame of the moving fluid

conductivity – assume constant but really depends on temperature

Eliminate $\boldsymbol{E}$:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times \left(\boldsymbol{v} \times \boldsymbol{B} - \frac{\boldsymbol{j}}{\sigma} \right) = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) - \frac{1}{\mu_0} \nabla \times \left(\frac{\nabla \times \boldsymbol{B}}{\sigma} \right)$$

$$\nabla \times (\nabla \times \boldsymbol{B}) = \nabla(\nabla \cdot \boldsymbol{B}) - \nabla^2 \boldsymbol{B}$$

$$\boxed{\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}}$$ **– induction equation**

magnetic diffusivity

$$\eta = \frac{1}{\mu_0 \sigma}$$

Implications of the induction equation

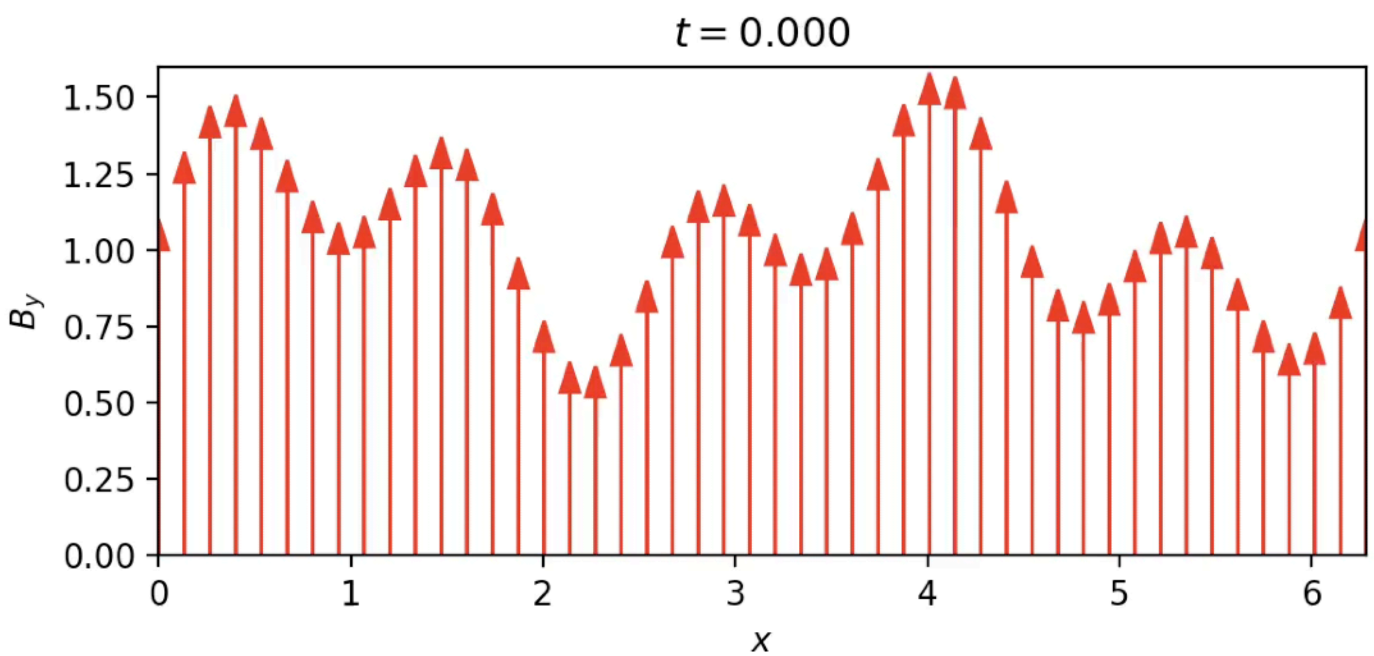
$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}$$

If $\boldsymbol{v} = \boldsymbol{0}$, we have **diffusive decay** of the magnetic field:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \eta \nabla^2 \boldsymbol{B}$$

– smooths out gradients

e.g. $$\boldsymbol{B} = \left(B_0 + \sum_k B_k \sin(kx) \mathrm{e}^{-\eta k^2 t} \right) \boldsymbol{e}_y$$



– decay time for each "mode" is

$$\tau_\eta \approx \frac{1}{k^2 \eta}$$

To measure when diffusion is important, compare the size of the terms:

typical values

$$|\boldsymbol{v}| \sim v_0$$
$$|\boldsymbol{B}| \sim B_0$$

$$\Longrightarrow \quad \frac{|\nabla \times (\boldsymbol{v} \times \boldsymbol{B})|}{|\eta \nabla^2 \boldsymbol{B}|} \sim \frac{v_0 B_0 / \ell_0}{\eta B_0 / \ell_0^2} = \frac{v_0 \ell_0}{\eta}$$

magnetic Reynolds' number

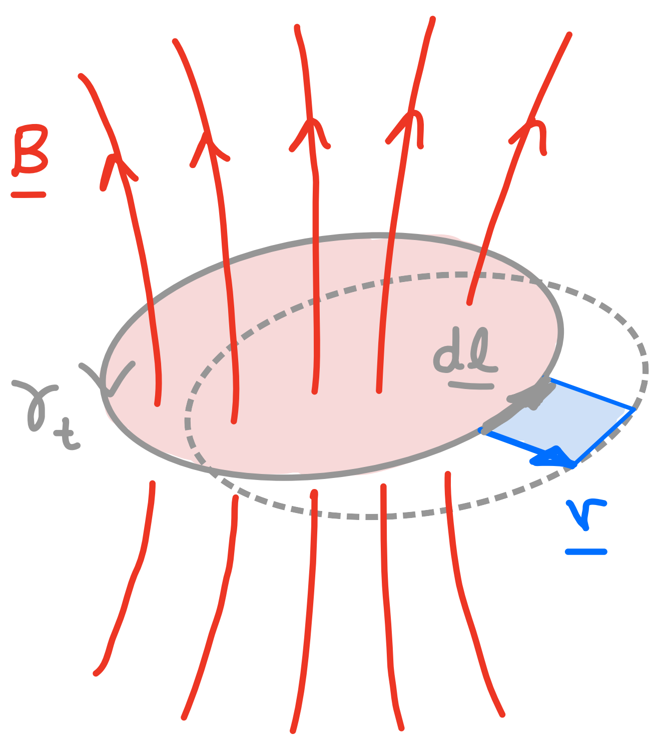
– for Sun, diffusion matters only when $\boldsymbol{B}$ varies on small scales **[see Gingell lecture…]**

Implications of the induction equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}$$

If $\eta = 0$, we have **ideal MHD**:

$$\boxed{\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B})}$$



flux through a material curve (moving with fluid)

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_{\gamma_t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{S_t} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}$$

$$= \int_{S_t} \frac{\partial \boldsymbol{B}}{\partial t} \cdot \mathrm{d}\boldsymbol{S} + \oint_{\gamma_t} \boldsymbol{B} \cdot (\boldsymbol{v} \times \mathrm{d}\boldsymbol{l})$$

$$= \int_{S_t} \frac{\partial \boldsymbol{B}}{\partial t} \cdot \mathrm{d}\boldsymbol{S} - \oint_{\gamma_t} (\boldsymbol{v} \times \boldsymbol{B}) \cdot \mathrm{d}\boldsymbol{l}$$

$$= \int_{S_t} \left(\frac{\partial \boldsymbol{B}}{\partial t} - \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) \right) \cdot \mathrm{d}\boldsymbol{S}$$

$$= 0$$

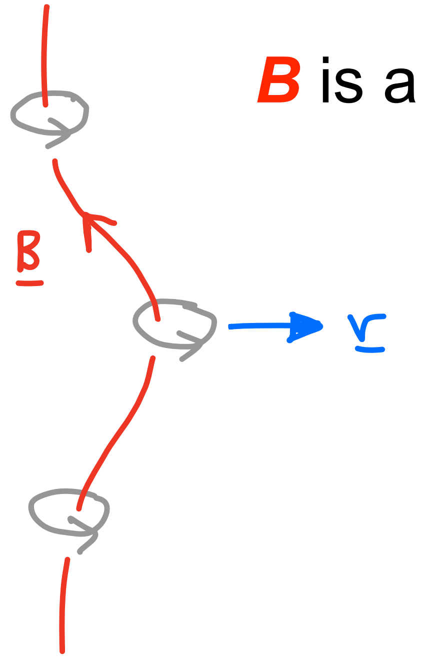
– known as **Alfvén's (frozen flux) Theorem**

Hannes Alfvén (1908–1995)

Nobel Prize for Physics, 1970 – "for fundamental work and discoveries in magnetohydro-dynamics with fruitful applications in different parts of plasma physics”

Ideal MHD solutions

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B})$$

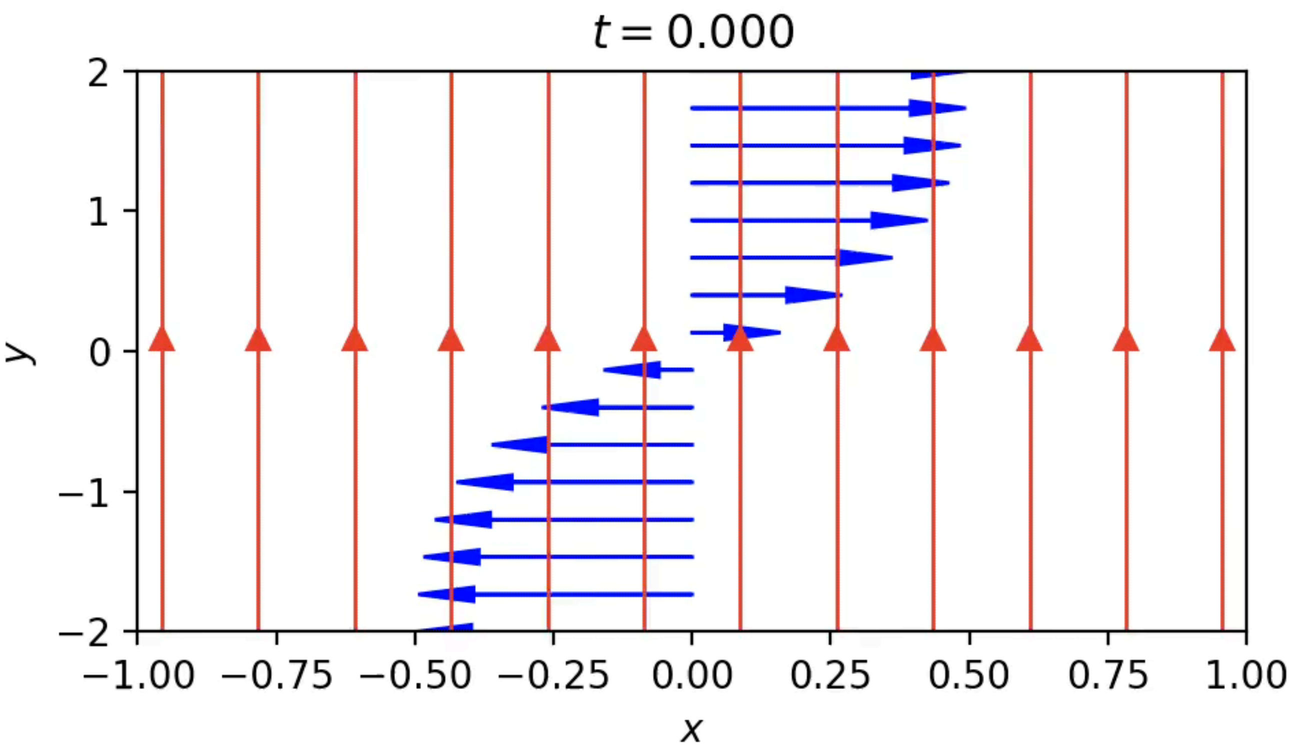


$\boldsymbol{B}$ is a "material vector"

e.g. (1)

$$\boldsymbol{v} = u(y)\boldsymbol{e}_x$$

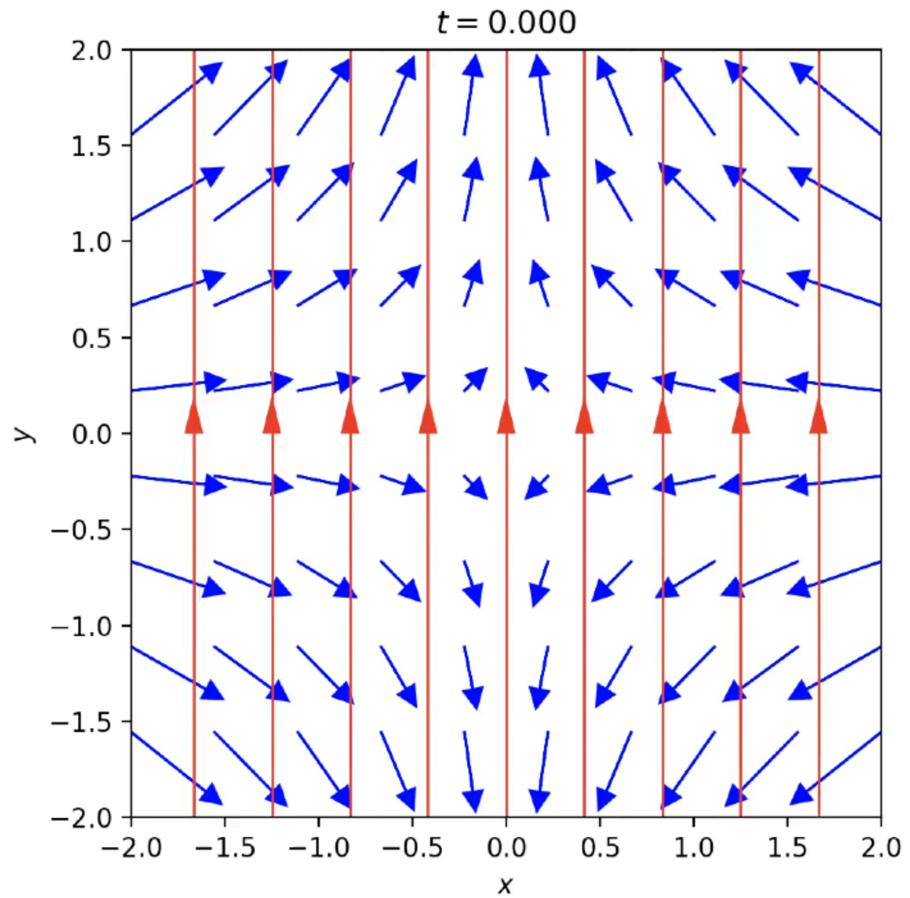
$$\boldsymbol{B} = B_0\left[u'(y)t\boldsymbol{e}_x + \boldsymbol{e}_y\right]$$



e.g. (2)

$$\boldsymbol{v} = v_0(-x\boldsymbol{e}_x + y\boldsymbol{e}_y)$$

$$\boldsymbol{B} = b(xe^{v_0 t})e^{v_0 t}\boldsymbol{e}_y$$

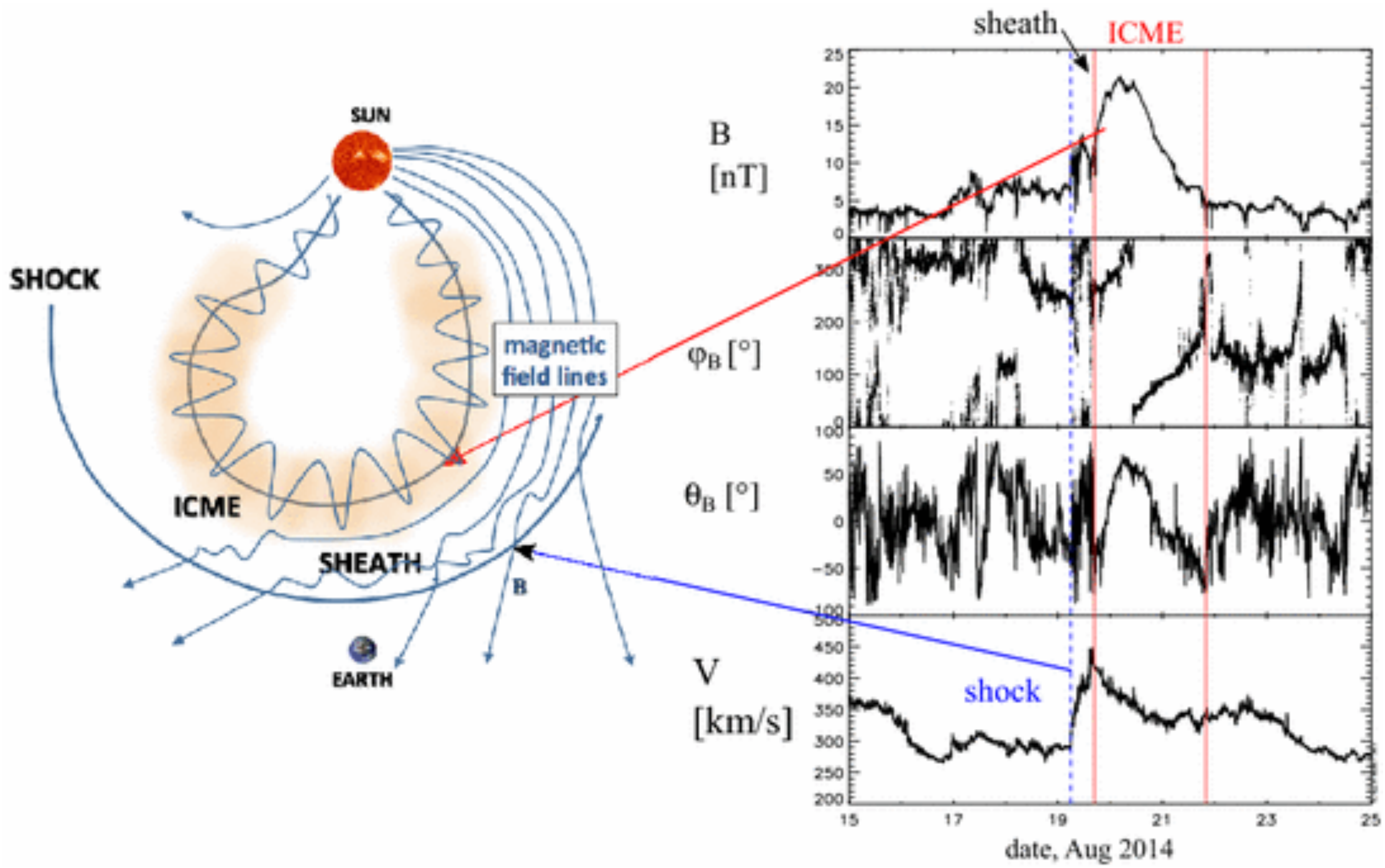


– exponential amplification!

[e.g. solar dynamo…]

Example – magnetic clouds

– twisted “magnetic flux ropes” expelled from the Sun.



Kilpua et al., *Living Rev Solar Phys* (2017)

[see Morgan lecture…]

Fluid equations

3. Momentum equation [usual Navier-Stokes with extra **Lorentz force** term]:

$$\boxed{\underbrace{\rho\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t}}_{\text{mass [per unit volume]} \times \text{acceleration}} = \underbrace{-\nabla p + \rho\boldsymbol{g} + \boldsymbol{F}_{\text{viscous}} + \boldsymbol{j}\times\boldsymbol{B}}_{\text{force [per unit volume]}}}$$

– material derivative: $\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + (\boldsymbol{v}\cdot\nabla)$

cf. force on a charged particle $\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})$

– neglect Coulomb force for sub-relativistic speeds

4. Continuity equation [mass conservation]:

$$\boxed{\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{v}) = 0} \iff \frac{\partial\rho}{\partial t} + \boldsymbol{v}\cdot\nabla\rho = -\rho\nabla\cdot\boldsymbol{v} \iff \frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho\nabla\cdot\boldsymbol{v}$$

5. Energy equation [of choice]:

constant sound speed

– e.g. **isothermal** $\boxed{p = c_s^2\rho}$

polytropic index $\gamma = \frac{5}{3}$ (hydrogen)

– e.g. **adiabatic ideal gas** $\boxed{\rho\frac{\mathrm{D}\varepsilon}{\mathrm{D}t} = -p\nabla\cdot\boldsymbol{v}}$ for **internal energy** $\varepsilon = \frac{p}{(\gamma-1)\rho}$

– e.g. non-adiabatic ideal gas…

Example – Parker solar wind

Assume Sun's corona is spherically symmetric, isothermal with $\boldsymbol{B} \equiv \mathbf{0}$ and $\boldsymbol{v} = v(r)\boldsymbol{e}_r$.

The (M)HD equations reduce to

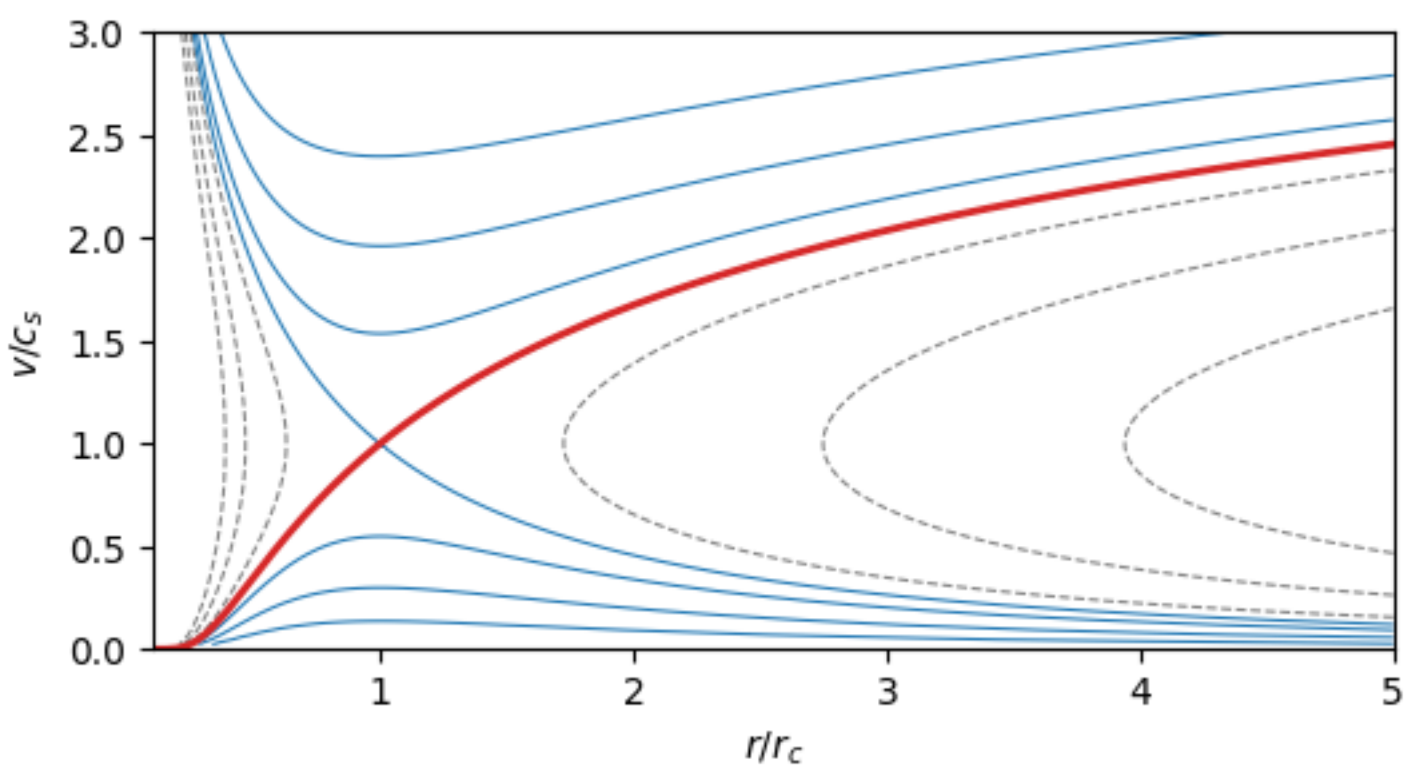
3. $\rho v v' = -p' - \frac{G M_\odot \rho}{r^2}$ 4. $r^2 \rho v = \text{const.}$ 5. $p = c_s^2 \rho$

$$\implies \left(v - \frac{c_s^2}{v}\right) v' = \frac{2c_s^2}{r} - \frac{G M_\odot}{r^2}$$

Solving this ODE gives the (implicit) solution

$$\left(\frac{v}{c_s}\right)^2 - \log\left(\frac{v}{c_s}\right)^2 = 4 \log \frac{r}{r_c} + \frac{4 r_c}{r} + C$$

"critical radius" $r_c \equiv \frac{G M_\odot}{2 c_s^2}$



– only physical solution [subsonic at Sun and no pressure at large distances] is $C = -3$.

Example – Parker solar wind

Assume the Sun's corona is axisymmetric, isothermal with $\boldsymbol{B} \equiv \mathbf{0}$ and $\boldsymbol{v} = v(r)\boldsymbol{e}_r$.

The (M)HD equations reduce to

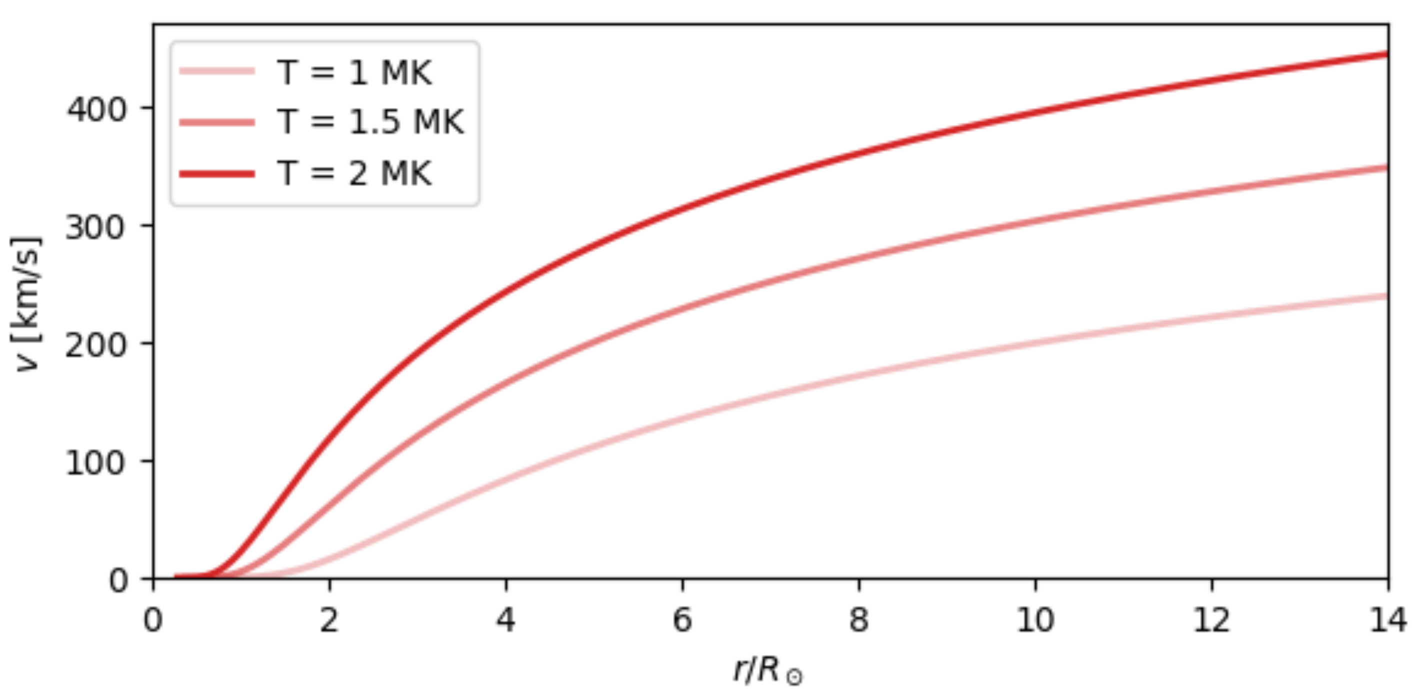
3. $\rho v v' = -p' - \dfrac{G M_\odot \rho}{r^2}$ 4. $r^2 \rho v = \text{const.}$ 5. $p = c_s^2 \rho$

$$\Longrightarrow \left(v - \frac{c_s^2}{v}\right) v' = \frac{2c_s^2}{r} - \frac{G M_\odot}{r^2}$$

Solving this ODE gives the (implicit) solution

$$\left(\frac{v}{c_s}\right)^2 - \log\left(\frac{v}{c_s}\right)^2 = 4\log\frac{r}{r_c} + \frac{4r_c}{r} + C$$

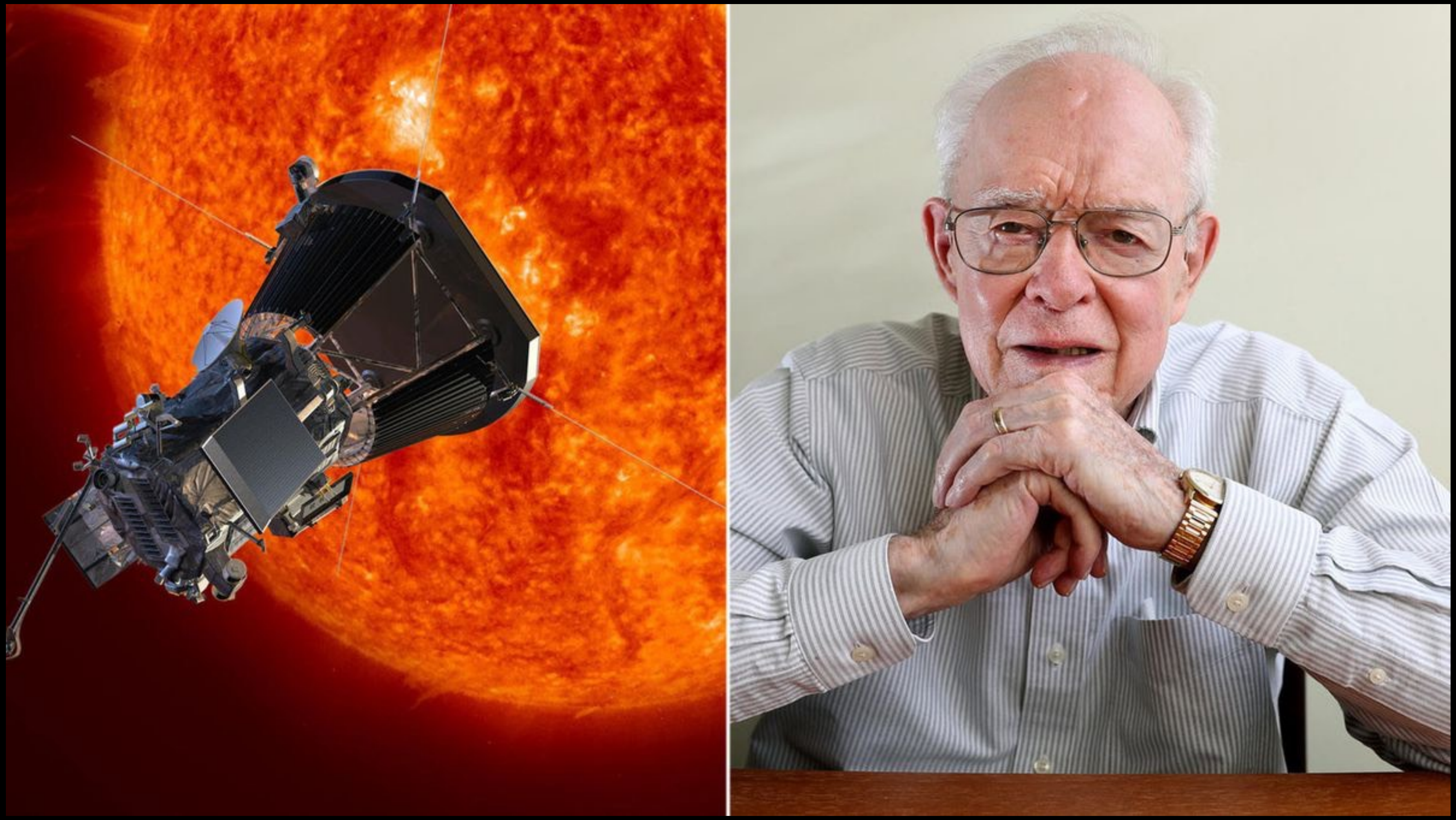
"critical radius" $r_c \equiv \dfrac{G M_\odot}{2c_s^2}$



– only physical solution [subsonic at Sun and no pressure at large distances] is C = -3.

[see Morgan lecture…]

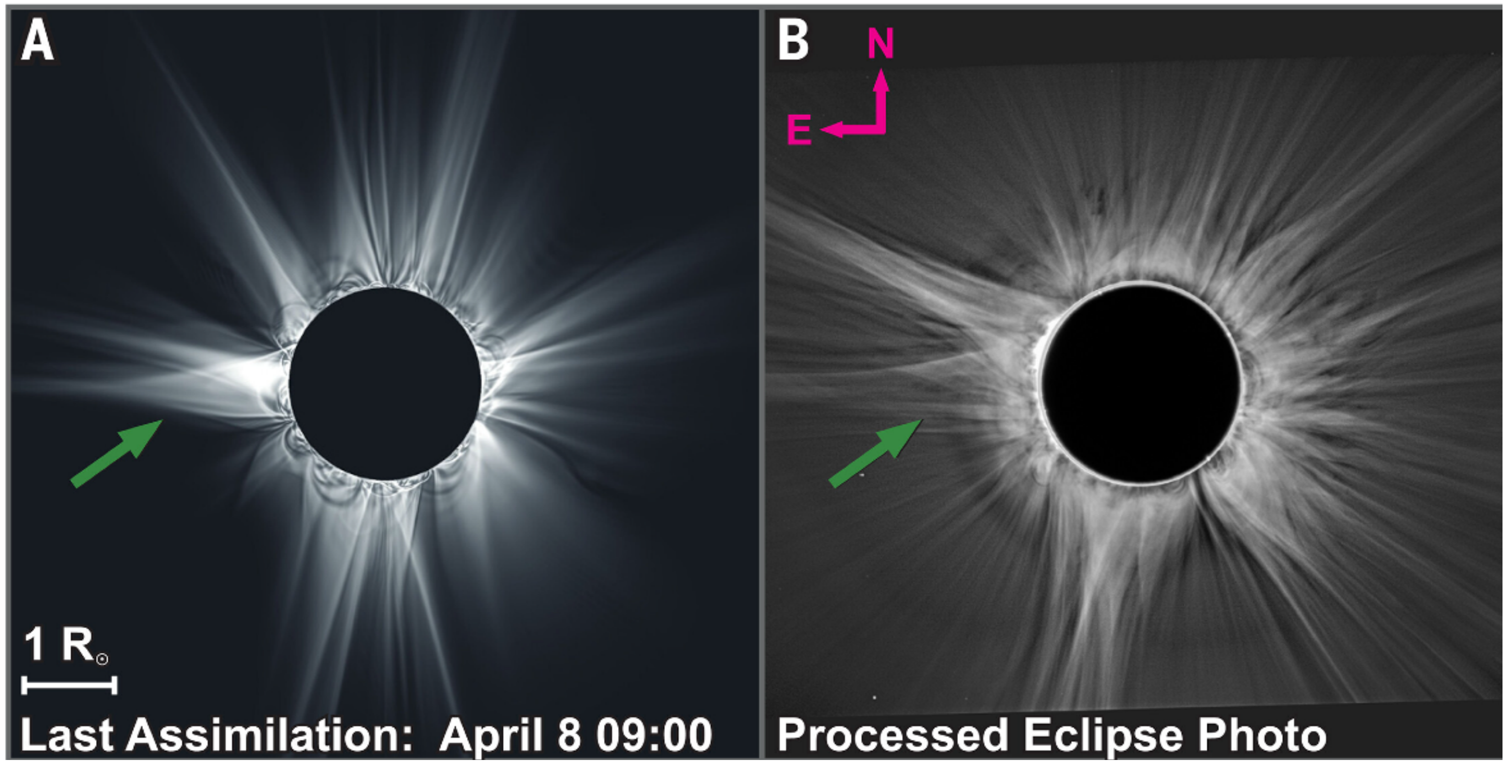
Eugene Parker (1927–2022)



Example – eclipse prediction

https://www.predsci.com/

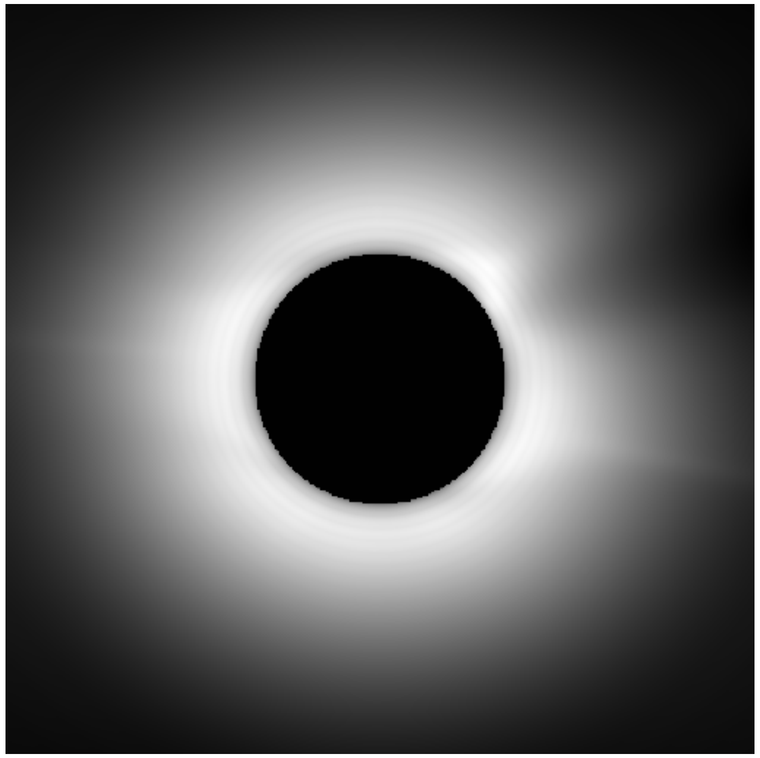
Downs et *al.*, *Science* (2025)

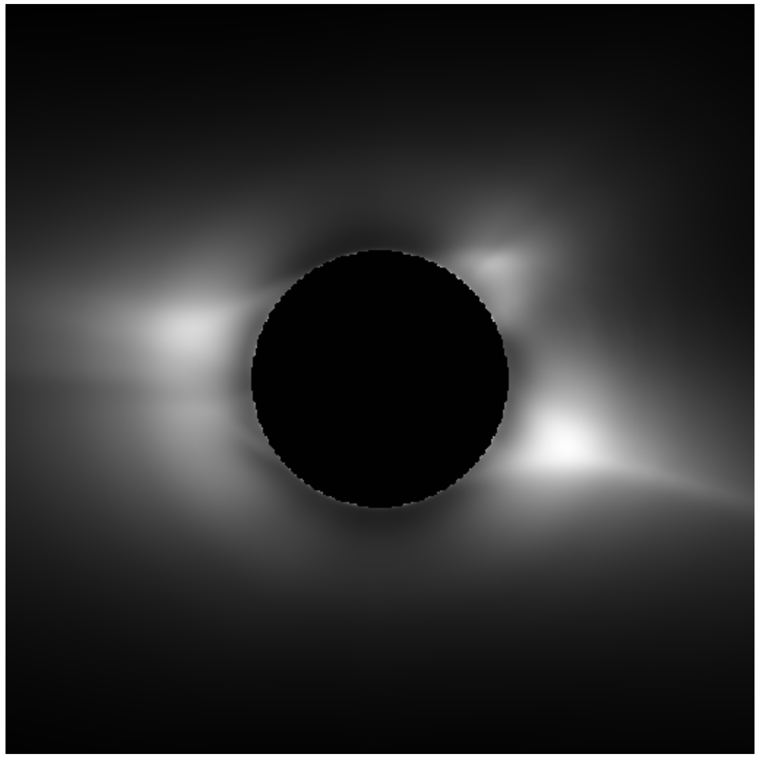


Thermodynamics matter!

from https://www.predsci.com/mhdweb/

"polytropic" vs. **"thermodynamic"**





$$\rho \frac{\mathrm{D}\varepsilon}{\mathrm{D}t} = -p \nabla \cdot \boldsymbol{v}$$

(adiabatic)

$$\rho \frac{\mathrm{D}\varepsilon}{\mathrm{D}t} = -p \nabla \cdot \boldsymbol{v} + \underbrace{\nabla \cdot (\kappa \nabla T)}_{\text{heat conduction}} - \underbrace{\rho^2 Q(T)}_{\text{radiation}} + \underbrace{\frac{|\boldsymbol{j}|^2}{\sigma}}_{\text{ohmic heating}} + \ldots$$

temperature $T = \dfrac{m\varepsilon}{k_B(\gamma - 1)}$

Magnetic equilibria

$$\rho \frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = -\nabla p + \rho \boldsymbol{g} + \boldsymbol{F}_{\text{viscous}} + \boldsymbol{j} \times \boldsymbol{B}$$

When is neglecting the left-hand side valid?

Typical values: $|\rho| \sim \rho_0$ $\quad |\boldsymbol{v}| \sim v_0$ $\quad t \sim \frac{\ell_0}{v_0}$ $\quad |\boldsymbol{B}| \sim B_0$ $\quad |p| \sim p_0$

$$\left|\rho \frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t}\right| \ll |\nabla p| \iff \frac{\rho_0 {v_0}^2}{\ell_0} \ll \frac{p_0}{\ell_0} \iff v_0 \ll \underbrace{\sqrt{\frac{p_0}{\rho_0}}}_{\text{sound speed}}$$

$$\left|\rho \frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t}\right| \ll |\boldsymbol{j} \times \boldsymbol{B}| \iff \frac{\rho_0 {v_0}^2}{\ell_0} \ll \frac{{B_0}^2}{\mu_0 \ell_0} \iff v_0 \ll \underbrace{\frac{B_0}{\sqrt{\mu_0 \rho_0}}}_{\text{Alfvén speed}}$$

– conversely, must be rough speed of ***B***-driven motions…

– magnetohydrostatic equilibrium

$$\boxed{\boldsymbol{j} \times \boldsymbol{B} - \nabla p + \rho \boldsymbol{g} = \boldsymbol{0}}$$

Force-free equilibria

$$\boldsymbol{j} \times \boldsymbol{B} - \nabla p + \rho \boldsymbol{g} = \boldsymbol{0}$$

– can neglect gravity if

$$|\rho \boldsymbol{g}| \ll |\nabla p| \iff \rho_0 g \ll \frac{p}{\ell_0} \iff \ell_0 \ll \underbrace{\frac{p_0}{\rho_0 g}}_{\text{pressure scale height}}$$

– can neglect pressure gradients if

$$|\nabla p| \ll |\boldsymbol{j} \times \boldsymbol{B}| \iff \frac{p_0}{\ell_0} \ll \frac{{B_0}^2}{\mu_0 \ell_0} \iff \underbrace{\frac{2\mu_0 p_0}{{B_0}^2}}_{\text{plasma beta}} \ll 1$$

– **force-free equilibrium** $\boxed{\boldsymbol{j} \times \boldsymbol{B} = \boldsymbol{0}} \iff \nabla \times \boldsymbol{B} = \alpha \boldsymbol{B}$

$0 = \nabla \cdot (\alpha \boldsymbol{B}) \iff \boldsymbol{B} \cdot \nabla \alpha = 0$ so α is constant along magnetic field lines

1. $\nabla \alpha \neq \boldsymbol{0}$

nonlinear force-free field

2. $\alpha(\boldsymbol{x}) = \alpha_0$

linear force-free field

3. $\alpha \equiv 0$

potential field
($\boldsymbol{j} = \boldsymbol{0}$)

Example – potential fields

$\boldsymbol{j} = \mathbf{0}$

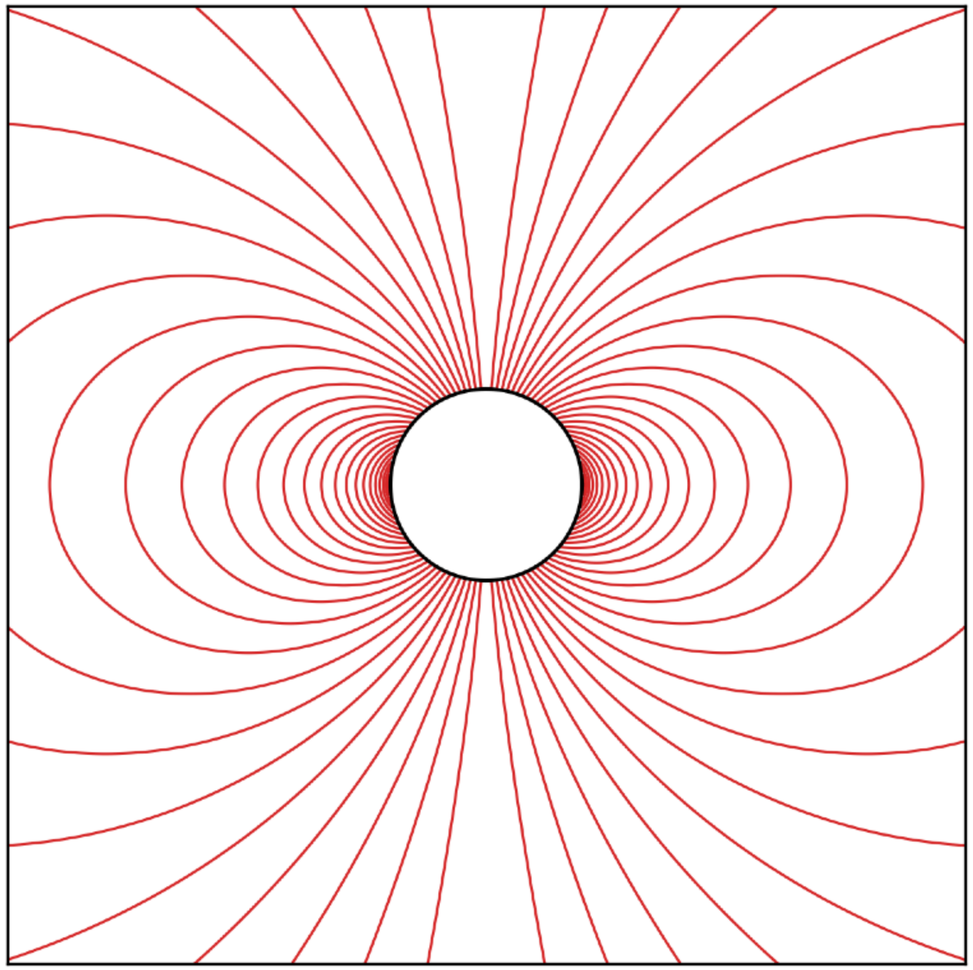
$\nabla \times \boldsymbol{B} = \mathbf{0} \quad \Longrightarrow \quad \boldsymbol{B} = \nabla \psi \qquad \nabla \cdot \boldsymbol{B} = 0 \quad \Longrightarrow \quad \boxed{\nabla^2 \psi = 0}$ ⟵ **Laplace equation**

– inner boundary: $\left.\dfrac{\partial \psi}{\partial r}\right|_{r=R_\odot} = \left. B_r \right|_{r=R_\odot} = b_\odot(\theta, \phi)$ [Neumann boundary condition]

– outer boundary: “infinite space” [decaying modes only]

e.g. dipole $b_\odot(\theta, \phi) = B_0 \cos\theta$

$$\boldsymbol{B} = B_0 \left[2 \left(\frac{R_\odot}{r} \right)^3 \cos\theta \, \boldsymbol{e}_r + \left(\frac{R_\odot}{r} \right)^3 \sin\theta \, \boldsymbol{e}_\theta \right]$$



Example – potential fields

$\boldsymbol{j} = \mathbf{0}$

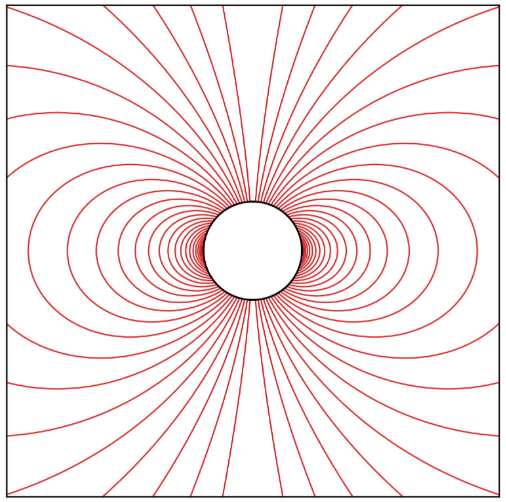
$\nabla \times \boldsymbol{B} = \mathbf{0} \quad \Longrightarrow \quad \boldsymbol{B} = \nabla \psi \qquad \nabla \cdot \boldsymbol{B} = 0 \quad \Longrightarrow \quad \boxed{\nabla^2 \psi = 0}$ ← **Laplace equation**

– inner boundary: $\left.\frac{\partial \psi}{\partial r}\right|_{r=R_\odot} = \left.B_r\right|_{r=R_\odot} = b_\odot(\theta, \phi)$ [Neumann boundary condition]

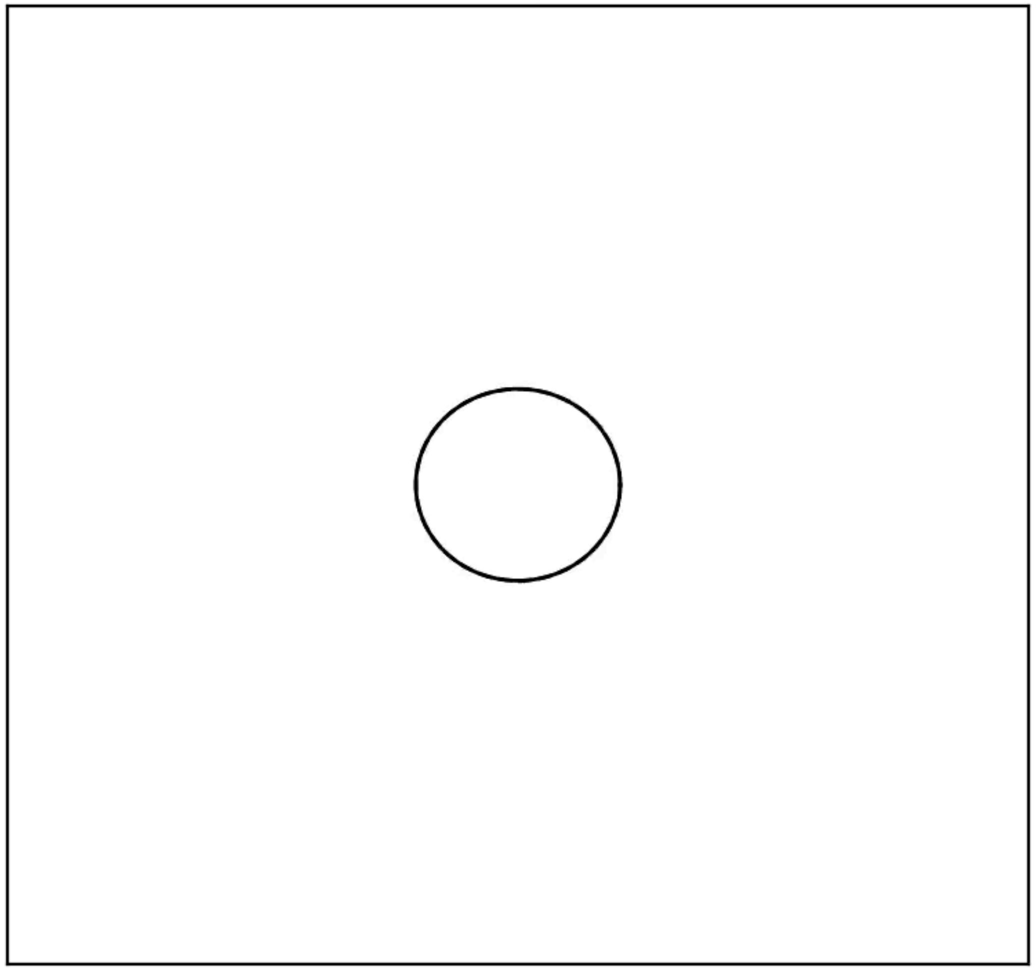
– outer boundary: $\left.\boldsymbol{e}_r \times \nabla \psi\right|_{r=R_1} = \mathbf{0}$ [**PFSS**/**potential field source surface** model]

e.g. dipole $b_\odot(\theta, \phi) = B_0 \cos\theta$

$$\boldsymbol{B} = B_0 \left[\frac{R_\odot^3 (r^3 + 2R_1^3)}{r^3 (R_\odot^3 + 2R_1^3)} \cos\theta \, \boldsymbol{e}_r + \frac{R_\odot^3 (R_1^3 - r^3)}{r^3 (R_\odot^3 + 2R_1^3)} \sin\theta \, \boldsymbol{e}_\theta \right]$$



$R_1 = 1.00 r_0$



Example – potential fields

$\boldsymbol{j} = 0$

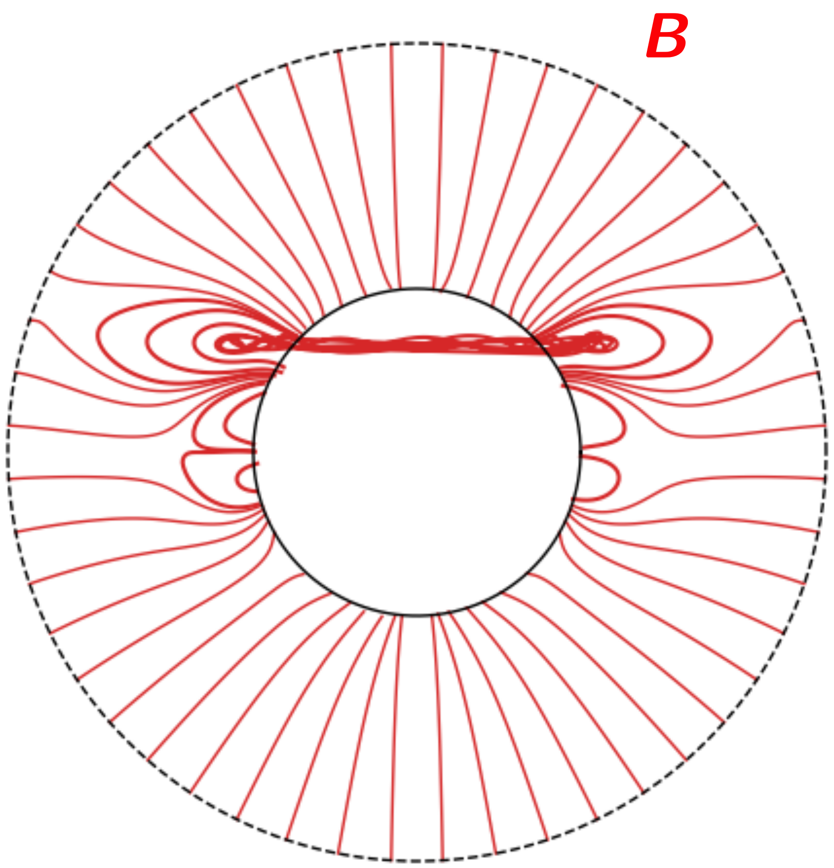
Claim: The potential field has minimum energy for given boundary conditions.

– any magnetic field can be written

$$\boldsymbol{B} = \boldsymbol{B}_{\mathrm{p}} + \boldsymbol{B}_{\mathrm{j}} \qquad \text{where} \qquad \boldsymbol{B}_{\mathrm{p}} = \nabla\psi \qquad \nabla^2\psi = 0$$

$$\left.\frac{\partial\psi}{\partial r}\right|_{r=R_\odot} = B_r(R_\odot, \theta, \phi)$$

$$\boldsymbol{e}_r \times \nabla\psi|_{r=R_1} = 0$$



Claim: The potential field has minimum energy for given boundary conditions.

– any magnetic field can be written

$$\boldsymbol{B} = \boldsymbol{B}_\mathrm{p} + \boldsymbol{B}_\mathrm{j} \qquad \text{where} \qquad \boldsymbol{B}_\mathrm{p} = \nabla\psi \qquad \nabla^2\psi = 0$$

$$\left.\frac{\partial\psi}{\partial r}\right|_{r=R_\odot} = B_r(R_\odot, \theta, \phi)$$

$$\boldsymbol{e}_r \times \nabla\psi|_{r=R_1} = 0$$

$$\underbrace{\int_V \frac{|\boldsymbol{B}|^2}{2\mu_0}\,dV}_{\text{magnetic energy}} = \frac{1}{2\mu_0}\int_V (\boldsymbol{B}_\mathrm{p} + \boldsymbol{B}_\mathrm{j})\cdot(\boldsymbol{B}_\mathrm{p} + \boldsymbol{B}_\mathrm{j})\,dV$$

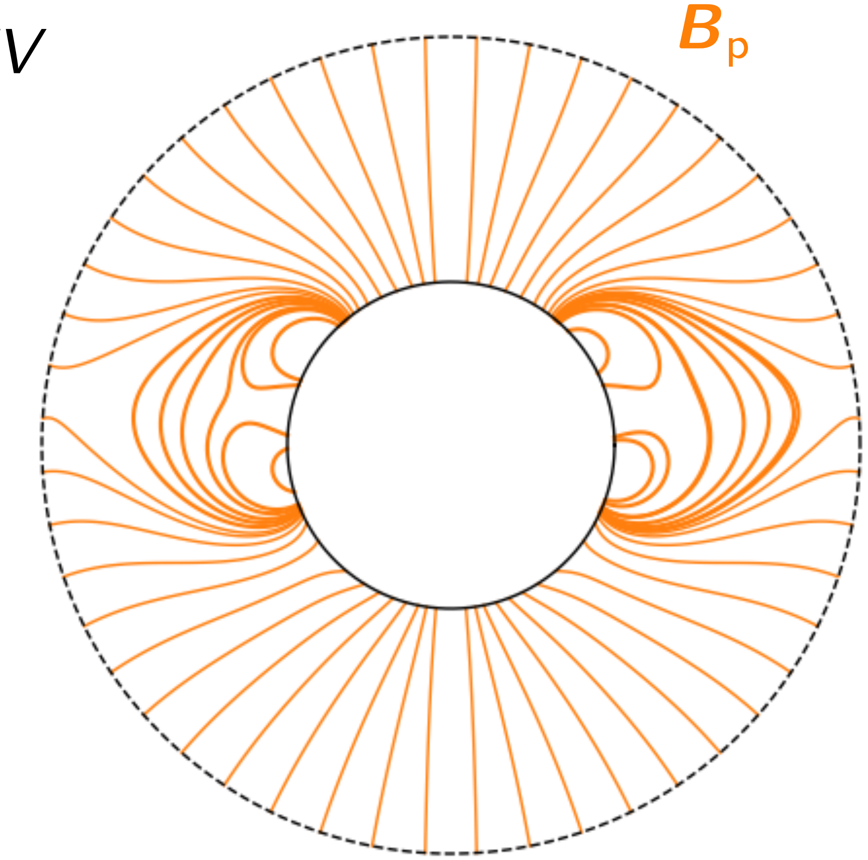
$$= \int_V \frac{|\boldsymbol{B}_\mathrm{p}|^2}{2\mu_0}\,dV + \frac{1}{\mu_0}\underbrace{\cancel{\int_V \boldsymbol{B}_\mathrm{p}\cdot\boldsymbol{B}_\mathrm{j}\,dV}} + \int_V \frac{|\boldsymbol{B}_\mathrm{j}|^2}{2\mu_0}\,dV$$

$$= \int_V \nabla\psi\cdot\boldsymbol{B}_\mathrm{j}\,dV$$

$$= \int_V \left[\nabla\cdot(\psi\boldsymbol{B}_\mathrm{j}) - \psi\nabla\cdot\boldsymbol{B}_\mathrm{j}\right] dV$$

$$= \oint_{\partial V} \psi\boldsymbol{B}_\mathrm{j}\cdot \mathrm{d}\boldsymbol{S}$$

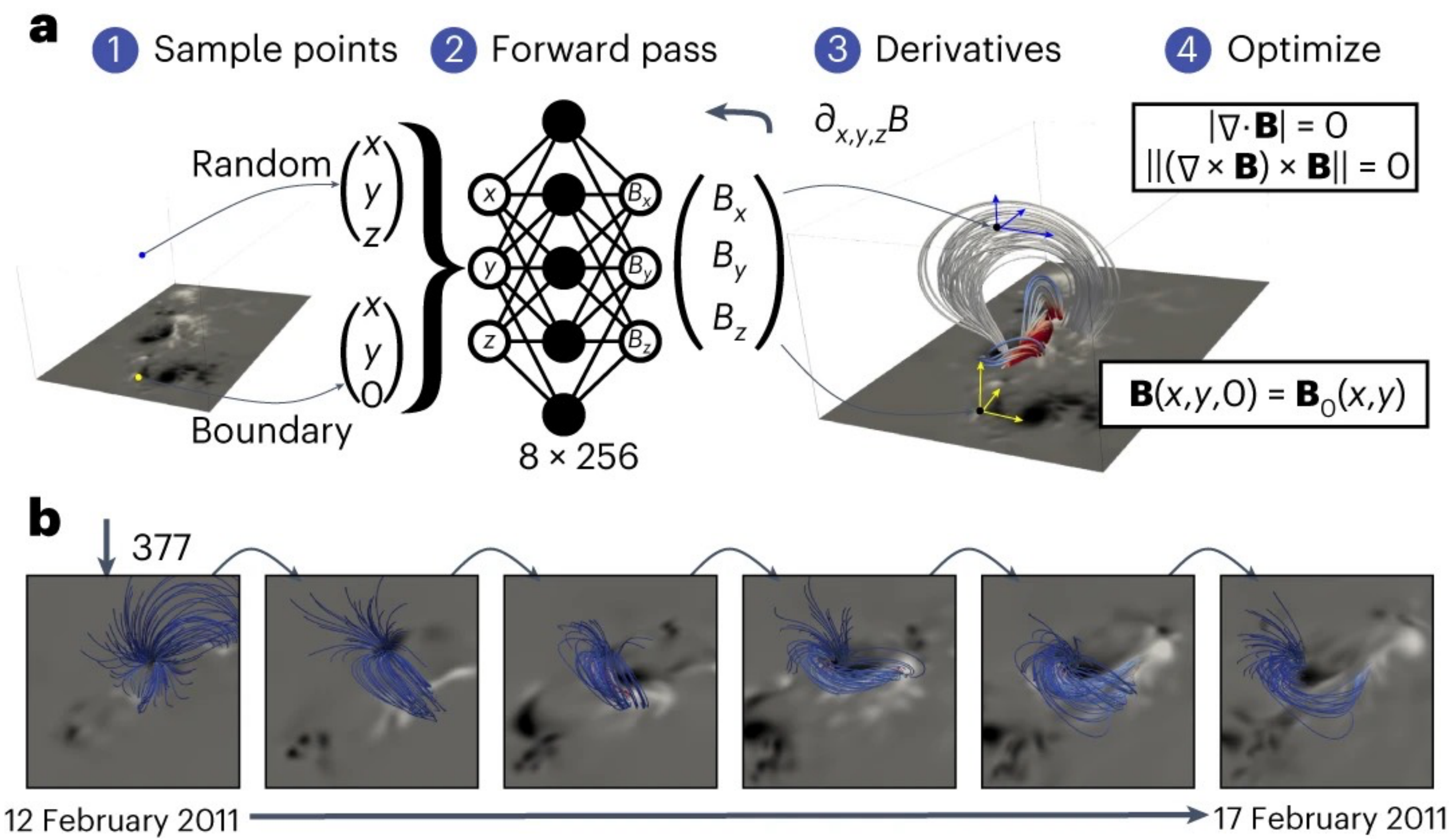
$$= \int_V \frac{|\boldsymbol{B}_\mathrm{p}|^2}{2\mu_0}\,dV + \int_V \frac{|\boldsymbol{B}_\mathrm{j}|^2}{2\mu_0}\,dV$$



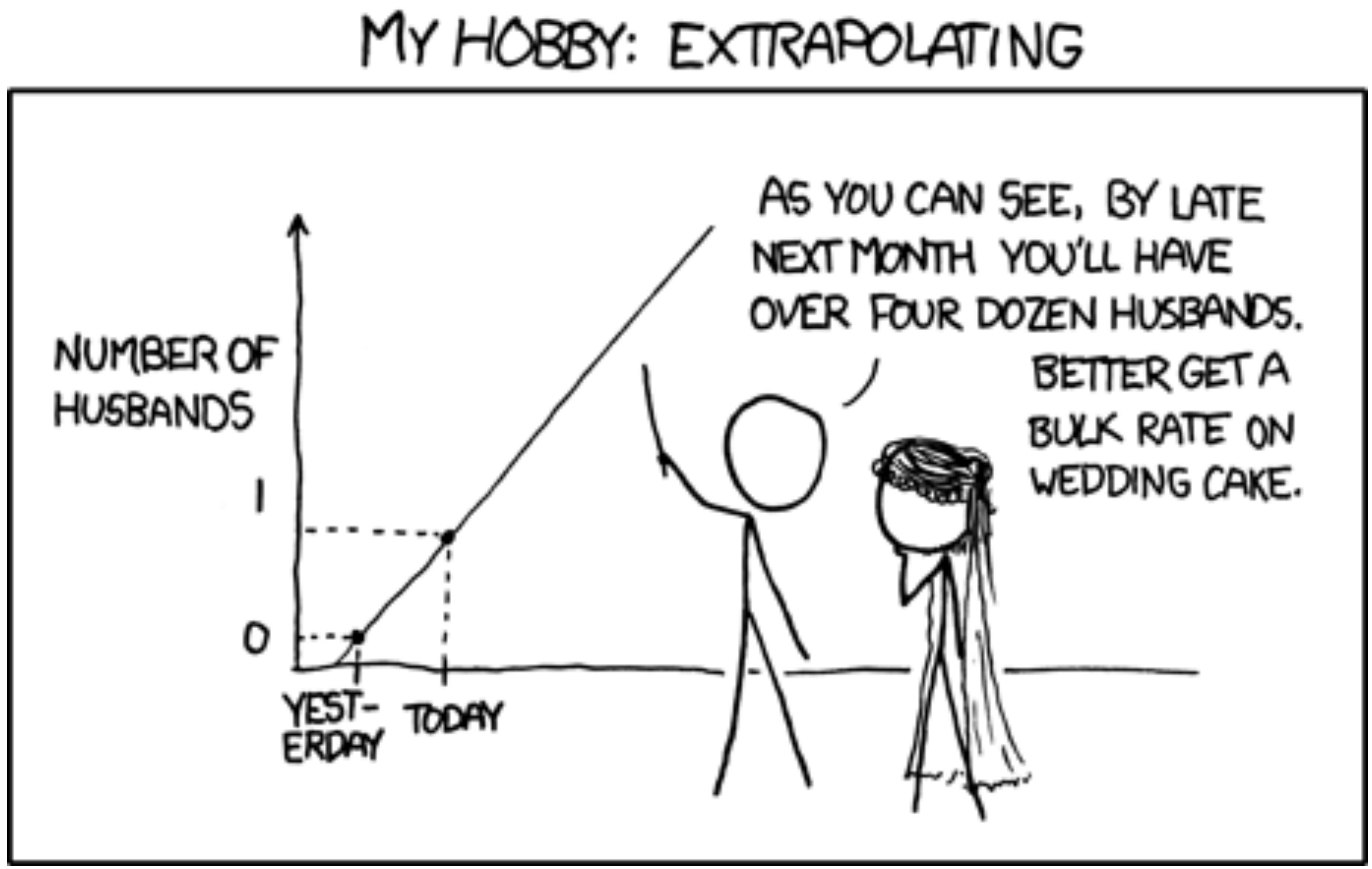
Example – NLFFF models for active regions

$\boldsymbol{j} \times \boldsymbol{B} = 0$

– challenge: observations don't give a unique solution



Jarolim et *al.*, *Nature Astronomy* (2023)



https://xkcd.com/1851/

Summary

MHD equations

$$\nabla \cdot \boldsymbol{B} = 0 \qquad \nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}$$

$$\rho \frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = -\nabla p + \rho \boldsymbol{g} + \boldsymbol{F}_{\mathrm{viscous}} + \boldsymbol{j} \times \boldsymbol{B}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \qquad \rho \frac{\mathrm{D}\varepsilon}{\mathrm{D}t} = -p \nabla \cdot \boldsymbol{v} + \ldots$$

Further reading:

- My lecture notes from the Durham 4th year course: https://www.maths.dur.ac.uk/users/anthony.yeates/NOTES/mhd_notes/_book/
- *Essential Magnetohydrodynamics for Astrophysics*, H. Spruit: https://arxiv.org/abs/1301.5572
- *Introduction to Modern Magnetohydrodynamics*, S. Galtier, Cambridge.
- *Magnetohydrodynamics of Laboratory and Astrophysical Plasmas*, H. Goedbloed, R. Keppens & S. Poedts, Cambridge.
- *Physics of Space Plasma Activity*, K. Schindler, Cambridge.
- *An Introduction to Magnetohydrodynamics*, P.A. Davidson, Cambridge.
- *Magnetohydrodynamics of the Sun*, E.R. Priest, Cambridge.