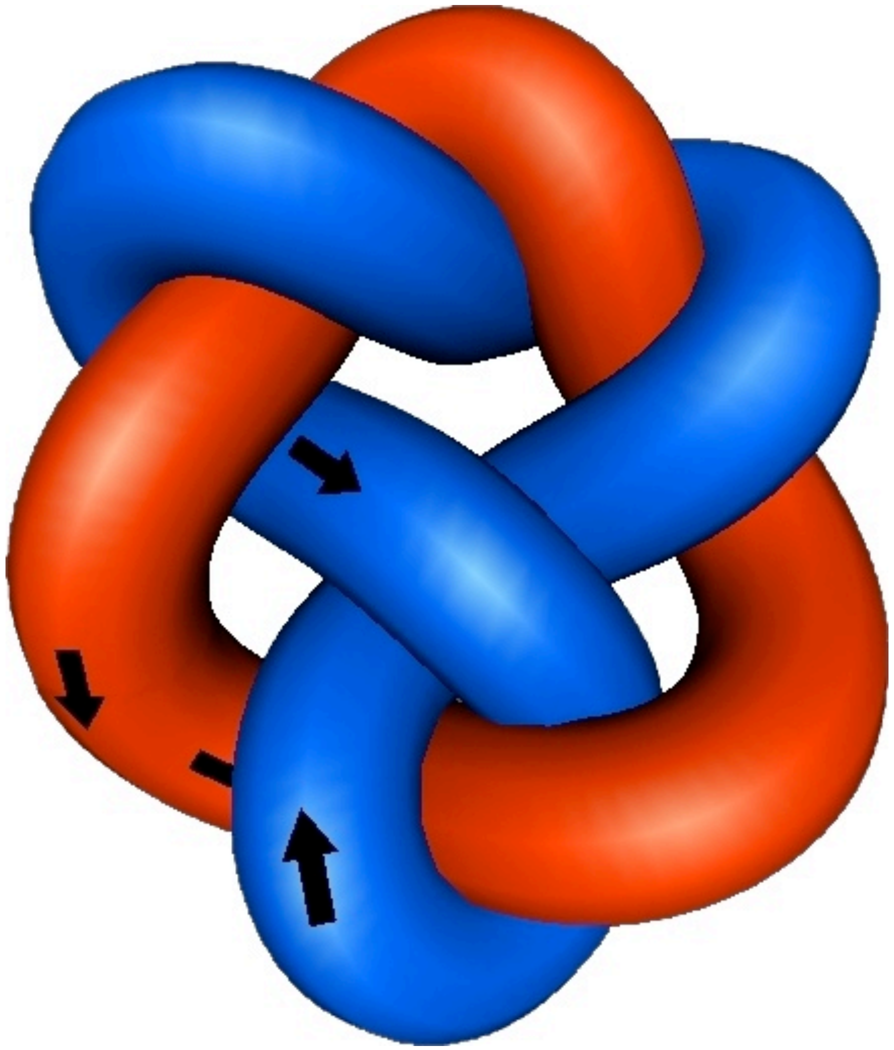


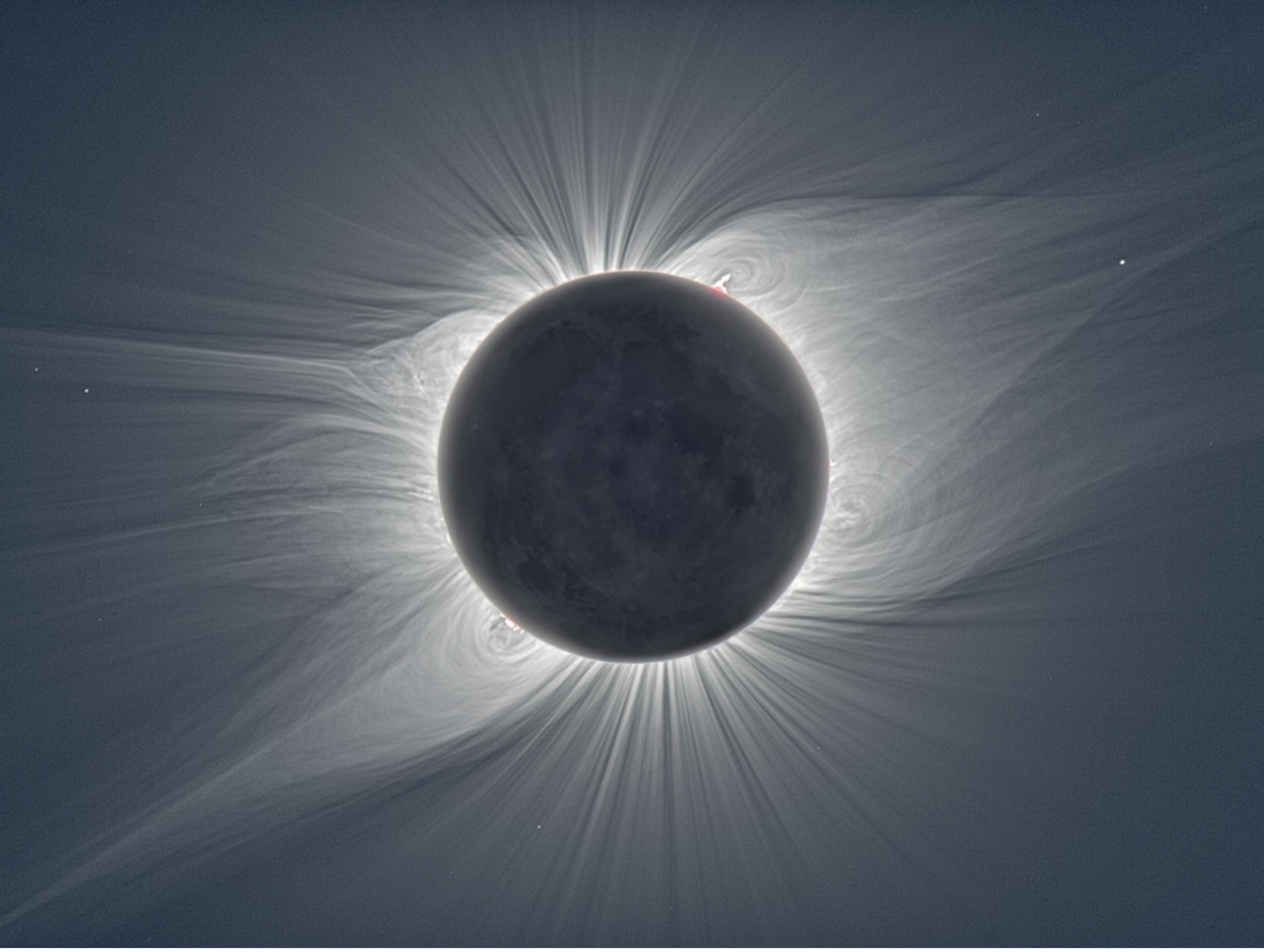
Magnetic Field Topology

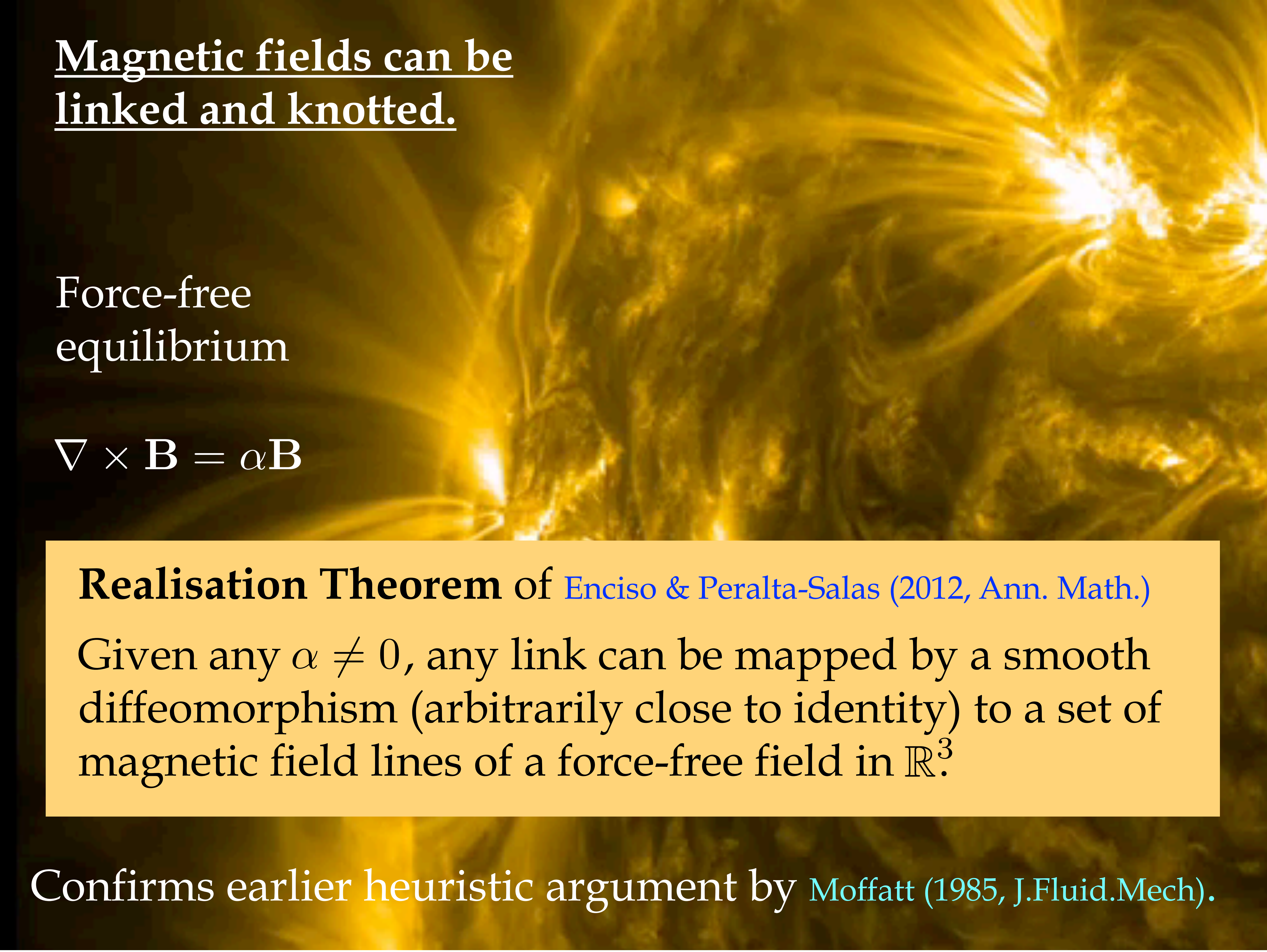
Anthony Yeates

Geometry & Topology Seminar, Durham, 02/05/13

Durham University

Science & Technology Facilities Council





Magnetic fields can be linked and knotted.

Force-free equilibrium

$$\nabla \times \mathbf{B} = \alpha \mathbf{B}$$

Realisation Theorem of Enciso & Peralta-Salas (2012, Ann. Math.)

Given any $\alpha \neq 0$, any link can be mapped by a smooth diffeomorphism (arbitrarily close to identity) to a set of magnetic field lines of a force-free field in $\mathbb{R}^3$.

Confirms earlier heuristic argument by Moffatt (1985, J.Fluid.Mech).

Importance of topology lies in the evolution.

Infinitely conducting plasma $\Rightarrow$ Vanishing electric field *in plasma frame* $\Rightarrow$ $$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0$$

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$$

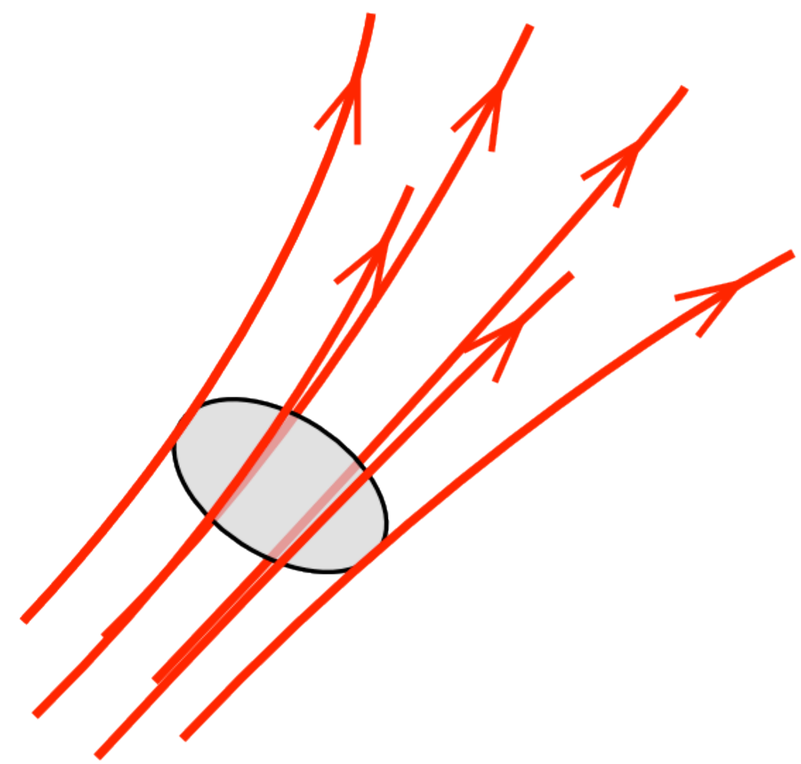
"MHD induction eqn"

Faraday's law

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

Alfvén's Theorem Magnetic flux through a comoving surface is conserved.

$$\frac{d}{dt} \int_V \mathbf{B} \cdot \mathbf{n} \, dA = 0$$



$\Rightarrow$ connectivity of field lines, knots and linkages preserved.

Can form an integral invariant called magnetic helicity.

$$\boxed{H = \int_V \mathbf{A} \cdot \mathbf{B}\, dV}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

1. Gauge invariant under $\mathbf{A} \to \mathbf{A} + \nabla\chi$ when $\mathbf{B} \cdot \mathbf{n}\big|_{\partial V} = 0\,.$

2. Ideal-MHD invariant   Woltjer (1958, Proc.Nat.Acad.Sci.)

$$\frac{dH}{dt} = -2\int_V \mathbf{E} \cdot \mathbf{B}\, dV + \oint_{\partial V} \big(\phi\mathbf{B} - (\mathbf{v} \times \mathbf{B}) \times \mathbf{A}\big) \cdot \mathbf{n} dA = 0$$

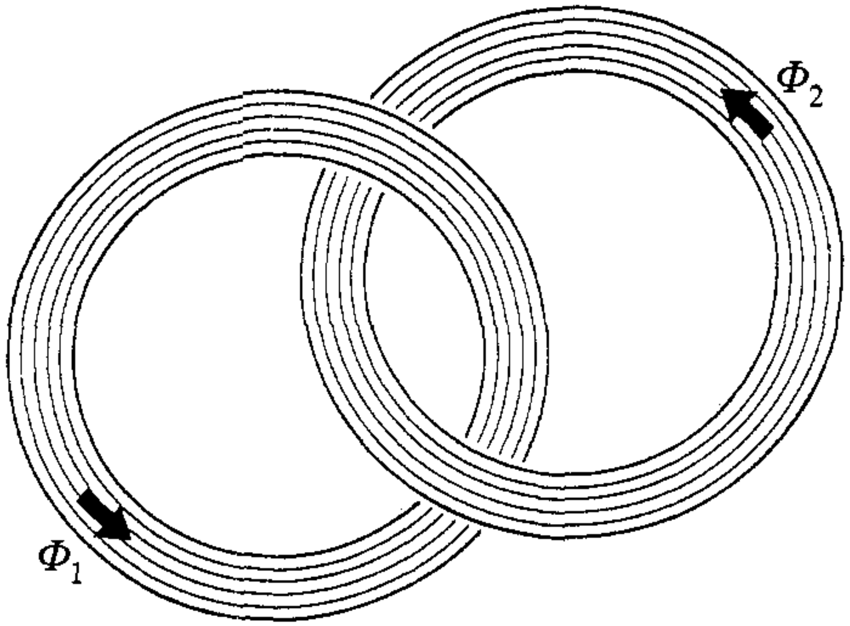
$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \quad \Rightarrow \quad \frac{\partial \mathbf{A}}{\partial t} = -\mathbf{E} + \nabla\phi$$

Topological interpretation of H.

1. Integral of linked magnetic flux

Moffatt (1969, J.Fluid.Mech.)

$$H = 2 \sum_{i<j} l(C_i, C_j) \Phi_i \Phi_j$$

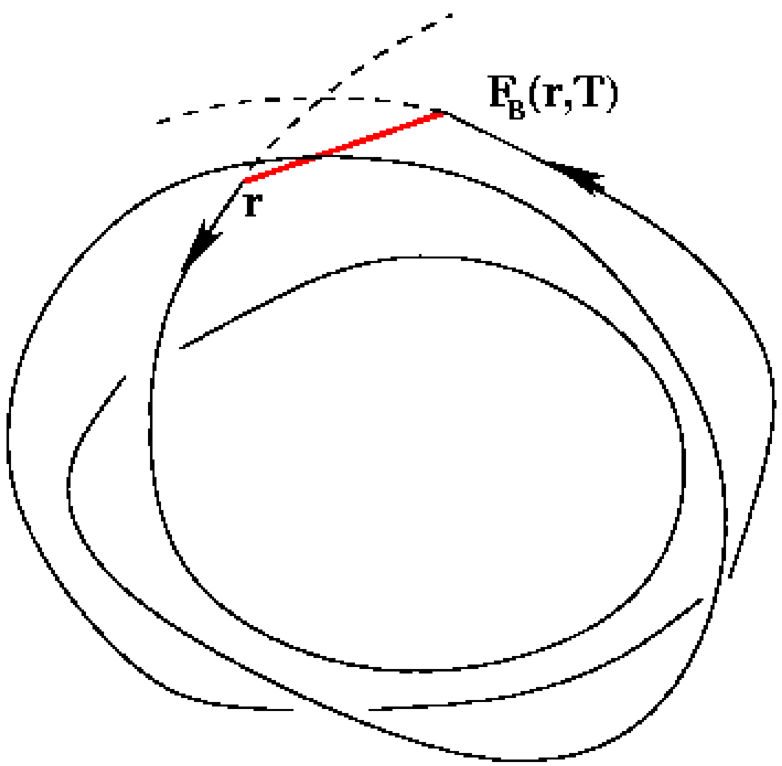


2. Asymptotic linking number

Arnol'd (1986, Sel.Math.Sov.)

$$H = \int_V \int_V \lambda(\mathbf{r}_1, \mathbf{r}_2)\, dV_1 dV_2$$

$$\lambda(\mathbf{r}_1, \mathbf{r}_2) = \lim_{T_1, T_2 \to \infty} \frac{l_A(\mathbf{r}_1, \mathbf{r}_2, T_1, T_2)}{T_1 T_2}$$



3. Algebraic topology definition Cantarella & Parsley (2012, J.Geom.Phys.)

Practical importance of H.

1. Almost invariant in **non-ideal** plasma $\quad \mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{j}$

Berger (1984, Geo.Astro.Fluid.Dyn.)

$$E = \frac{1}{2\mu_0}\int_V B^2\, dV \Rightarrow \frac{dE}{dt} = -\int_V \eta j^2\, dV$$

$$\left|\frac{dH}{dt}\right| \leq \sqrt{8\eta\mu_0\left|\frac{dE}{dt}\right|}$$

$$\frac{dH}{dt} = -2\int_V \mathbf{j}\cdot\mathbf{B}\, dV$$

2. H conservation determines end-state in reversed-field pinch. Taylor (1974, PRL)

3. Inverse cascade (to large scales) in turbulence.
Frisch et al. (1975, J.Fluid.Mech.)

4. Lower bound on magnetic energy. Arnol'd (1986, Sel.Math.Sov.)

$$E \geq \frac{1}{L_V}|H|$$

Ricca (2008, Proc.Roy.Soc.A) - explicit L_V for "magnetic links"

Limitations of H.

1. Not gauge invariant if $\mathbf{B}\cdot\mathbf{n}\big|_{\partial V} \neq 0$

The **relative helicity** Berger & Field (1984, J.Fluid.Mech.)

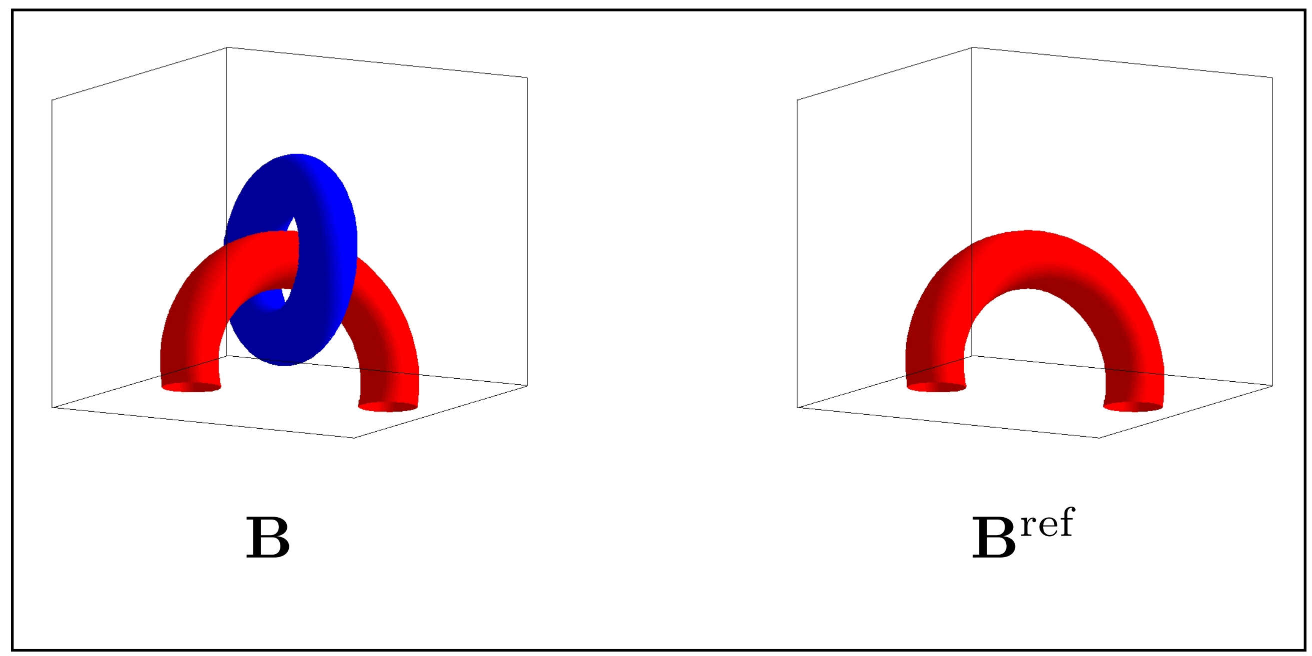
$$H_r = \int_V (\mathbf{A} + \mathbf{A}^{\mathrm{ref}})\cdot(\mathbf{B} - \mathbf{B}^{\mathrm{ref}})\,dV$$

where

$$\mathbf{B}\cdot\mathbf{n}\big|_{\partial V} = \mathbf{B}^{\mathrm{ref}}\cdot\mathbf{n}\big|_{\partial V}$$

and

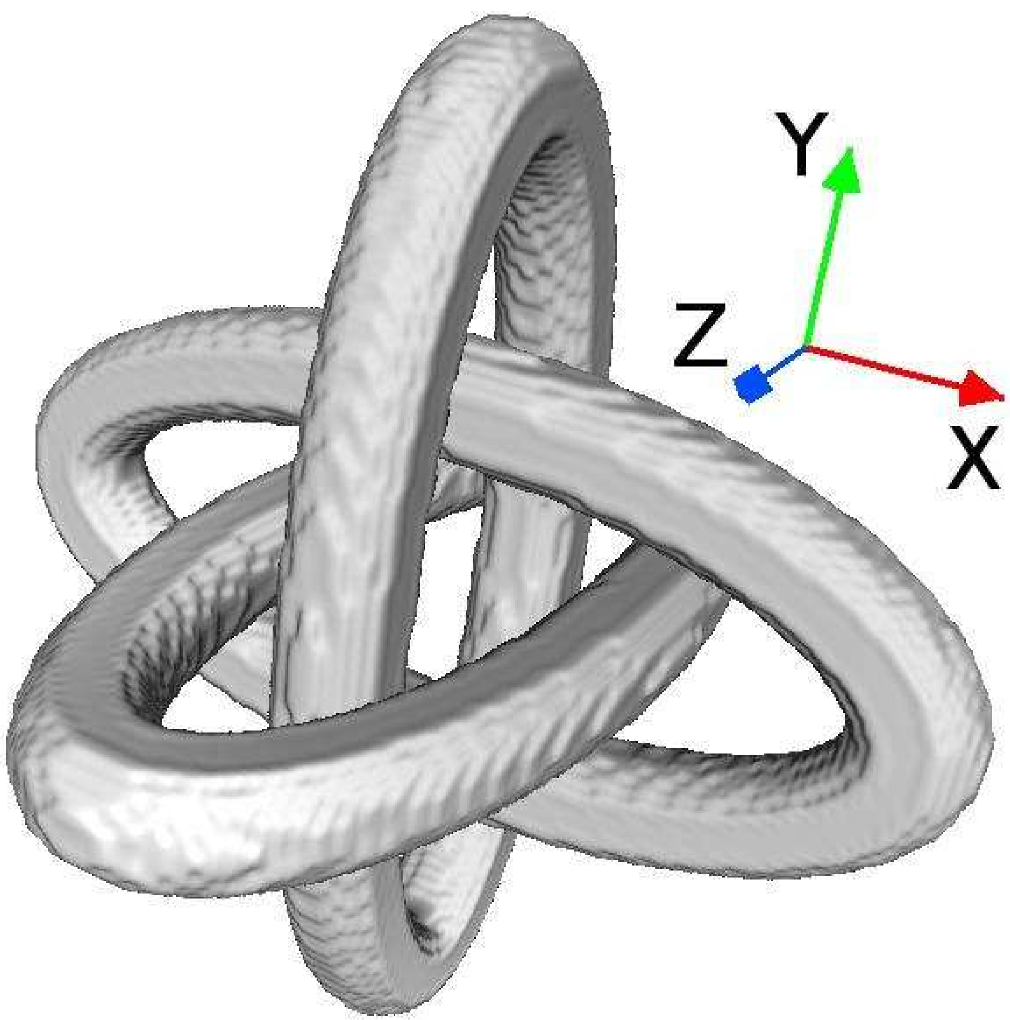
$$\mathbf{n}\times\mathbf{A}\big|_{\partial V} = \mathbf{n}\times\mathbf{A}^{\mathrm{ref}}\big|_{\partial V}$$

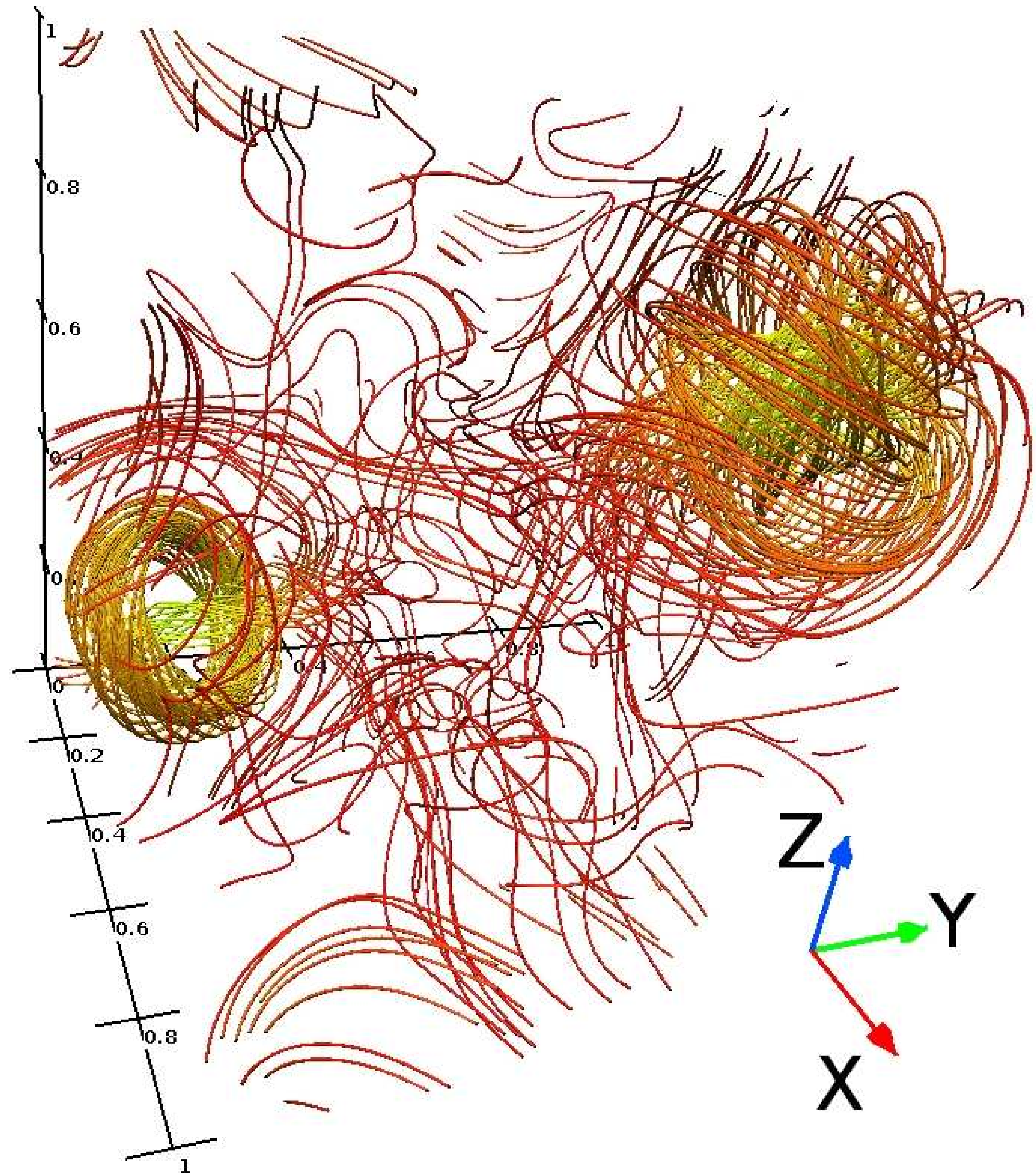


is gauge invariant and ideal invariant for $\mathbf{v}\big|_{\partial V} = 0$.

2. H/H_r may not be only relevant constraint...

e.g. Borromean rings Candelaresi & Brandenburg (2011, PRE)

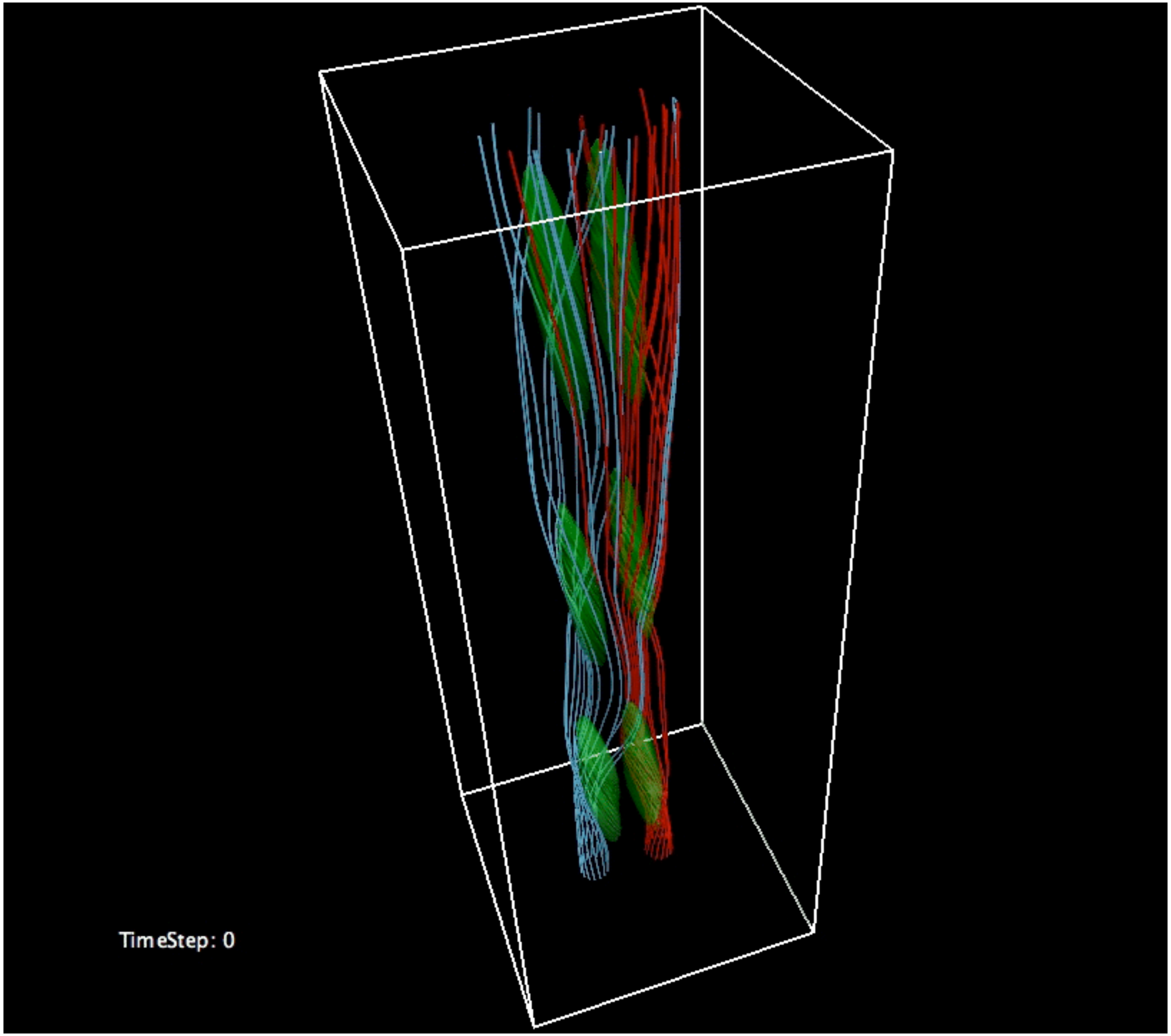




? Organises into two regions of opposite helicity.

e.g. Dundee numerical unbraiding experiment

Pontin et al. (2011, Astron.Astrophys)



? Organises into two tubes of opposite helicity.

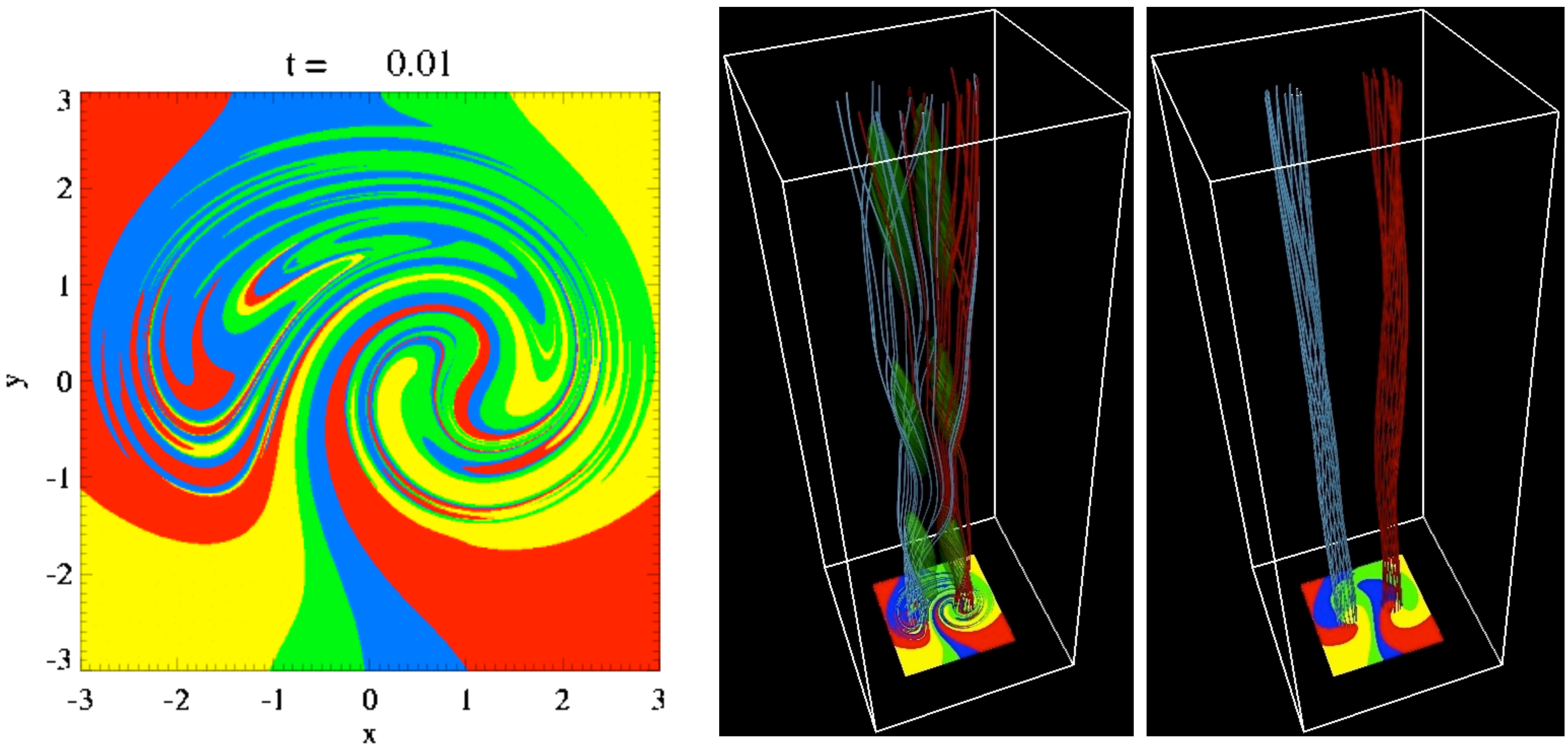
- Explanation of Dundee experiment -

Look at the field line mapping.

Total Poincaré index of fixed points (Lefschetz number) is invariant if boundary is ideal.

$\Rightarrow$ explains why two tubes.

Yeates, Hornig & Wilmot-Smith (2010, PRL)
Yeates & Hornig (2011, J.Phys.A)

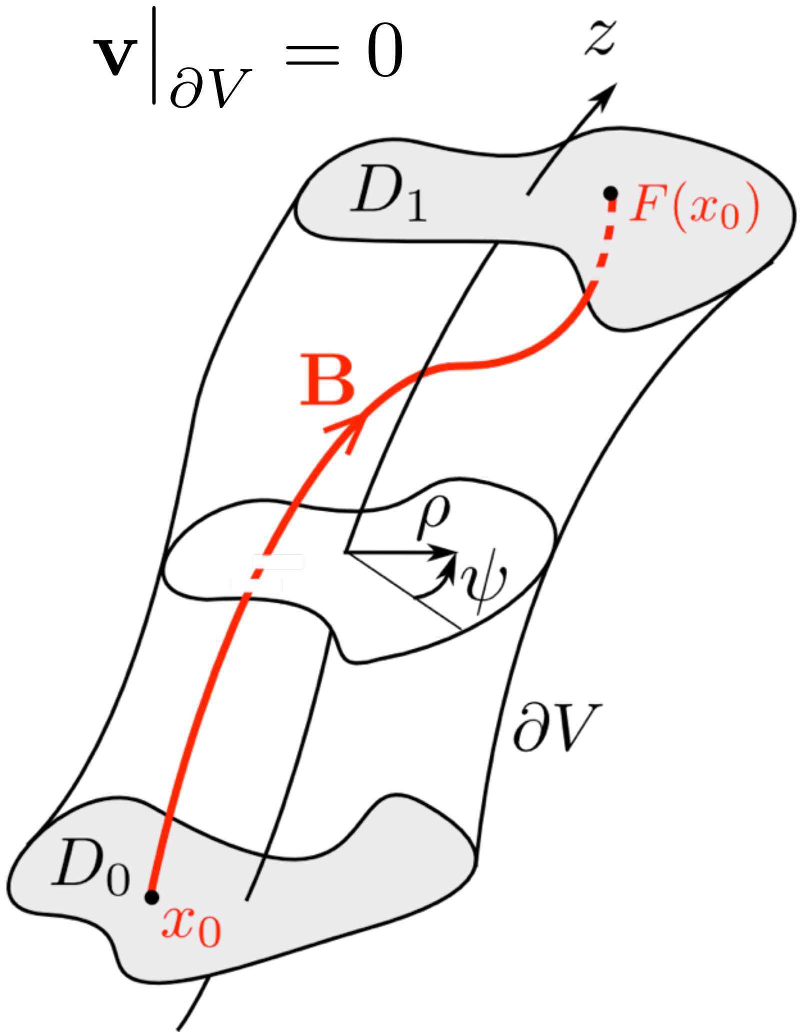


Demonstration of explicit invariant other than H.

- Topological Flux Function -

A scalar function can characterise the field line topology.

$\mathbf{v}\big|_{\partial V} = 0$



"Magnetic braid" (flux tube where $\mathbf{B} \neq 0$).

F is symplectic (preserves magnetic flux).

Topological flux function on D_0

$$\mathcal{A}(x_0) = \int_{x_0}^{F(x_0)} \mathbf{A}\big(f(x_0;z)\big)\cdot d\mathbf{l}$$

With gauge condition $\mathbf{n}\times\mathbf{A}\big|_{\partial V} = \mathbf{n}\times\mathbf{A}^{\mathrm{ref}}\big|_{\partial V}$ we get ideal invariance:

$$\frac{d\mathcal{A}}{dt} = \int_{x_0}^{F(x_0)}\left(\frac{\partial\mathbf{A}}{\partial t} - \mathbf{v}\times\nabla\times\mathbf{A} + \nabla(\mathbf{v}\cdot\mathbf{A})\right)\cdot d\mathbf{l} = \big(\phi + \mathbf{v}\cdot\mathbf{A}\big)_{x_0}^{F(x_0)}$$

The TFF is a "helicity per field line".

For our gauge condition,

$$H_r = \int_V \mathbf{A}\cdot\mathbf{B}\,dV - \underbrace{\int_V \mathbf{A}^{\mathrm{ref}}\cdot\mathbf{B}^{\mathrm{ref}}\,dV}_{H^{\mathrm{ref}}}$$

Changing coordinates gives

$$H_r - H^{\mathrm{ref}} = \int_V \mathbf{A}\big(f(x_0;z)\big)\cdot\mathbf{B}\left(f(x_0;z)\right)\frac{B_z(x_0)}{B_z\big(f(x_0;z)\big)}d^2x_0dz$$

$$= \boxed{\int_{D_0} \mathcal{A}(x_0)B_z(x_0)\,d^2x_0}$$

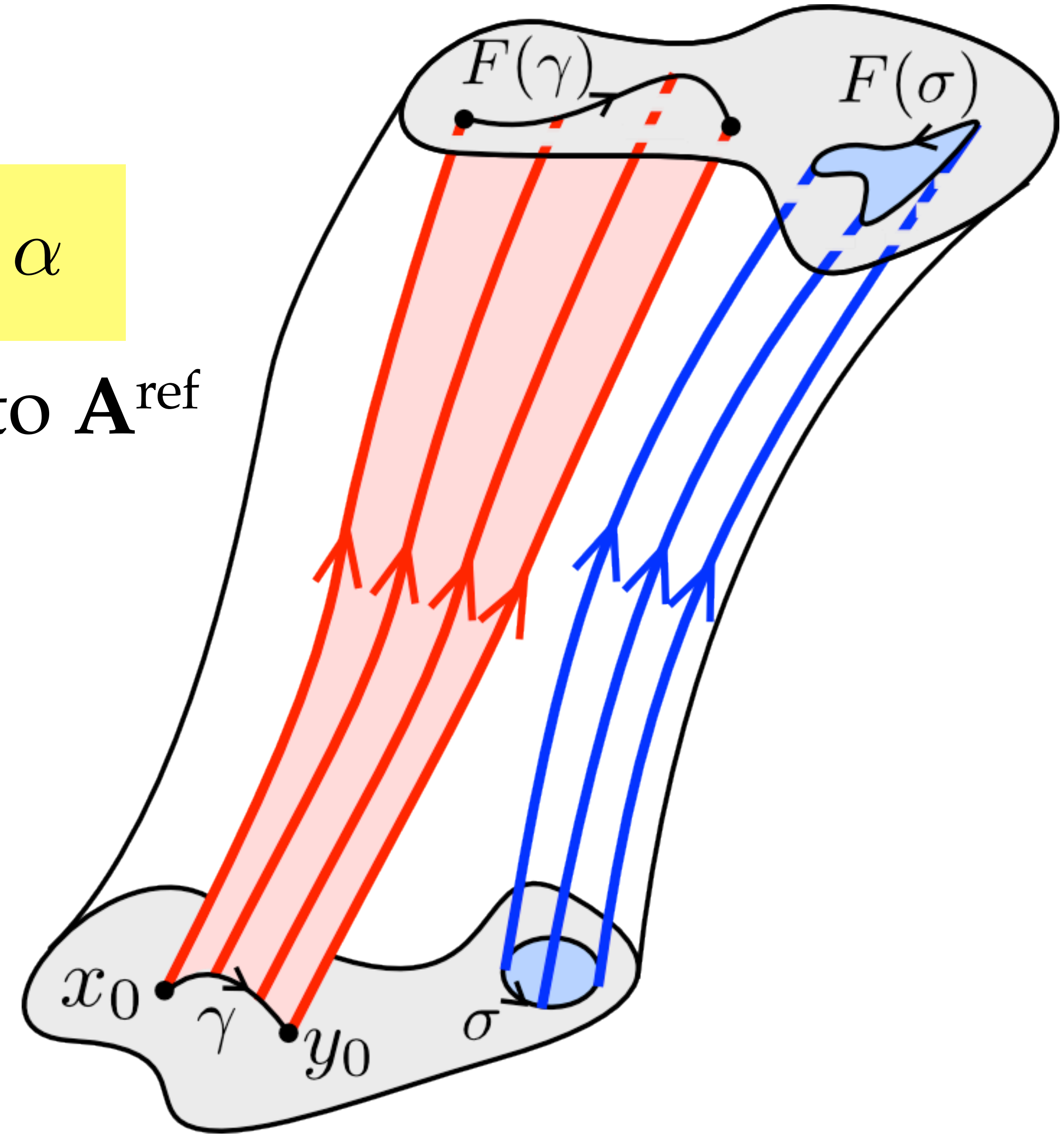
c.f. Berger (1988, Astron.Astrophys.) defines TFF as limiting helicity of infinitessimal tube around field line.

Why "topological"?

Can show that $\boxed{d\mathcal{A} = F^*\alpha - \alpha}$

where α is 1-form associated to $\mathbf{A}^{\text{ref}}$

$$\alpha = A_1^{\text{ref}}\,dx_1 + A_2^{\text{ref}}\,dx_2$$



Integrate along a curve γ

$$\int_\gamma d\mathcal{A} = \int_\gamma F^*\alpha - \int_\gamma \alpha$$

i.e.

$$\mathcal{A}(x_0) - \mathcal{A}(y_0) = \int_{F(\gamma)} \mathbf{A}^{\text{ref}} \cdot d\mathbf{l} - \int_\gamma \mathbf{A}^{\text{ref}} \cdot d\mathbf{l}.$$

- open curve: "field lines form a flux surface"
- closed curve: "conservation of vertical flux"

Theorem Yeates & Hornig (2013, Phys.Plasmas)

If two magnetic braids have field line mappings $F, \widetilde{F}$ that agree on ∂D_0 with the same winding number, then

$$\widetilde{\mathcal{A}} = \mathcal{A} \iff \widetilde{F} = F.$$

Idea of proof (hard direction):

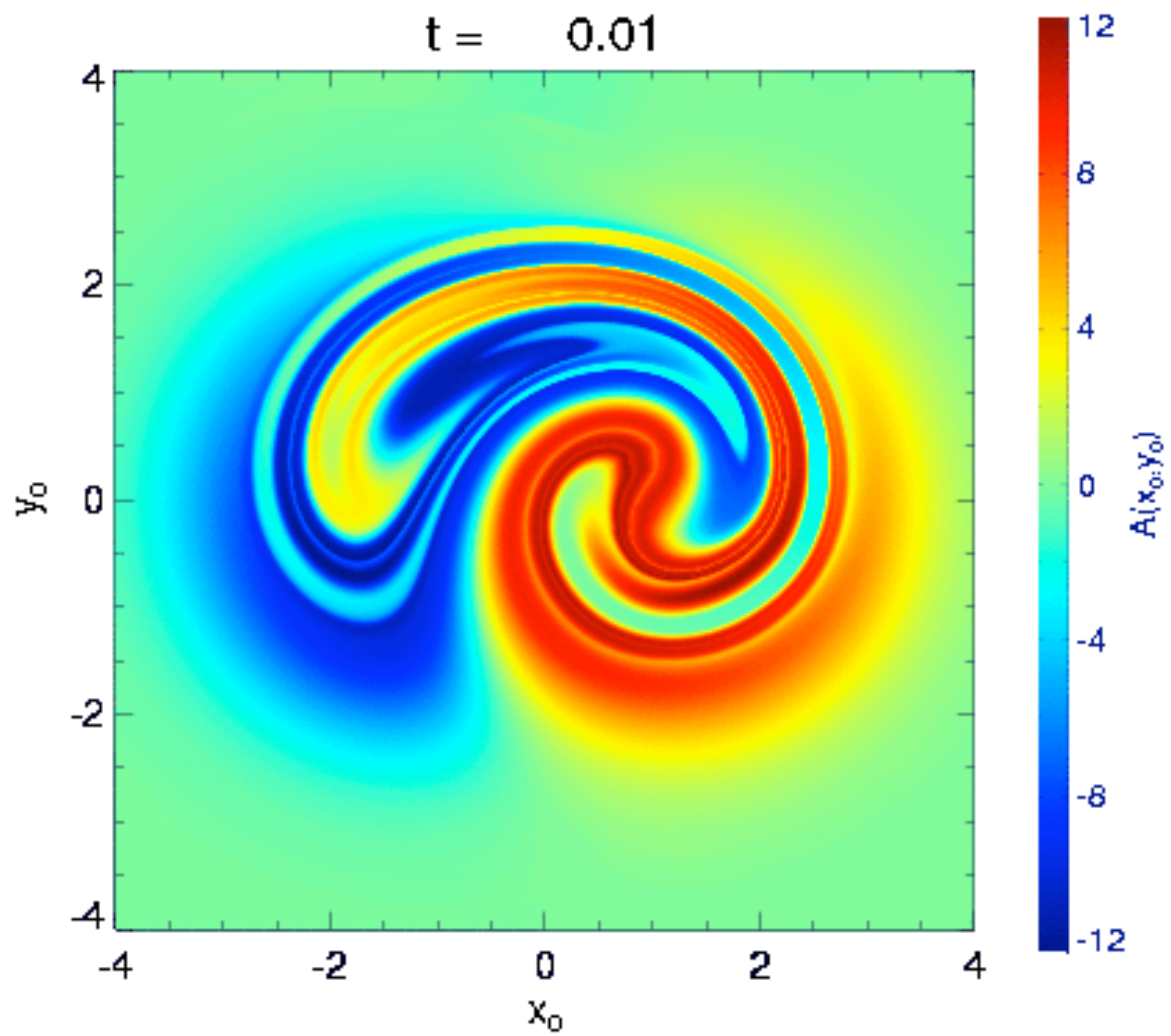
Assume $\widetilde{\mathcal{A}} = \mathcal{A}$ and let $G = \widetilde{F} \circ F^{-1}$ then

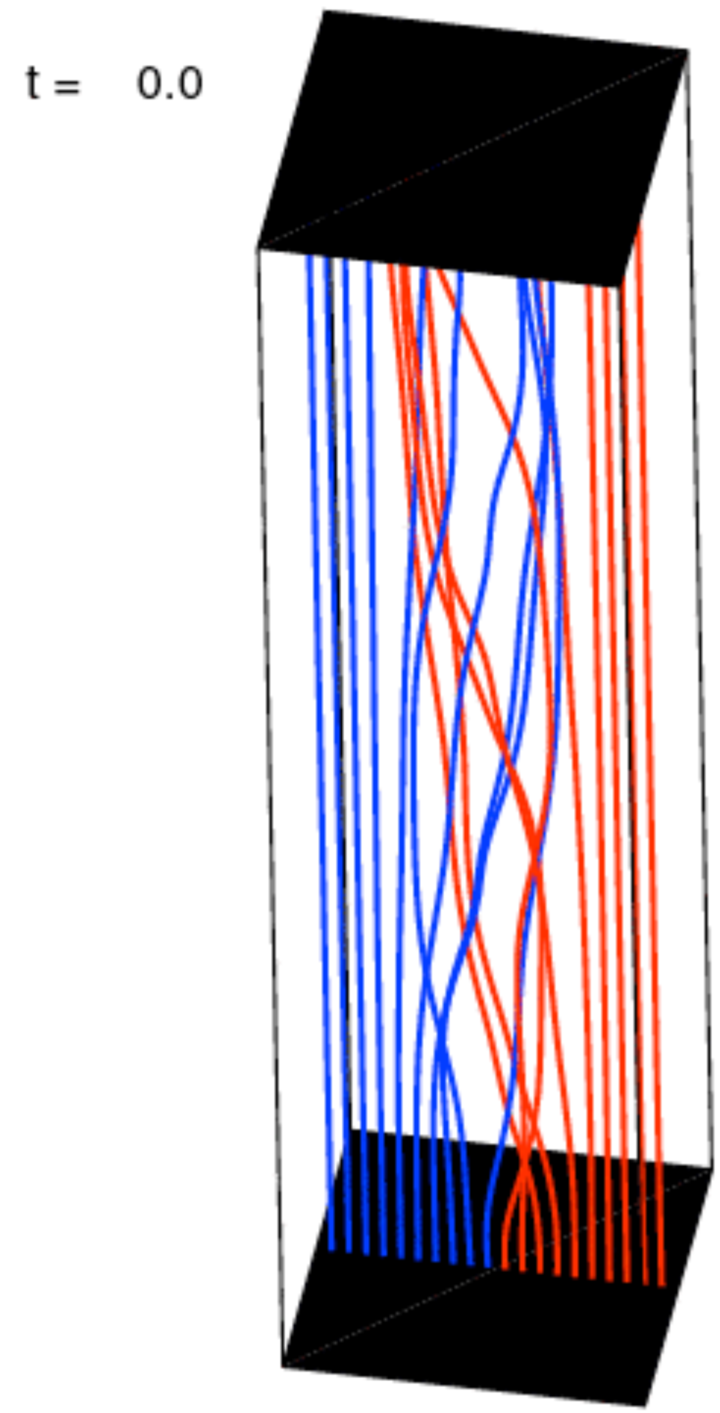
$$d\mathcal{A} = F^*\alpha - \alpha \quad \Rightarrow \quad G^*\alpha - \alpha = 0.$$

If we impose the additional gauge condition $A_1^{\rm ref} = 0$ then the maps preserving α are known to have certain form, and our boundary conditions lead to G=*id*. $\square$

In this gauge, $\mathcal{A}$ is the **action** for the system of field line equations in canonical Hamiltonian form.

e.g. Dundee experiment



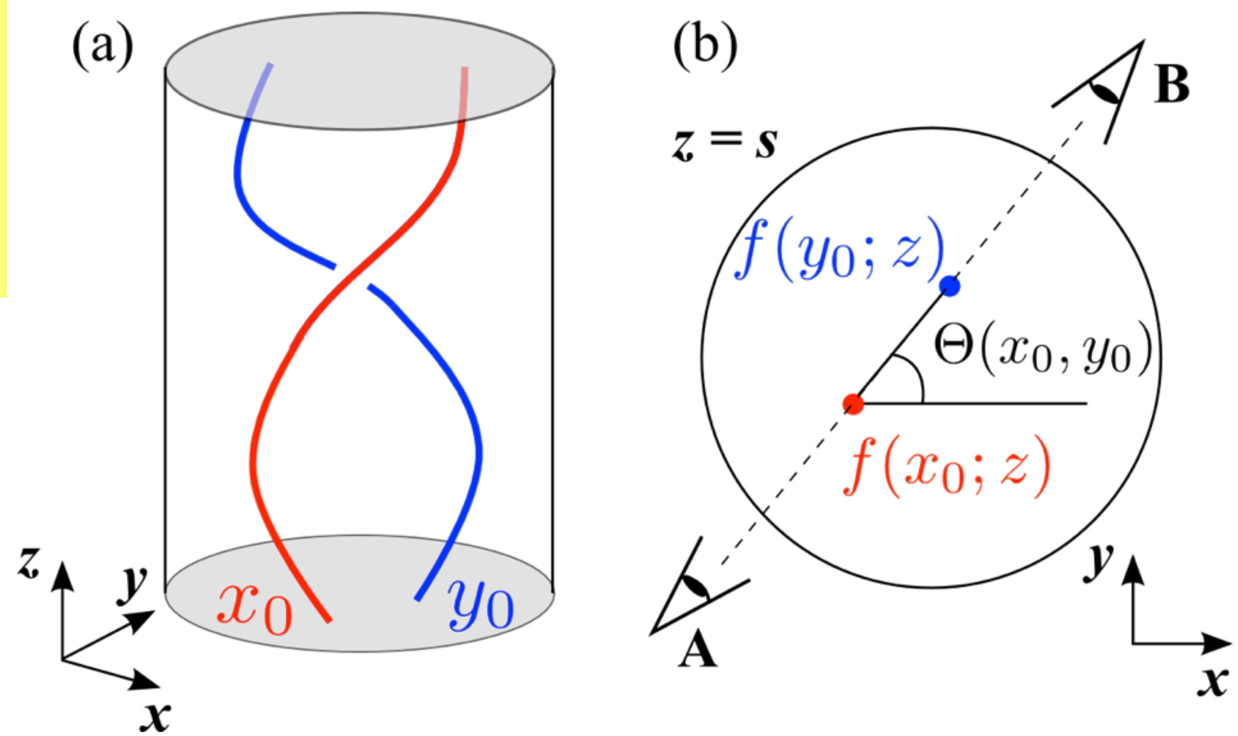


Why "flux function"?

$\mathcal{A}$ is average pairwise linking number Y & H (2013, Phys.Plasmas)

$$\mathcal{A}(x_0) = \int_{D_0} c_{x_0,y_0} B_z(y_0)\, d^2y_0$$

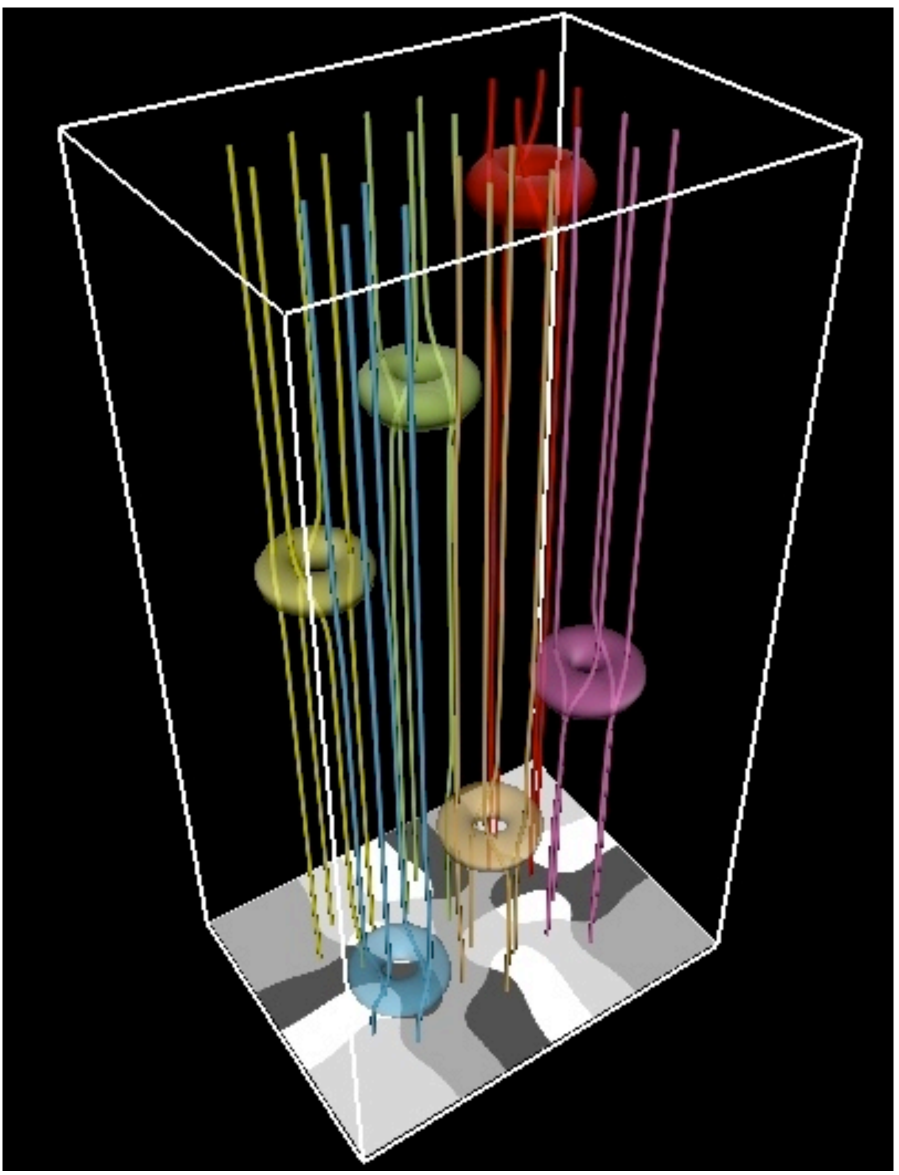
$$c_{x_0,y_0} = \frac{1}{2\pi} \int_{x_0}^{F(x_0)} \frac{d\Theta(x_0,y_0)}{dz}\, dz$$

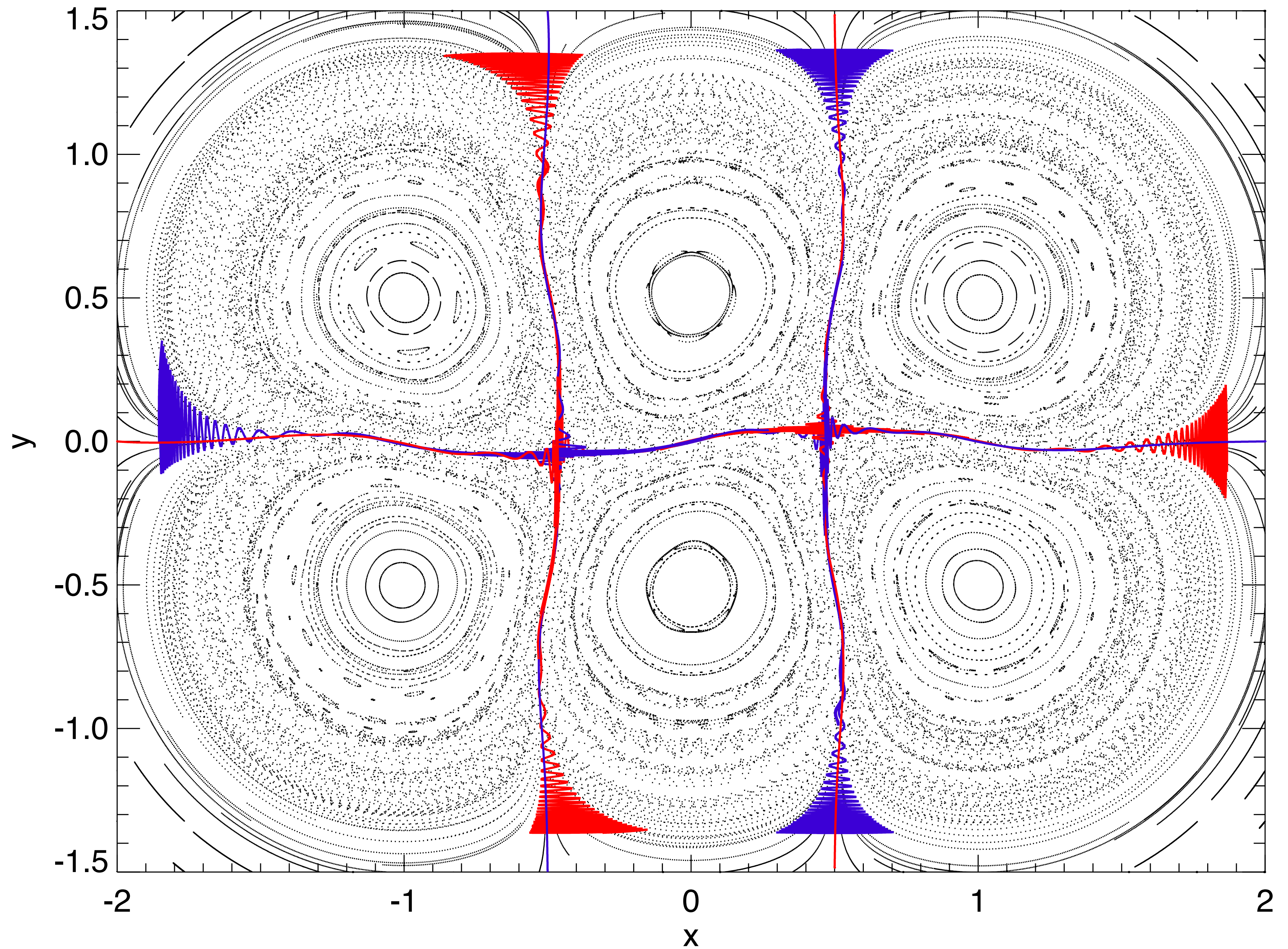


For any pair of field lines $c_{x0,y0}$ is a topological invariant
Berger (1986, Geo.Astro.Fluid.Dyn.)

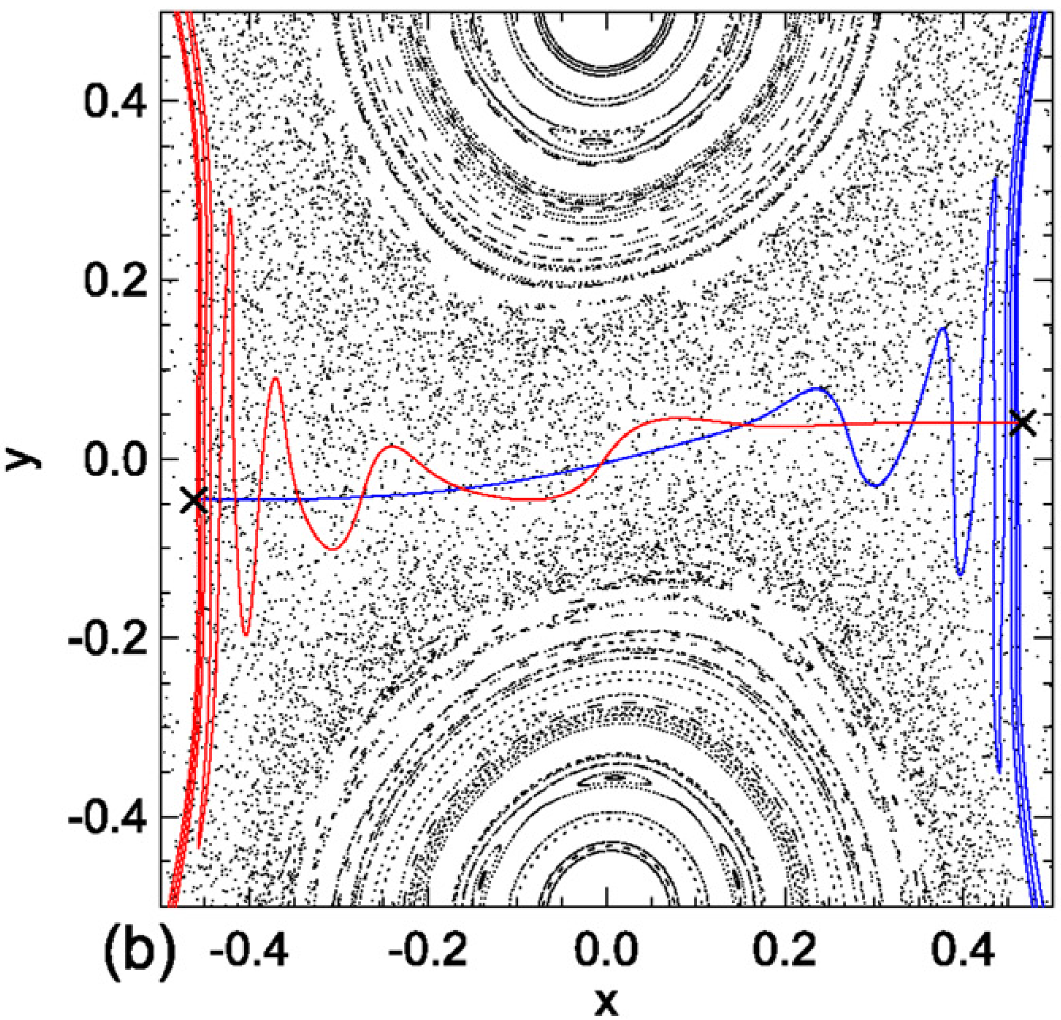
Not to be confused with unsigned crossing number
Freedman & He (1991, J.Fluid.Mech.), Berger (1993, PRL).

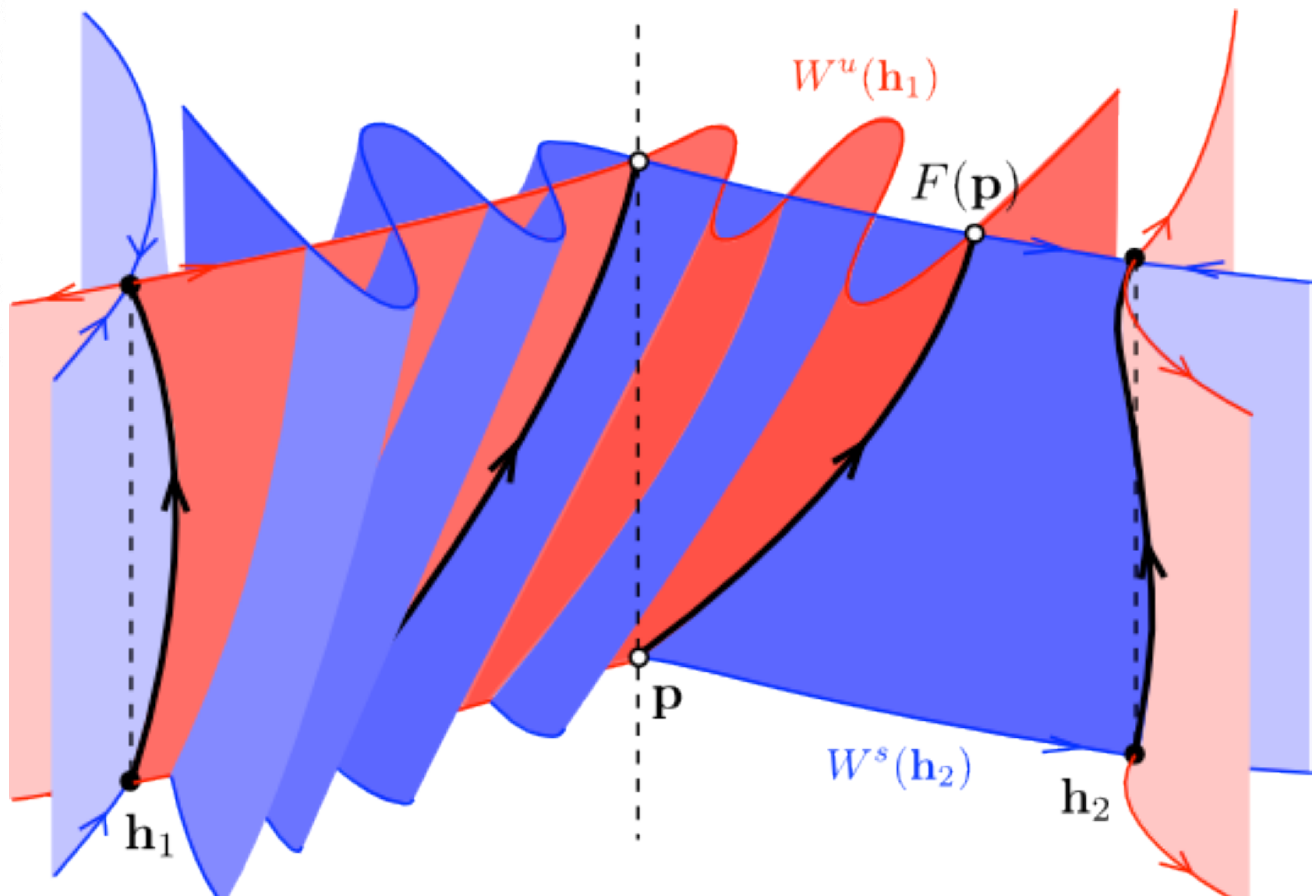
Can use values on fixed point field lines to partition flux (in periodic magnetic braid). Yeates & Hornig (2011, Phys.Plasmas)





Chaotic field lines ⇒ leakage of flux





Summary

1. Topology of magnetic field lines is important for their evolution.

2. Magnetic helicity is a well-known robust invariant.

3. Our work: complete invariant for "magnetic braids".

Yeates & Hornig (arXiv: 1304.8064) **- conference proceedings from ICMS workshop**

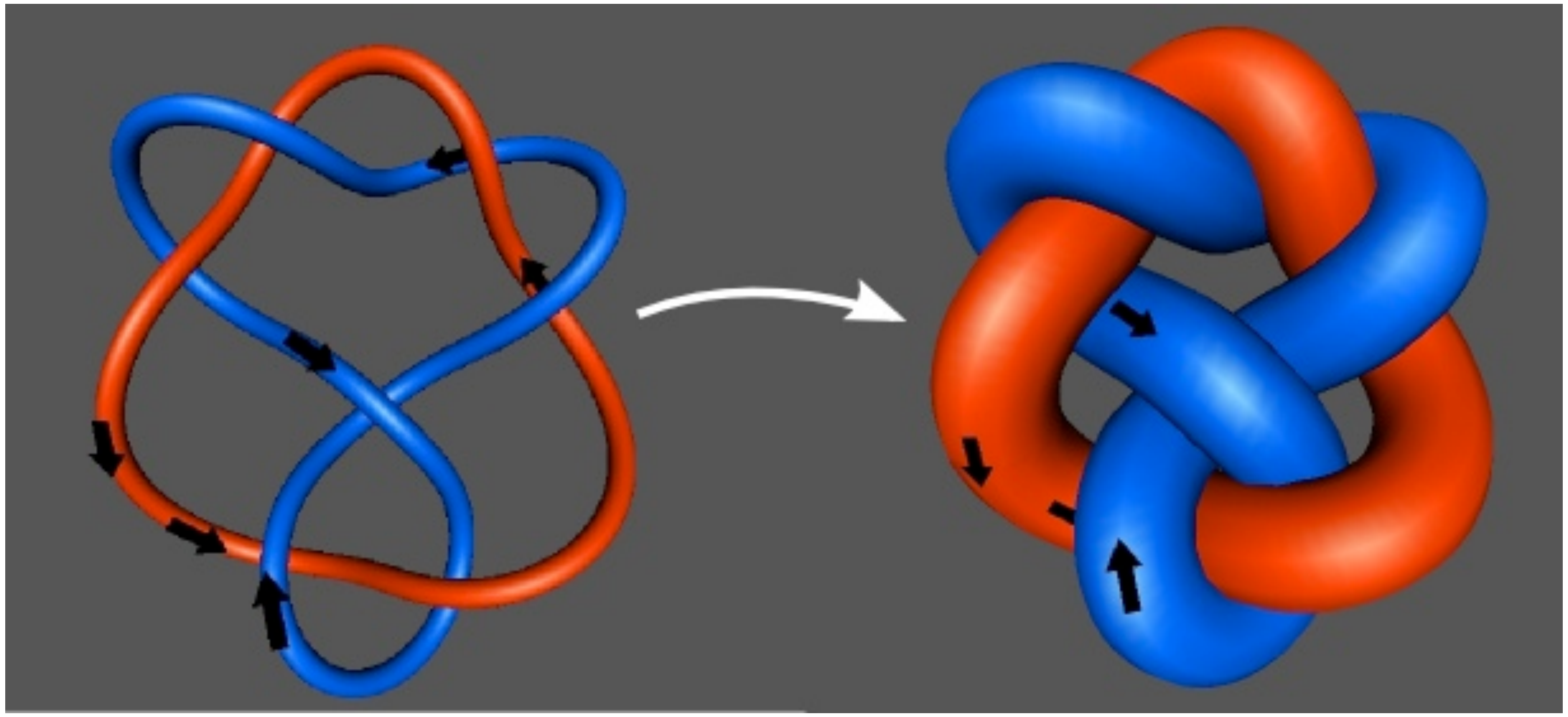
Ongoing/future work:

- Time evolution of the topological flux function (**non-ideal MHD**).
    - Measuring rate and location of **reconnection**.
    - Identifying other robust properties.

http://www.maths.dur.ac.uk/~bmjg46/

Durham University

Moffatt (1990, Nature) *define* a link invariant via ideal-MHD relaxation of a **magnetic link**.



Use bounds for magnetic energy in terms of asymptotic *unsigned* crossing number (not invariant but has lowest value).

Freedman & He (1991, J.Fluid.Mech.)
Ricca (2008, Proc.Roy.Soc.A)