Problems 0 - Polynomial interpolation

Approximation Theory (MATH3081/4221) — Epiphany 2015 — anthony.yeates@dur.ac.uk

The problem marked $\star$ should be handed in for marking at the lecture on **Thursday 22nd January**. I use † to indicate (what I consider to be) trickier problems.

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1. *Interpolation error.*

   (a) Write down the Lagrange interpolation polynomial of degree 1 for the function $f(x) = x^3$, using the points $x_0 = 0$, $x_1 = a$.

   (b) Verify Theorem 0.1 by direct calculation, and show that in this case ξ is unique and has the value $\xi = \frac{1}{3}(x + a)$.

2. *Equally-spaced versus Chebyshev nodes.* Consider the problem of finding a degree 4 polynomial interpolant $p_4(x)$ for the function $f(x) = e^x$ on the interval $[-1, 1]$.

   (a) Suppose we choose the equally-spaced nodes $-1, -\frac{1}{2}, 0, \frac{1}{2}, 1$. Find an upper bound for the difference between f and p_4 at (i) $x = \frac{1}{4}$ and (ii) $x = \frac{3}{4}$.

   (b) Without computing the locations of the Chebyshev nodes, find an upper bound on the error in the Chebyshev interpolant q_4, and comment on how this compares to part (a).

$\star$ **3.** *Chebyshev nodes.* Consider the problem of fitting a linear polynomial to the function $f(x) = x^2$ on $[-1, 1]$.

   (a) Find the Chebyshev nodes and the linear polynomial p_1 that interpolates f at these nodes.

   (b) Compute $\|f - p_1\|_\infty$. Is it possible to find a linear polynomial q_1 (not necessarily interpolating f at the same nodes) such that $\|f - q_1\|_\infty$ is any smaller?

   (c) Find the linear polynomial that interpolates $g(x) = x^3$ at the same Chebyshev nodes. Is this the polynomial that minimises $\|g - p_1\|_\infty$ among all $p_1 \in \mathcal{P}_1$? Explain why or why not.

4. *Barycentric formulae.*

   (a) Let $\lambda_i = 1/w'_{n+1}(x_i)$, where w_{n+1} is the usual error polynomial from Theorem 0.1 and x_i for $i = 0, \ldots, n$ are the interpolation nodes. Show that the polynomial interpolating f may be written as

   $$p_n(x) = w_{n+1}(x) \sum_{i=0}^{n} \frac{\lambda_i}{x - x_i} f_i.$$

   Remark: This was derived by Jacobi in his 1825 PhD thesis. Once the weights λ_i are known, it allow you to compute each value $p_n(x)$ with only $O(n)$ operations.

   † (b) Show that $p_n(x)$ may alternatively be written as the *barycentric interpolation formula*

   $$p_n(x) = \sum_{i=0}^{n} \frac{\lambda_i f_i}{x - x_i} \Bigg/ \sum_{i=0}^{n} \frac{\lambda_i}{x - x_i}.$$

   Remark: This formula is an efficient and stable way to compute Chebyshev interpolants (although it should not be used at the nodes themselves because both numerator and denominator are infinite).