Problems 1 - Piecewise Polynomial Interpolation

Approximation Theory (MATH3081/4221) — Epiphany 2015 — anthony.yeates@dur.ac.uk

The problem marked ⋆ should be handed in for marking at the lecture on **Thursday 5th February**.
There will be a problem class on this chapter on Monday 2nd February.
I use † to indicate (what I consider to be) trickier problems.

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5. *General splines.* An interpolating spline of degree N is required to have continuous derivatives up to and including order $N-1$. How many additional conditions are required to specify the spline uniquely?

6. *Flatness of linear splines.* Let $f(x) = x^3$.

   (a) Compute the linear spline s which interpolates f at the knots 0, 1, and 2.

   (b) Compute the quadratic polynomial p_2 which interpolates f at the same knots.

   (c) Verify that $\|s'\|_2 \leq \|f'\|_2$ and $\|s'\|_2 \leq \|p_2'\|_2$, where the norm is defined on the interval $[0,2]$.

7. *Quadratic splines.* Let $a = x_0 < x_1 < \ldots < x_n = b$ be a sequence of equally spaced knots on an interval $[a,b]$, and let $s \in C^1[a,b]$ be a quadratic spline that interpolates a function f at the knots.

   (a) Define the moments $M_i = s'(x_i)$ for $i = 0, \ldots, n$. Construct the spline s in terms of these M_i and derive a system of linear equations for the M_i. For what values of i must they hold? How many extra conditions are necessary?

   † (b) If the function to be interpolated is periodic, we might try to introduce the extra condition $s'(a) = s'(b)$. Show, by considering the resulting system of equations, that this "periodic" quadratic spline exists for certain conditions on the knots.

   (c) Use the above to approximate $f(x) = \sin(2\pi x)$ on the interval $[0,1]$ with four knots.

8. *Is it a spline?* Is the following function a cubic spline? Why or why not?

$$s(x) = \begin{cases} 0, & x < 0, \\ x^3, & 0 \leq x < 1, \\ x^3 + (x-1)^3, & 1 \leq x < 2, \\ -(x-3)^3 - (x-4)^3, & 2 \leq x < 3, \\ -(x-4)^3, & 3 \leq x < 4, \\ 0, & 4 \leq x. \end{cases}$$

9. *Natural cubic spline.* Compute the natural cubic spline interpolating $(0,0)$, $(1, \frac{1}{2})$, $(2,0)$.

⋆ **10.** *Not-a-knot cubic spline.* Let s be a cubic spline interpolating a function f at the evenly-spaced knots $a = x_0 < x_1 < \ldots < x_n = b$, with spacing h, and suppose that s satisfies the so-called "not-a-knot" conditions that s''' is continuous at the two knots x_1 and x_{n-1}.

   (a) Derive the system of linear equations satisfied by the moments $M_i := s''(x_i)$ for $n+1$ knots.

   (b) Suppose we try to find the not-a-knot cubic spline through the data $(0,0)$, $(1,1)$ and $(2,8)$. Write down the system of linear equations in this case, show that the solution is not unique, and write the resulting spline in terms of M_0 as a free parameter.

   (c) Explain why you expected the solution in (b) to be non-unique.

   (d) By considering the two pieces of the cubic spline derived in (b), or otherwise, explain why this type of cubic spline is called "not-a-knot".

† 11. By reducing the linear system found in Problem 10(a) to a strictly diagonally dominant form, or otherwise, show that there is a unique not-a-knot cubic spline for $n \geq 3$.

12. *Holladay's Theorem for complete cubic splines.* The cubic spline interpolating the points (x_0, f_0), (x_1, f_1), $\ldots (x_n, f_n)$ and satisfying the end conditions $s'(x_0) = c$, $s'(x_n) = d$ for fixed constants c, d is known as the *complete* cubic spline. Adapt the proof of Theorem 1.3 to show that the *complete* cubic spline minimises $\|f''\|_2$ among all functions $f \in C^2[x_0, x_n]$ that satisfy $f'(x_0) = c$ and $f'(x_n) = d$.

13. *General B-splines.* The B-spline of degree N for equally-spaced knots is defined as

$$B_N(x) = \sum_{k=0}^{N+1} (-1)^k \binom{N+1}{k} (x - kh)_+^N,$$

where h is the knot spacing, the subscript $+$ means the *positive part*

$$f(x)_+ = \begin{cases} f(x), & f(x) > 0, \\ 0, & f(x) \leq 0, \end{cases}$$

and

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

is the usual binomial coefficient.

(a) Find and sketch the linear B-spline $B_1(x)$ and verify that it is indeed a linear spline function.

(b) Find the cubic B-spline $B_3(x)$ and verify that it is the same function as given in the lecture.

† (c) Show that the function $B_N(x)$ is always a spline of degree N.

14. *Applying linear B-splines.* Any linear spline may be expressed as

$$s(x) = \sum_{i=0}^{n} \alpha_i \phi_i(x)$$

where the basis functions are linear B-splines $\phi_i(x) = h^{-1} B_1(x - x_{i-1})$, with B_1 as defined in Problem 13. For the data $(0, 1)$, $(1, 2)$, $(2, 3)$ and $(3, 4)$, write down and sketch the linear spline basis functions $\phi_k(x)$, and hence form the linear spline $s(x)$. Use this to evaluate the spline at $x = \frac{3}{2}$.

15. *Applying cubic B-splines.* If cubic B-splines are used to compute the *complete* cubic spline that interpolates the function $f(x) = \sin(\pi x/2)$ at the knots $x_0 = 0$, $x_1 = 1$ and $x_2 = 2$, find the linear system that has to be solved.