Problems 2 - Minimax Approximation

Approximation Theory (MATH3081/4221) — Epiphany 2015 — anthony.yeates@dur.ac.uk

The problem marked $\star$ should be handed in for marking at the lecture on **Thursday 19th February**.
There will be a problem class on this chapter on Monday 16th February.
I use † to indicate (what I consider to be) trickier problems.

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16. *Bernstein polynomial approximation.* Compute the approximations using Bernstein polynomials of degree $n = 1$ and $n = 2$ to the function $f(x) = 1 - \left|x - \frac{1}{3}\right|$ on $[0, 1]$. Verify that the approximation is converging in the ∞-norm.

17. *Recursive definition of Bernstein polynomials.* Let $b_{n,k}$ for $k = 0, \ldots, n$ be the Bernstein basis functions, as defined in the lecture. Show that these basis functions satisfy the recursion relation

$$b_{n,k}(x) = (1 - x)b_{n-1,k}(x) + xb_{n-1,k-1}(x).$$

Remark: This is the basis of de Casteljau's fast algorithm for drawing Bézier curves.

18. *Derivatives of Bernstein polynomials.* Show that the derivatives of the Bernstein basis functions $b_{n,k}(x)$ for $k = 0, \ldots, n$ satisfy

$$\frac{d}{dx}b_{n,k}(x) = n\Big(b_{n-1,k-1}(x) - b_{n-1,k}(x)\Big).$$

19. *Cubic Bézier curves.* Verify that the cubic Bézier curve $\mathbf{B}_3(t)$ with control points $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ is tangent (i) at $\mathbf{x}_0$ to the line joining $\mathbf{x}_0$ and $\mathbf{x}_1$, and (ii) at $\mathbf{x}_3$ to the line joining $\mathbf{x}_2$ and $\mathbf{x}_3$.

20. *A Bézier curve.* Find the parametric equations of the Bézier curve with control points $(0, 1)$, $(\frac{1}{5}, \frac{3}{2})$, $(\frac{3}{5}, 2)$ and $(1, 0)$. Find the slope of the curve at each of its end-points and make a rough sketch of the curve.
As a check, you could try drawing the curve in Postscript.

21. *Minimax approximation.* Find the minimax linear approximation to $f(x) = \sinh(x)$ on $[0, 1]$.

22. *Minimax approximation to a polynomial.* Find the minimax approximation of degree 4 to the polynomial $f(x) = x^5 + 2x^2 - x$.

23. *Non-monic polynomials.* Prove that, if p_m^* is the minimax polynomial of degree m for a polynomial $f \in \mathcal{P}_{m+1}$, then αp_m^* is the minimax approximation for αf.

$\star$ **24.** *De la Vallée Poussin Theorem.* Let $f(x) = -\cos(x)$ and $q_1(x) = 0.5x - 1.1$.

(a) Show that $\{0, \frac{1}{2}, 1\}$ is a non-uniform alternating set for f and q_1 on $[0, 1]$.

(b) Use the De la Vallée Poussin Theorem with these points to find a lower bound for $\|f - p_1^*\|_\infty$, where p_1^* is the minimax degree 1 polynomial for f on $[0, 1]$.

(c) Use q_1 to find an upper bound for $\|f - p_1^*\|_\infty$.

(d) By postulating a suitable alternating set, or otherwise, find p_1^*.

25. *The Equioscillation Theorem.* In light of the Chebyshev Equioscillation Theorem, explain why the function $q_1(x)$ in Problem 24 could not possibly be the minimax degree 1 polynomial.

26. *Every minimax polynomial is an interpolant.* Let $p_n^* \in \mathcal{P}_n$ be a minimax approximation to $f \in C[a,b]$. Show that there exist $n+1$ distinct points $a < x_0 < x_1 < \ldots < x_n < b$ such that p_n^* is the polynomial interpolant in $\mathcal{P}_n$ to f at these $n+1$ points.

† 27. *Minimax polynomials of even functions.* Let $f \in C[-1,1]$ be even, i.e. $f(-x) = f(x)$.

   (a) Use the Equioscillation Theorem to prove that the minimax polynomial p_n^* is even for any $n \geq 0$.

   (b) Prove that for any $n \geq 0$, $p_{2n}^* = p_{2n+1}^*$.

   (c) Find the minimax polynomial of degree 1 for $f(x) = |x|$ on $[-1,1]$.

28. *Remez algorithm.* Use the Remez Exchange algorithm to compute the linear minimax approximation to $f(x) = x^2$ on $[0,3]$, using the initial reference set $\{0,1,3\}$. Comment on the convergence of the algorithm.